Electromagnetic Mass Splittings of the Baryon Octet*

S. L. COHEN AND C. R. HAGEN
Department of Physics and Astronomy, University of Rochester, Rochester, New York
(Received 12 December 1966)

It is assumed that the electromagnetic mass splittings within the baryon octet can be self-consistently calculated from the one-photon elastic form-factor contribution and the electromagnetic mass differences within the pseudoscalar-meson octet. The baryon mass differences are shown to be expressible as functions of three parameters: a cutoff Λ, the f/d ratio, and the pion-nucleon coupling constant. A unique set of real values for these parameters is found which correctly predicts both the sign and magnitude of all the electromagnetic baryon mass splittings. The pion-nucleon coupling constant is found to be equal to 16.7, in good agreement with experiment.

I. INTRODUCTION

THE problem of calculating the electromagnetic mass splittings within isotopic-spin multiplets has been of interest since charge independence was first proposed. Although one generally expects the charged components of an isomultiplet to be more massive than the neutral component because of the inertia of the electrostatic field of a charged particle, there exist several examples where this is not the case, e.g., the p-n, Σ^+-Σ^0, and $K^\pm$-K^0 mass differences. It will be convenient to refer to this latter situation as a sign reversal.

The early attempts to account for the electromagnetic mass differences involved calculating the mass splittings within each isomultiplet independently, assuming that purely electromagnetic forces generate the observed mass splittings. Using a one-photon-exchange approximation to the elastic form factor merely confirms the classical result stated above. Since the elastic form factor is always positive and is coupled only to the charged component, one necessarily obtains the wrong signs for mass splittings such as $\delta N=p-n$. However, it was pointed out be Feynman and Speisman[1] that if one takes into account the anomalous magnetic moment of the proton and neutron, it is possible to arrive at the correct sign for δN. These authors assumed that the proton and neutron could be described by a Dirac equation with a Pauli moment term, which has the effect of modifying the expression for the proton self-energy by the addition of a term opposite in sign to the elastic-form-factor contribution. Since this latter effect is quadratically divergent (in contrast to the logarithmic divergence of the elastic-form-factor contribution) it is clear that, for a sufficiently large cutoff, one will find $\delta N<0$. (Feynman found that a cutoff of 1.3 nucleon masses gave the correct sign and magnitude.) However, Sunakawa and Tanaka[2] showed that if one uses the experimentally observed electric and magnetic form factors (assuming that they can be smoothly extrapolated to high momentum transfer) instead of a cutoff in the Feynman expression, one obtains $\delta N>0$.[3]

It has thus become increasingly evident that not only the purely electromagnetic effects (which we will refer to as the "driving forces") but also the electromagnetic corrections to strong interactions (acalled the "feedback") must be included in order to understand the sign reversals.

The crucial interplay between driving force and feedback terms has been discussed by Barton and Dare[4] within the framework of a simple potential model. They assume that the unperturbed system is described by a wave function ψ satisfying the Schrödinger equation with an eigenvalue-dependent potential

$$[-\nabla^2+V(E,\mathbf{x})]\psi(\mathbf{x})=E\psi(\mathbf{x}). \qquad (1)$$

Upon applying a small perturbation δV, the energy eigenvalue is shifted by an amount δE given by

$$\delta E=\frac{\langle\psi|\delta V|\psi\rangle}{1-\langle\psi|\partial V/\partial E|\psi\rangle} \qquad (2)$$

to lowest order. In this equation, the numerator represents the driving force while $\langle\psi|\partial V/\partial E|\psi\rangle$ represents the feedback contribution. A sign reversal can thus occur if (a) $\langle\psi|\delta V|\psi\rangle<0$ and $\langle\psi|\partial V/\partial E|\psi\rangle<1$, which is called a sign reversal of the driving type, or (b) $\langle\psi|\delta V|\psi\rangle>0$ and $\langle\psi|\partial V/\partial E|\psi\rangle>1$, which is called a sign reversal of the feedback type. Barton[4] points out, however, that if condition (b) occurs in an elementary-particle theory, δE will go through a singularity as the potential V is turned off (i.e., the strong coupling constant is allowed to decrease to zero) because the term $\langle\psi|\partial V/\partial E|\psi\rangle$ must become equal to unity at some point. This is physically repugnant because it implies that the mass shift δE, as a function of the strong coupling constant g^2, has a pole at some value of g^2 less then g_s^2 (g_s being the physical value of the strong coupling constant) and a power series in g^2 for δE about $g^2=0$ cannot be expected to give acceptable results beyond the singularity. Thus considerable doubt is cast on the validity of any calculation involving a sign reversal of

* Research supported in part by the U. S. Atomic Energy Commisssion.

[1] R. P. Feynman and G. Speisman, Phys. Rev. **94**, 500 (1954).

[2] S. Sunakawa and K. Tanaka, Phys. Rev. **115**, 754 (1959).

[3] One might still get $\delta N<0$ by assuming the existence of a hard core at high momentum transfer. Cf. M. Cini, E. Ferrari, and R. Gatto, Phys. Rev. Letters **2**, 7 (1959); H. Katsumari and M. Shimada, Phys. Rev. **124**, 1203 (1961); A. I. Soloman, Nuovo Cimento **27**, 748 (1962).

[4] G. Barton and D. Dare, Phys. Rev. **150**, 1220 (1966).

the feedback type in a single-channel model (an argument which we shall henceforth refer to as Barton's criticism).

