Convergence of Padé Approximants for the Bethe-Salpeter Amplitude*

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We extend some earlier work on the Bethe-Salpeter equation to show that the sequence of $[N,N]$ Padé approximants to $\tan\delta_l$ converges to the correct result if the scattered particles are of equal mass. The proof includes a demonstration that the symmetrized kernel of the Bethe-Salpeter equation after a coordinate-space Wick rotation is L^2. An interesting connection between Padé approximants and the Schwinger variational principle is given.

INTRODUCTION

IN this paper we give a more detailed account of some work described briefly by Nuttall[1] concerning bounds on the phase shifts from the Bethe-Salpeter (BS) equation and the convergence of the Padé approximants. In addition to supplying derivations for the results stated by Nuttall, we also make some new observations.

The paper begins with a discussion of the R-matrix version of the BS equation, and it is shown that the coordinate-space Wick rotation employed by Schwartz and Zemach[2] (SZ) is also useful in this case. Throughout, we study the scattering of a pair of spinless particles of equal mass interacting via the exchange of another spinless particle, although it may be possible to generalize a number of our results in several directions.

The equation now has a close formal similarity to the Lippmann-Schwinger equation with a potential operator of definite sign, and we show explicitly that the methods of Gailitis[3] and Sugar and Blankenbecler[4] may be applied to obtain lower bounds on the phase shifts. In calculating numerical values for the phase shifts, SZ used the Schwinger variational principle, but we show that in the way it was employed, it amounted to an application of the bound mentioned above. This explains the observation of SZ that their trial values always approached a limit from below. The germ of these results may be found in the work of Kato[5] and in the remarks of the authors referred to above.

We then give an interesting form for the $[N,N]$ Padé approximant[6] which makes it clear that the $[N,N]$ approximant to each partial-wave part of the on-energy-shell R matrix is the result of solving the BS equation exactly for a particular choice of trial potential in the Sugar-Blankenbecler method. This enables us to deduce that the phase shifts given by the $[N,N]$ Padé approximants form an increasing sequence bounded above by the exact result, and consequently they must converge to a limit. By using an argument based on the compactness of the symmetrized kernel of the Euclidean BS equation, we demonstrate that this limit is indeed the correct phase shift.

Our work has interesting consequences. It brings to light a suggestive connection between the Schwinger variational principle and Padé approximants in scattering theory. The results about bounds appear to be true only for the scattering of equal-mass scalar particles via the exchange of another scalar particle (possibly of different mass). It is quite possible that the Padé approximants still converge when these restrictions are lifted, but this question has not been resolved.

R-MATRIX EQUATION

The derivation of bounds on the phase shift in potential theory follows most naturally from a study of the equation for the R matrix, and the same is the case for the BS equation. The Heitler integral equation giving R in terms of the BS amplitude T is

$$\langle k'|R(E)|k\rangle=\langle k'|T(E)|k\rangle+\frac{i}{8\pi^2}\int d^4q\langle k'|T(E)|q\rangle \times\delta^+[(P+q)^2-m^2]\delta^+[(P-q)^2-m^2] \times\langle q|R(E)|k\rangle, \quad (1)$$

where $2P=(E,\mathbf{0})$ is the total momentum and k,k'',q are relative momenta. The BS equation for T is

$$\langle k'|T(E)|k\rangle=\langle k'|V(E)|k\rangle+\frac{i}{(2\pi)^4}\int d^4q \times\frac{\langle k'|V(E)|q\rangle\langle q|T(E)|k\rangle}{[(P+q)^2-m^2+i\epsilon][(P-q)^2-m^2+i\epsilon]}, \quad (2)$$

where the inhomogeneous term $\langle k'|V(E)|k\rangle$ represents the sum of all two-particle irreducible diagrams. To obtain the equation discussed by SZ we set[7]

$$\langle k'|V(E)|k\rangle=\frac{g^2}{(k-k')^2-M^2+i\epsilon}. \quad (3)$$

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[1] J. Nuttall, Phys. Letters 23, 492 (1966).
[2] C. Schwartz and C. Zemach, Phys. Rev. 141, 1454 (1966); hereafter referred to as SZ.
[3] M. Gailitis, Zh. Eksperim. i Teor. Fiz. 47, 160 (1964) [English transl.: Soviet Phys.—JETP 20, 107 (1965)].
[4] R. Sugar and R. Blankenbecler, Phys. Rev. 136, B472 (1964).
[5] T. Kato, Phys. Rev. 80, 475 (1950).
[6] See for instance the review by G. A. Baker, Advan. Theoret. Phys. 1, 5 (1965).
[7] As far as possible we use the notation of SZ except that $k^2=k_0^2-\mathbf{k}^2$.