It must be emphasized, however, that this argument only holds in the single-channel case considered by Barton. In the multichannel case, the situation is considerably more complicated as we now demonstrate. Let us assume that there are several isospin multiplets involved which are described by a multicomponent wave function ψ_i. Then the coupled Schrödinger equation with an energy-dependent potential is

$$[-\nabla^2\delta_{ij}+V_{ij}-E^{\alpha'}\delta_{ij}]\psi_j^{\alpha'}=0\,, \tag{3}$$

where the summation convention on repeated indices applies to unprimed indices but *not* to primed indices. Multiplication by $\psi_i^{\alpha'*}$ gives

$$\psi_i^{\alpha'*}[-\nabla^2\delta_{ij}+V_{ij}-E^{\alpha'}\delta_{ij}]\psi_j^{\alpha'}=0\,. \tag{4}$$

Turning on the perturbation δV_{ij}, we have $\psi_i^{\alpha'}\to\psi_i^{\alpha'}+\delta\psi_i^{\alpha'}$, $V_{ij}\to V_{ij}+\delta V_{ij}+(\partial V_{ij}/\partial E^{\beta})\delta E^{\beta}$, and $E^{\alpha'}\to E^{\alpha'}+\delta E^{\alpha'}$, which gives, upon insertion into Eq. (4), the result

$$\psi_i^{\alpha'*}\delta V_{ij}\psi_j^{\alpha'}+\psi_i^{\alpha'*}\frac{\partial V_{ij}}{\partial E^{\beta}}\delta E^{\beta}\psi_j^{\alpha'}-\psi_i^{\alpha'*}\delta E^{\alpha'}\delta_{ij}\psi_j^{\alpha'}=0\,. \tag{5}$$

By defining

$$A_{\alpha'}=\psi_i^{\alpha'*}\delta V_{ij}\psi_j^{\alpha'} \quad\text{and}\quad B_{\alpha'\beta}=\psi_i^{\alpha'*}\frac{\partial V_{ij}}{\partial E^{\beta}}\psi_j^{\alpha'}\,, \tag{6}$$

one can rewrite Eq. (5) in the form

$$A_{\alpha'}+B_{\alpha'\beta}\delta E_{\beta}-\delta E_{\alpha'}=0\,. \tag{7}$$

We may take as a simple example the two-channel case with just two eigenvalues. Hence α', $\beta'=1$, 2 and Eq. (7) may be used to solve for the energy shifts,

$$\delta E_1=\frac{(1-B_{22})A_1+B_{12}A_2}{1-B_{11}-B_{22}+B_{11}B_{22}-B_{12}B_{21}}\,, \tag{8a}$$

$$\delta E_2=\frac{(1-B_{11})A_2+B_{21}A_1}{1-B_{11}-B_{22}+B_{11}B_{22}-B_{12}B_{21}}\,. \tag{8b}$$

In cases of interest, the potentials V_{ij} depend on a coupling constant g^2, so we set $B_{11}=ag^2$, $B_{12}=bg^2$, $B_{21}=cg^2$, $B_{22}=dg^2$ with the physical value of g^2 being set equal to unity. Equations (8) then become

$$\delta E_1=\frac{A_1+g^2(bA_2-dA_1)}{1-(a+d)g^2+(ad-bc)g^4}\,, \tag{8$'$a}$$

$$\delta E_2=\frac{A_2+g^2(cA_1-aA_2)}{1-(a+d)g^2+(ad-bc)g^4}\,. \tag{8$'$b}$$

As an illustration, we let $A_1=A_2=+1$, so that in the absence of feedback ($g^2=0$) there are no sign reversals. When the coupling constant is set equal to its physical value, $g^2=1$, we want δE_1 to be negative, δE_2 to be positive (e.g. the case $\delta E_1=p-n$ and $\delta E_2=\Xi^- -\Xi^0$) and the denominators such that when g^2 is allowed to vanish, δE_1 and δE_2 go through no singularities. This may be easily accomplished by setting $a=-2$, $b=-1$, $c=1$, and $d=1$, so that Eqs. (8′a) and (8′b) become

$$\delta E_1=\frac{1-2g^2}{1+g^2-g^4}\,, \tag{8$''$a}$$

$$\delta E_2=\frac{1+3g^2}{1+g^2-g^4}\,, \tag{8$''$b}$$

which at $g^2=1$ yield $\delta E_1=-1$, $\delta E_2=+4$, at $g^2=0$ yield $\delta E_1=\delta E_2=+1$, and remain finite for all g^2 between $g^2=0$ and $g^2=1$. Hence we have demonstrated that in the multichannel case, Barton's criticism does not necessarily apply to feedback-induced sign reversals (although it still applies, of course, if any of the δE's should happen to have a singularity between $g^2=0$ and $g^2=1$ in our example).

The effect of including electromagnetic corrections to strong interactions has also been discussed by Muta[5] within the framework of an SU_2 invariant version of the Lee model. Taking the V and N particles as two isodoublets and the θ particle as an isotriplet, he regards the $I_3=+\frac{1}{2}$ component of the V and N particles as a *proton* and the $I_3=-\frac{1}{2}$ component as a *neutron*. The symmetry-breaking interaction is then given by

$$H_{\rm SB}=\delta m[V^{\dagger}(\tfrac{1}{2})V(\tfrac{1}{2})+N^{\dagger}(\tfrac{1}{2})N(\tfrac{1}{2})]\,, \tag{9}$$

where $\delta m\propto e^2/4\pi$. Solving for the physical particle states, one arrives at a relation for δN which is negative, regardless of the size of δm, provided only that g^2, the coupling constant of the Lee model is sufficiently large.

Two examples which illustrate Barton's criticism in the one-channel case are found in the papers of Pagels,[6] and Fried and Truong.[7] Pagels has done a calculation utilizing the mechanical form factors for the nucleons and has arrived at a result $\delta N=-1.7$ MeV with a cutoff equal to 3 (nucleon masses).[2] His result is due to the effect of a very large nucleon mass shift acting back on itself together with corrections to the SU_2 predicted pion-nucleon coupling constants which together give a contribution equal to 1.3 δN. However, one can easily see from Pagels' Eq. (34) that the solution for δM will go through a singularity as g^2 is decreased from its physical value to zero.[8] Fried and Truong have utilized

[5] T. Muta, Nuovo Cimento **49**, 307 (1966).

[6] H. Pagels, Phys. Rev. **144**, 1261 (1966).

[7] H. M. Fried and T. N. Truong, Phys. Rev. Letters **16**, 559 (1966).

[8] Professor Okubo has brought to the author's attention a paper by H. Miyazawa, Y. Oi, and M. Suzuki, Progr. Theoret. Phys. (Kyoto), Suppl. (1965), which contains a calculation similar to that of Pagels.

the Lehmann representation for the nucleon two-point function in a calculation somewhat related to Pagels's. One can see from Fried's Eq. (15) that Barton's criticism also applies here.[9]

The composite-model, one-channel case has been considered by Dashen and Frautschi.[10] They assume that, before the electromagnetic interaction is turned on, the nucleon consists of a pion bound to a nucleon in a $P_{1/2}$ state (i.e., the nucleon appears as a pole in the direct channel of the pion-nucleon scattering amplitude)

$$|p\rangle = -(\sqrt{\tfrac{1}{3}})|p\pi^0\rangle + (\sqrt{\tfrac{2}{3}})|n\pi^+\rangle, \qquad (10a)$$

$$|n\rangle = -(\sqrt{\tfrac{2}{3}})|p\pi^-\rangle + (\sqrt{\tfrac{1}{3}})|n\pi^0\rangle, \qquad (10b)$$

and that the pion-nucleon scattering amplitude has already been obtained in the usual N/D form. Upon turning on the electromagnetic interaction, they assert that the binding energy of the neutron is increased relative to that of the proton, resulting in a negative value for δN. The mechanism for this is supposed to be the *repulsive* interaction of the proton Dirac moment with the magnetic field of the orbiting π^- [cf. Eq. (10b)]. Barton,[11] however, has shown that this mechanism gives an *attractive* interaction and hence lowers the neutron mass relative to the proton.[12]

It is the purpose of this work to consider the problem of the baryon mass differences in a multichannel approach, thereby avoiding Barton's criticism. This is accomplished by considering a model in which electromagnetic interactions provide the symmetry-breaking effects in a theory which is otherwise SU_3-invariant. The importance of utilizing SU_3 invariance in the evaluation of electromagnetic mass differences has been demonstrated in the calculation of Wojtaszek, Marshak, and Riazuddin[13] and also by Barton.[4,14] Wojtaszek *et al.* include electromagnetic effects such as π^0-η and Σ^0-Λ mixing with the driving forces (in addition to the usual elastic-form-factor contribution) and succeed in obtaining the correct signs for the electromagnetic mass splittings within the baryon and pseudoscalar octets, provided that the f/d ratio and cutoff are properly chosen. Barton has shown that a composite model in which the baryons are taken to be bound states of baryons and mesons and the mesons bound states of baryon-antibaryon pairs can also generate the correct signs for the electromagnetic mass splittings.[15]

Our calculation is based on a model developed in a previous paper[16] (henceforth referred to as I) which was applied to the problem of calculating the medium-strong mass splittings. Here, we assume that the electromagnetic mass splittings within the baryon octet can similarly be understood as arising, as a low-energy effect, from the electromagnetic mass splittings within the pseudoscalar octet together with the usual elastic-form-factor contribution. We take the view that the meson mass differences are a more complicated effect (requiring either the consideration of three meson intermediate states or vector-meson–pseudoscalar-meson intermediate states) so that both the electromagnetic meson mass splittings and the elastic form factor constitute the driving forces for purposes of this calculation. The feedback, therefore, consists of the effect of the mass splittings of a given baryon multiplet acting back on itself as well as its effect on the mass splittings within all the other baryon multiplets. Although we have chosen to include the meson mass differences with the driving terms, we remark that some authors[14] find it more convenient to include them in the feedback.

As in I, we take the SU_3 values for the coupling constants, neglecting any electromagnetic corrections.[17] Since the electromagnetic interaction affects the U-spin multiplets in the same way that medium-strong interactions affect the isospin multiplets, we can proceed as in I to express the three independent baryon mass splittings as functions of the pion-nucleon coupling constant $g^2/4\pi$, the f/d ratio, and a cutoff Λ. Using the observed values of the electromagnetic mass splittings as input, we are able to calculate g^2, f/d, and Λ arriving at the value $g^2/4\pi = 16.7$. The interplay of feedback and driving force is discussed by considering the effect of reversing the input sign of the $K^{\pm}-K^0$ mass difference.

For the sake of comparison, an attempt was also made to calculate the meson mass splitting by assuming that the baryon-antibaryon intermediate states represent the dominant contribution to the inverse two-point functions of the meson fields. The results were indifferent at best, a fact which is not surprising since there are a number of two-particle intermediate states which are less massive than $\bar{B}B$ (e.g., the vector-meson–pseudoscalar-meson states).

[9] We emphasize here that the mechanism proposed in Ref. 7 is quite distinct from that presented in the present paper. It will be shown here that one can account for both the sign and magnitudes of the baryon mass differences without the occurrence of singularities (such as required by Fried and Truong) in the solutions of the eigenvalue equations for the mass shift.

[10] R. F. Dashen and S. C. Frautschi, Phys. Rev. **135**, B1190 (1964); R. F. Dashen, *ibid.* **135**, B1196 (1964).

[11] G. Barton, Phys. Rev. **146**, 1149 (1966).

[12] Barton (Ref. 11) further remarks that Dashen's result is incorrect due to an improper method employed in subtracting off a spurious infrared divergence. See also Y. S. Kim, Phys. Rev. **142**, 1150 (1966).

[13] J. Wojtaszek, R. E. Marshak, and Riazuddin, Phys. Rev. **136**, B1053 (1964).

[14] G. Barton, Phys. Rev. **153**, 1673 (1967).

[15] It is well to remark here that Barton's "composite-particle model" is basically a nonrelativistic version of the elementary-particle approach presented in this paper. In particular, we note that the concept of baryons as baryon-meson composites is expressed in field theory by means of the lowest-order self-energy diagrams representing the virtual transition of the baryons into a baryon-meson pair. We draw particular attention to this point in order to display the fact that neither the present calculation nor that of Barton can be taken to imply any support for the bootstrap hypothesis.

[16] S. L. Cohen and C. R. Hagen, Phys. Rev. **149**, 1138 (1966).

[17] This is shown to be fairly small in Ref. 6, amounting to ~20% of the total feedback contribution.

II. THE MODEL

The SU_3-invariant interaction can be written as

$$\mathcal{L}_I=(1-f)g_0\,\mathrm{Tr}[\bar{B}\gamma_5BP+\bar{B}\gamma_5PB]+fg_0\,\mathrm{Tr}[\bar{B}\gamma_5BP-\bar{B}\gamma_5PB], \quad (11)$$

where B and P are 3×3 matrices representing the baryon and pseudoscalar octets, respectively. Performing the indicated matrix multiplication and taking the trace, one can rewrite Eq. (11) in the more explicit isospin-invariant form

$$\begin{aligned}\mathcal{L}_I=&g_{N\pi}\bar{N}\gamma_5\boldsymbol{\tau}N\cdot\boldsymbol{\pi}+g_{\Lambda\pi}(\bar{\Lambda}\gamma_5\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}+\text{H.a.})+ig_{\Sigma\pi}\boldsymbol{\Sigma}\gamma_5\times\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}\\&+g_{\Xi\pi}\bar{\Xi}\gamma_5\boldsymbol{\tau}\Xi\cdot\boldsymbol{\pi}+g_{\Lambda K}(\bar{N}\gamma_5K\Lambda+\text{H.a.})\\&+g_{\Sigma K}\bar{N}\gamma_5\boldsymbol{\tau}K\cdot\boldsymbol{\Sigma}+h_{\Lambda K}(\Xi\gamma_5\bar{K}\Lambda+\text{H.a.})\\&+h_{\Sigma K}(\bar{\Xi}\gamma_5\boldsymbol{\tau}\bar{K}\cdot\boldsymbol{\Sigma}+\text{H.a.})+g_{N\eta}\bar{N}\gamma_5N\eta+g_{\Lambda\eta}\bar{\Lambda}\gamma_5\Lambda\eta\\&+g_{\Sigma\eta}\boldsymbol{\Sigma}\gamma_5\cdot\boldsymbol{\Sigma}\eta+g_{\Xi\eta}\bar{\Xi}\gamma_5\Xi\eta, \quad (12)\end{aligned}$$

where $\bar{K}=-i\tau_2K^*$ (H.a.$\equiv$Hermitian adjoint). The interaction term $\mathcal{L}_I$ of Eq. (12) is, of course, SU_3-invariant if one uses the appropriate SU_3 values for the coupling constants (as given for example in I). To transform $\mathcal{L}_I$ into the U-spin-invariant form, we use the Weyl transformation[18]

$$W=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix},$$

which leaves $\mathcal{L}_I$ invariant but takes us from I spin to U spin. Applying W to Eq. (11), we get for a typical term $W\bar{B}BPW^{-1}=W\bar{B}W^{-1}WBW^{-1}WPW^{-1}=\bar{B}'B'P'$. Writing out B' gives

$$B'=WBW^{-1}=W\begin{bmatrix}B_1{}^1&B_2{}^1&B_3{}^1\\B_1{}^2&B_2{}^2&B_3{}^2\\B_1{}^3&B_2{}^3&B_3{}^3\end{bmatrix}W^{-1}=\begin{bmatrix}B_2{}^2&B_3{}^2&B_1{}^2\\B_2{}^3&B_3{}^3&B_1{}^3\\B_2{}^1&B_3{}^1&B_1{}^1\end{bmatrix}. \quad (13)$$

Substituting in the definitions of the $B_j{}^i$'s gives

$$B'=\begin{bmatrix}\frac{1}{\sqrt6}\Lambda^0-\frac{1}{\sqrt2}\Sigma^0&\Xi^0&\Sigma^+\\n^0&-\frac{2}{\sqrt6}\Lambda^0&p^+\\\Sigma^-&\Xi^-&\frac{1}{\sqrt6}\Lambda^0+\frac{1}{\sqrt2}\Sigma^0\end{bmatrix}, \quad (14a)$$

which upon comparison with B,

$$B=\begin{bmatrix}\frac{1}{\sqrt6}\Lambda^0+\frac{1}{\sqrt2}\Sigma^0&\Sigma^-&\Xi^-\\\Sigma^+&\frac{1}{\sqrt6}\Lambda^0-\frac{1}{\sqrt2}\Sigma^0&\Xi^0\\p^+&n^0&-\frac{2}{\sqrt6}\Lambda^0\end{bmatrix}, \quad (14b)$$

enables us to write down the well-known definitions of the U-spin multiplets:

$$N'=\begin{pmatrix}\Sigma^-\\\Xi^-\end{pmatrix}, \quad (15a)$$

$$\Sigma'=\begin{bmatrix}n^0\\\tilde{\Sigma}^0\\\Xi^0\end{bmatrix}, \quad (15b)$$

$$\Xi'=\begin{pmatrix}p^+\\\Sigma^+\end{pmatrix}, \quad (15c)$$

$$\Lambda'=(\tilde{\Lambda}), \quad (15d)$$

where the $\tilde{\Lambda}$ and $\tilde{\Sigma}^0$ operators are given by

$$\tilde{\Lambda}=-\tfrac12\Lambda-\tfrac12\sqrt3\Sigma^0, \quad (16a)$$

$$\tilde{\Sigma}^0=\tfrac12\sqrt3\Lambda-\tfrac12\Sigma^0. \quad (16b)$$

Going through the identical procedure for the meson matrix P gives the meson U-spin multiplets

$$\pi'=\begin{bmatrix}K^0\\\tilde{\pi}^0\\\bar{K}^0\end{bmatrix}, \quad (17a)$$

$$K'=\begin{pmatrix}\pi^-\\K^-\end{pmatrix}, \quad (17b)$$

$$\bar{K}'=\begin{pmatrix}K^+\\\pi^+\end{pmatrix}, \quad (17c)$$

and

$$\eta'=(\tilde{\eta}), \quad (17d)$$

with the $\tilde{\eta}$ and $\tilde{\pi}^0$ operators defined by

$$\tilde{\eta}=-\tfrac12\eta-\tfrac12\sqrt3\pi^0, \quad (18a)$$

and

$$\tilde{\pi}^0=\tfrac12\sqrt3\eta-\tfrac12\pi^0. \quad (18b)$$

It is clear that merely replacing the operators $\bar{B}$, B, and P by the operators $\bar{B}'$, B', and P' in Eq. (12) gives the U-spin invariant interaction Lagrangian. The definitions of g^2 and f/d remain unchanged, so that one has $g_{N\pi}{}^2=g_{N'\pi'}{}^2$, etc.

---

[18] A. J. Macfarlane, E. C. G. Sudarshan, and C. Dullemond, Nuovo Cimento 30, 845 (1963).

If one neglects medium-strong effects, the lowest-order electromagnetic-mass-splitting term transforms as $A_{11}{}^{11}$ which commutes with U_3 and Q (in direct analogy to the medium strong $A_3{}^3$ which commutes with I_3 and Y). Using the tensor properties of $A_{11}{}^{11}$, one can show[19] that the electromagnetic mass splittings are given by

$$N': \Xi^- = \Sigma^- = M_0 + b', \tag{19a}$$

$$\Sigma': n = \tilde{\Sigma}^0 = \Xi^0 = M_0, \tag{19b}$$

$$\Xi': p = \Sigma^+ = M_0 + c', \tag{19c}$$

and

$$\Lambda': \tilde{\Lambda} = M_0 + \tfrac{2}{3}(b' + c' + d'), \tag{19d}$$

together with

$$\Sigma^0 = M_0 + \tfrac{1}{2}(b' + c' + d'), \tag{19e}$$

$$\Lambda = M_0 + \tfrac{1}{6}(b' + c' + d'). \tag{19f}$$

The constants b', c', and d' may be evaluated by using the physically observed values for the three independent mass splittings $\Xi^- - \Xi^0$, $\Sigma^+ - \Sigma^0$, and $n - p$. ($\Sigma^- - \Sigma^0$ is related to the other three by the Coleman-Glashow relation.[20]) From the values obtained for b', c', and d', one can readily evaluate $N' - \Lambda'$, $\Sigma' - \Lambda'$, and $\Xi' - \Lambda'$. Proceeding in an identical fashion, the two independent meson mass differences $K'^2 - \eta'^2$ and $\pi'^2 - \eta'^2$ may be calculated.[21] The results are $N' - \Lambda' = 4.66$ MeV, $\Sigma' - \Lambda' = -1.84$ MeV. $\Xi' - \Lambda' = -3.13$ MeV, $\pi'^2 - \eta'^2 = 7020$ MeV^2, and $K'^2 - \eta'^2 = 3020$ MeV^2.

We use the same labeling conventions as in I (that is, i, j, $k = 1, 2, 3, 4$ stand for the N', Σ', Ξ', and Λ', respectively, while $\alpha, \beta = 1, 2, 3$ stand for the π', K', and η', respectively). It is assumed that the baryon masses M_i' may be written as a function of the meson masses $\mu_\alpha'^2$, $M_i' = M_i'(\mu_1'^2 \cdots \mu_3'^2)$, so that upon expanding M_i in a Taylor series about the SU_3 central mass μ'^2, we get, to lowest order in $\delta\mu_\alpha'^2$,

$$M_i' = M' + \sum_{\alpha=1}^{3} \left[\frac{\partial M_i'(\mu_1'^2 \cdots \mu_3'^2)}{\partial \mu_\alpha'^2}\right]_{\mu', M'} (\mu_\alpha'^2 - \mu'^2). \tag{20}$$

From the usual integral representation of the inverse baryon two-point function $S^{-1}(\gamma \cdot p)$, together with the requirement $S^{-1}(\gamma \cdot p = -M) = 0$, one derives (upon neglect of medium-strong effects) the result

$$M_i' = M_0' - \sum_{j,\beta} \int_{M_j' + \mu_\beta'}^{\infty} dm \left[\frac{r^+(M_j', \mu_\beta', m)}{m - M_i'} - \frac{r^-(M_j', \mu_\beta', m)}{m + M_i'}\right], \tag{21}$$

where $r^\pm = r_{\text{vs}}{}^\pm + r_{\text{em}}{}^\pm$, $r_{\text{vs}}{}^\pm$ representing the SU_3-invariant part of the inverse two-point function and $r_{\text{em}}{}^\pm$ being the contribution from the electromagnetic correction to strong interactions. Using the second-order expression for $r^\pm$ given in I, we apply $\partial/\partial\mu_\alpha'^2$ to Eq. (21) which, when evaluated at the SU_3 central masses μ' and M', gives

$$\left[\frac{\partial M_i'}{\partial \mu_\alpha'^2}\right]_{\mu', M'} = \frac{1}{16\pi^2} \left\{ \mathcal{g}_1 \sum_j a_{ij\alpha} g_{ij\alpha}{}^2 - \mathcal{g}_2 \sum_{j,\beta} a_{ij\beta} g_{ij\beta}{}^2 \left[\frac{\partial M_j}{\partial \mu_\alpha'^2}\right]\right\}_{\mu', M'}. \tag{22}$$

Here, $\mathcal{g}_1$ is a convergent integral resulting from the differentiation of the intermediate meson mass and $\mathcal{g}_2$ is a logarithmically divergent integral resulting from the differentiation of the intermediate baryon mass. The expressions for $\mathcal{g}_1$ and $\mathcal{g}_2$ are given in I, where $\mathcal{g}_1$ was found to be equal to -0.000551 MeV^{-1} for $\mu' = 549$ MeV and $M' = 1115$ MeV. The $a_{ij\alpha}$'s are the U-spin factors (which are identical to the isospin factors of I), and the indices may be understood by referring to Fig. 1 of I.

Making the substitutions $Q_{i\alpha} = [\partial M_i'/\partial\mu_\alpha'^2]_{\mu', M'}$, $P_{i\alpha} = \sum_j a_{ij\alpha} g_{ij\alpha}{}^2$, $R_{ij} = \sum_\beta a_{ij\beta} g_{ij\beta}{}^2$, and $a = \mathcal{g}_2/\mathcal{g}_1$, we rewrite Eq. (22) as

$$Q_{i\alpha} = \frac{\mathcal{g}_1}{16\pi^2}(P_{i\alpha} - a \sum_j R_{ij} Q_{j\alpha}), \tag{22'}$$

where the P's, Q's, and R's will have the same functional form as in I. One can then write Eq. (20) as

$$M_i' = M' + \textstyle\sum_\alpha Q_{i\alpha}(\mu_\alpha'^2 - \mu'^2). \tag{23}$$

Upon expressing Eq. (22′) in matrix form and solving for the matrix Q, one has

$$Q = \frac{\mathcal{g}_1}{16\pi^2}(1 + bR)^{-1} P, \tag{24}$$

where $b = \mathcal{g}_1 a/16\pi^2$. In Eq. (24), P is the contribution from the intermediate meson and hence represents the driving force; R is the contribution from the intermediate baryon and consequently represents the feedback. With no loss of generality one can choose $\mu'^2 = \mu_3'^2$, so that one needs only the two mass differences $\mu_1'^2 - \mu_3'^2$ and $\mu_2'^2 - \mu_3'^2$, the values for which were given above. Upon insertion of $(\mu_1'^2 - \mu_3'^2)/(\mu_2'^2 - \mu_3'^2) = 2.3$ and the definition $c' = \mu_2'^2 - \mu_3'^2$ into Eq. (23), one obtains

$$M_i' = M' + c'(2.3Q_{i1} + Q_{i2}). \tag{23'}$$

In Eq. (23′), we have not yet included the elastic-form-factor contribution (which of course has a lower threshold than the baryon-meson intermediate state). Setting this contribution equal to B_i, where $B_i > 0$, we may most easily take account of this by adding it di-

---

[19] S. Okubo (unpublished).

[20] S. Coleman and S. L. Glashow, Phys. Rev. Letters **6**, 423 (1961).

[21] The numerical values of the mass splittings are taken from the tables of A. H. Rosenfeld *et al.*, Lawrence Radiation Laboratory Report No. UCRL-8030 rev., 1966 (unpublished).

rectly to Eq. (23′):

$$M_i'-(\delta_{i1}+\delta_{i3})B_i=M'+c'(2.3Q_{i1}+Q_{i2}), \qquad (23'')$$

where we have used the fact that the photon only couples to the charged particles (N' and Ξ'). The integral representation for B_i is

$$B_i=-\int_{M_i'}^{\infty} dm\left\{\frac{\rho^+(M_i',m)}{m-M_i'}-\frac{\rho^-(M_i',m)}{m+M_i'}\right\}, \qquad (25)$$

where the spectral weights $\rho^\pm$ are given to lowest order (in a particular gauge) by

$$\rho^\pm=\frac{e^2}{16\pi^2}\frac{m^2+M_i'^2\mp 4mM_i'}{m^3}(m^2-M_i'^2). \qquad (26)$$

The integral in Eq. (25) may be easily performed, provided it is cut off at $m=\lambda M_i$, and one gets

$$B_i=\frac{M_i'e^2}{8\pi^2}\left[3\ \ln\lambda-\frac{1}{2}\frac{1}{\lambda^2}(\lambda^2-1)\right]. \qquad (27)$$

Since we choose to disregard medium-strong mass differences, neglect of the dependence of B_i on i involves only an error of order e^4 (as $M_1'-M_3'$ is of the order of e^2), so that one can consistently take $M_i'=M'$ and hence $B_1=B_3=B$.

Setting $\gamma=g^2b$, we subtract M_4' from Eq. (23″) for $i=1$, 2, 3 and by inserting the expressions for the $Q_{i\alpha}$'s, obtain the three independent mass difference equations:

$$M_1'-M_4'=\frac{\mathcal{g}_1c'g^2}{48\pi^2D}\Big[-27.6f^2+31.2f-0.9+\tfrac{4}{3}\gamma(-427.2f^4+492f^3-208.8f^2+139.2f-12.7)$$
$$+\frac{8}{9}\gamma^2(-759f^6-5107f^5+10042f^4-8587.6f^3+3373.8f^2-89.4f-112.5)$$
$$+\frac{16}{27}\gamma^3(10752f^8-422f^7-31383f^6+7786f^5+16311f^4-16723f^3+4596f^2+636f-300)\Big], \qquad (28a)$$

$$M_2'-M_4'=\frac{\mathcal{g}_1c'g^2}{48\pi^2D}\Big[20.8f^2+20.8f-10.4+\tfrac{4}{3}\gamma(734.4f^4-691.2f^3-21.6f^2+367.2f-91.8)$$
$$+\frac{8}{9}\gamma^2(10470f^6-30375f^5+28882f^4-9958.8f^3-2180.4f^2+2802f-467)$$
$$+\frac{16}{27}\gamma^3(-54298f^8+43623f^7+49324f^6-108960f^5+63480f^4-6288f^3-11832f^2+5520f-690)\Big], \qquad (28b)$$

$$M_3'-M_4'=\frac{\mathcal{g}_1c'g^2}{48\pi^2D}\Big[31.2f^2-27.6f-0.9+\tfrac{4}{3}\gamma(631f^4-919.2f^3+320.4f^2-37.2f-12.7)$$
$$+\frac{8}{9}\gamma^2(4415.8f^6-11223f^5+5338f^4+3172f^3-4270.2f^2+1439.4f-112.5)$$
$$+\frac{16}{27}\gamma^3(-28762f^8-11712f^7+61756f^6-23261f^5-33081f^4+42547f^3-20100f^2+4164f-300)\Big], \qquad (28c)$$

where

$$D=1+\tfrac{4}{3}\gamma(22f^2-14f+7)+\frac{2}{9}\gamma^2(544f^4-448f^3+456f^2-232f+58)$$
$$+\frac{4}{27}\gamma^3(-9088f^6+22656f^5-29184f^4+20096f^3-7824f^2+1680f-280)$$
$$+\frac{4}{81}\gamma^4(43008f^8-159744f^7+26180f^6+183552f^5-228288f^4+136320f^3-45120f^2+9600f-1200). \qquad (28d)$$

The three mass differences are now given as functions of the four parameters g^2, f, Λ, and λ. We treat λ as an adjustable parameter and attempt to find the values of g^2 and f which yield $\lambda=\Lambda$. Starting from the independent ratios $(M_1'-M_4'-B)/(M_2'-M_4')$ and $(M_1'-M_4'-B)/(M_3'-M_4'-B)$, we insert a value for

TABLE I. Values of B for selected values of λ.

λ	B (MeV)	λ	B (MeV)
1.2	+0.51	2.6	+3.16
1.4	+0.987	2.7	+3.30
1.9	+2.03	3.6	+4.38
2	+2.21	3.8	+4.58
2.4	+2.86	4	+4.78
2.5	+3.01	4.1	+4.87

TABLE II. Solutions for f and $g^2/4\pi$ for various values of λ.

λ	B (MeV)	f	$g^2/4\pi$
1.9	+2.03	0.175	10.8
2.2	+2.50	0.167	16.6
2.4	+2.86	0.167	16.7
2.5	+3.01	0.167	16.8
2.6	+3.16	0.167	18.0
4.1	+4.88	0.167	19.6

B given by Table I and calculate f and γ. From this Eq. (28a) yields $g^2/4\pi$ which, using the definition of γ, gives Λ. We iterate this procedure until $\lambda=\Lambda$.

III. RESULTS

Following the procedure described above, we arrive at the set of values[22] $g^2/4\pi=16.7$, $f=0.167$, and $\Lambda=\lambda=2.4$ ($\gamma=-0.7$) which compare favorably with the results of I ($g^2/4\pi=7.3$, $f=0.185$, and $\Lambda=2.7$), although in the present case the pion-nucleon coupling constant is in considerably better agreement with experiment.

From Table I, one sees that the elastic-form-factor contribution is 2.86 MeV, which we interpret as an average value for the baryon octet. This is not unreasonably large, considering that it accounts for only 44% of the $\Xi^- - \Xi^0$ mass difference, although it is somewhat larger than the estimates given in Ref. 13.

Although we have neglected the magnetic-form-factor contributions due to the baryon anomalous moments, one can simulate this effect by varying the cutoff λ and recalculating the values of the parameters f and $g^2/4\pi$. The results are given in Table II and show that the variation of the parameters f and $g^2/4\pi$ with λ is slow enough to justify the neglect of the magnetic-moment contribution (which is expected to be considerably smaller than that associated with the elastic form factor[3]).

IV. DISCUSSION OF FEEDBACK

One can investigate the importance of feedback by neglecting the term bR in Eq. (24) and attempting to find a solution including only the driving terms B and P. The result is that one arrives at the wrong signs for the mass differences, thereby leading to the conclusion that feedback is in some way responsible for the sign reversals. However, on reversing the input signs of the $K^\pm - K^0$ mass difference (but including feedback) we are not able to find a set of solutions f, g^2, and Λ which give the correct signs for the baryon mass differences. Thus one cannot generate the desired sign reversals within the baryon octet without the occurrence of a sign reversal within the pseudoscalar-meson octet. We thus conclude that the effect of the feedback is merely to weight the contributions of the imput $\pi^\pm - \pi^0$ and $K^\pm - K^0$ mass differences in such a way as to induce the observed sign reversals within the baryon octet.

Finally, we examine the feedback contribution $\mathfrak{F}$, which may be defined as the correction to the $\gamma=0$ expression for the mass splitting. In the case of Eq. (28a) this leads one to consider the form

$$\mathfrak{F}=\frac{1}{D}\left[1+\tfrac{4}{3}\gamma\frac{-427.2f^4+492f^3-208.8f^2+139.2f-12.7}{-27.6f^2+31.2f-0.9}+\frac{8}{9}\gamma^2\frac{-759f^6-5107f^5+10042f^4-8587.6f^3+3373.8f^2-89.4f-112.5}{-27.6f^2+31.2f-0.9}+\frac{16}{27}\gamma^3\frac{10752f^8-422f^7-31383f^6+7786f^5+16311f^4-16723f^3+4596f^2+636f-300}{-27.6f^2+31.2f-0.9}\right]. \quad (29)$$

As a consequence of our previous discussion we must require that $\mathfrak{F}$ go to zero smoothly as the strong interactions are turned off, i.e., in the limit $\gamma=0$ (if one allows g^2 to vanish the meson mass-splitting driving terms would also vanish). Although this case is complicated by the fact that $\mathfrak{F}$ is a function of two variables, f and γ, we may impose the condition that one of the mass-difference ratios, say $R_{12}=(M_1'-M_4'-B)/(M_2'-M_4')$, be maintained, at least at the endpoints $\gamma=-0.7$ and $\gamma=0$ (the other ratio R_{13} is determined by γ). The requirement that R_{12} be equal to -0.928 for $\gamma=0$ implies that $f=0.221$, so that we need only demand the existence of some path in the γ-f plane between the points $(-0.7, 0.167)$ and $(0, 0.221)$ such that $\mathfrak{F}$ have no singu-

[22] This is the only set of values with $0.1\leqslant f\leqslant 0.5$ which yields $g^2>0$ and therefore represents the desired solution to Eq. (28).

larities.[23] One such path is from (−0.7, 0.167) to (−0.7, 0.221) along the line $\gamma=-0.7$ and thence from (−0.7, 0.221) to (0, 0.221) along the line $f=0.221$.

[23] Although it is true that not all paths in the γ-f plane can avoid the occurrence of singularities in the mass difference equations (28), it is only necessary to demonstrate the existence of one such path. This allows one to "turn off" the strong interactions in such a way that all the mass differences will remain finite, thereby enabling one to avoid Barton's criticism.

ACKNOWLEDGMENT

The authors are indebted to Professor Marshak for useful remarks.

PHYSICAL REVIEW VOLUME 157, NUMBER 5 25 MAY 1967

Bootstrap and the Symmetry of the Mesons*

Y. C. Leung

Department of Physics, Northeastern University, Boston, Massachusetts

(Received 18 August 1966; revised manuscript received 16 January 1967)

We have considered the problem of symmetry induction among a set of vector and pseudoscalar mesons along the line of the Cutkosky model. The assumptions that enter into the model are discussed, and it is shown that within this set of assumptions the model can be extended to induce higher symmetry of the type commonly known as $SU(6)$ by explicitly enumerating the spin and orbital components of the particles in a nonrelativistic way. It is found that, within the bootstrap hypothesis, an extension can be achieved only if the internal symmetry group is a unitary group, $SU(m)$, and the extended group is $SU(2m)$. On the other hand, if the internal symmetry group is either an orthogonal group or symplectic group, no such extension is possible. If the internal symmetry group is a simple Lie group—for example, $SU(3)$—the concept of unitary parity (the analog of G parity) is found to be incompatible with $SU(6)$, i.e., they are mutually exclusive.

I. INTRODUCTION

THE history and philosophy of the bootstrap hypothesis in dealing with strongly interacting particles have by now been reviewed at great length by several authors.[1,2] The idea is a very intuitive one, simply being that "all particles are dynamical entities composed of each other and bound by forces produced by the exchange of the particles themselves."[2] However, in the absence of a truly workable dynamical theory, the bootstrap hypothesis serves mainly as a convenient mechanism in the qualitative understanding of the existence of certain particles, their masses, and the strengths of their interactions with other particles. On the other hand, it provides us with an extremely useful tool for the understanding of the internal symmetries. For example, on the basis of a proposed symmetry scheme, the bootstrap hypothesis can be employed to great advantage in obtaining an insight into the nature of bound states by simply examining the crossing matrices of the symmetry group involved. Cutkosky[3] has proposed a model (or a method) based on the bootstrap hypothesis, from which the internal symmetry of the constituent particles can be directly induced. In other words, the results of the bootstrap model can be formalized in the language of the algebra of a symmetry group. Therefore, within this model world we are given a very clear-cut reason why certain types of symmetry should emerge. The attractiveness of the Cutkosky model was recognized immediately and the method was subsequently extended by several authors.[4–6]

In this paper, we wish to present a further extension of the Cutkosky model along the same line. We shall first give a summary of the results already obtained, and in Sec. III we will show that the model can be extended to induce symmetry of the type commonly known as $SU(6)$, which we shall refer to as spin–unitary-spin independence, by enumerating the spin states of the particles explicitly in a nonrelativistic fashion. It is found that an extension can be achieved relatively easily if the internal symmetry group is a unitary group $SU(m)$. On the other hand, if the internal-symmetry group is the orthogonal or symplectic group, the extended group is again a unitary group. This actually leads to an inconsistency with the bootstrap hypothesis.

In Sec. IV, we discuss the problem of unitary parity as related to spin–unitary-spin independence. In Sec. V, we give a summary and a discussion of the physical states represented in the $SU(2m)$ group obtained here.

* Supported in part by the National Science Foundation and by the U. S. Army Research Office (Durham).

[1] B. M. Udgaonkar, *High Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1965), p. 791.

[2] F. Zachariasen, in *High Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1965), p. 823.

[3] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963).

[4] Chan Hong-Mo, P. C. DeCelles, and J. E. Paton, Nuovo Cimento **33**, 70 (1964).

[5] J. C. Polkinghorne, Ann. Phys. (N. Y.) **34**, 153 (1965).

[6] R. C. Hwa and S. Patil, Phys. Rev. **140**, B1586 (1965).