From (1) and (2) we may deduce the BS equation for R, which reads

$$\langle k'|R(E)|k\rangle=\langle k'|V(E)|k\rangle + i\int d^4q\langle k'|V(E)|q\rangle G_r(q,E)\langle q|R(E)|k\rangle\,, \quad (4)$$

with

$$G_r(q,E)=\frac{1}{(2\pi)^4}\times\{([(P+q)^2-m^2+i\epsilon][(P-q)^2-m^2+i\epsilon])^{-1} + 2\pi^2\delta^+[(P+q)^2-m^2]\delta^+[(P-q)^2-m^2]\}\,. \quad (5)$$

In coordinate space, instead of the Green's function $G(x,x')$ used by SZ, we now need $G_r(x,x')$, determined by the Fourier transform of (5),

$$G_r(x,x')=G(x,x') + \frac{i}{16\pi E|\mathbf{r}-\mathbf{r}'|}(e^{-ik|\mathbf{r}-\mathbf{r}'|}-e^{ik|\mathbf{r}-\mathbf{r}'|})\,. \quad (6)$$

Since the extra term in (6) does not depend on time, the Wick rotation may be performed just as in SZ, and we must now replace the Euclidean metric Green's function $H(x,x')$ by $H_r(x,x')$. Using the notation of SZ Eq. (2.36), we find that

$$H_r(x,x')=\frac{\cos(k|\mathbf{r}-\mathbf{r}'|)}{8\pi E|\mathbf{r}-\mathbf{r}'|}-\frac{1}{8\pi^2E}\left(\int_{-\infty}^{-\omega_2}+\int_{\omega_1}^{\infty}\right)d\beta \times e^{\beta(\tau-\tau')}K_0(Q|R-R'|)\,. \quad (7)$$

In the equal-mass case $\omega_1=\omega_2=(\mathbf{k}^2+m^2)^{1/2}$ and $H_r(x,x')$ is a Hermitian operator.

To express physical matrix elements of R in terms of Euclidean quantities, we again parallel the discussion of SZ, and introduce a function $\phi_r(x)$ satisfying

$$\phi_r(x)=e^{i\mathbf{k}\cdot\mathbf{r}}+\int d^4x'H_r(x,x')V(x')\phi_r(x')\,, \quad (8)$$

using the Euclidean metric. For the remainder of this article we will restrict ourselves to the case when the "potential" is given by (3), leading to a local $V(x)$ in (8), which is given explicitly by SZ Eq. (2.35):

$$V(x)=(4M\lambda/R)K_1(MR)\,,\quad R=(r^2+\tau^2)^{1/2}. \quad (9)$$

Note that $V(x)>0$ for all real x. We must stress that the Wick rotation in this form can only be valid for $\omega<m+\frac{1}{2}M$, which corresponds to the threshold for production of an M particle, and the remainder of the paper is subject to this condition.

For physical values of the relative momenta ($k_0=k_0'=0$), the R matrix may be written in terms of an integral in Euclidean space

$$R(\mathbf{k}',\mathbf{k},E)=-\int d^4x\, e^{-i\mathbf{k}'\cdot\mathbf{r}}V(x)\phi_r(x)\,. \quad (10)$$

To bring out the similarity of the Euclidean equations to potential theory, we shall write (8) and (10) in operator form, working in the Hilbert space of Euclidean functions $\psi(x)$. Using $\phi_\mathbf{k}(x)=e^{i\mathbf{k}\cdot\mathbf{r}}$, (8) becomes

$$|\phi_r\rangle=|\phi_\mathbf{k}\rangle+H_rV|\phi_r\rangle \quad (11)$$

and (10) becomes

$$R(\mathbf{k}',\mathbf{k},E)=-\langle\phi_{\mathbf{k}'}|V|\phi_r\rangle\,. \quad (12)$$

BOUNDS ON THE PHASE SHIFTS

To obtain bounds on the phase shifts from the BS equation we write, using (11) and (12),

$$R(\mathbf{k}',\mathbf{k},E)=-\langle\phi_{\mathbf{k}'}|V(1-H_rV)^{-1}|\phi_\mathbf{k}\rangle\,. \quad (13)$$

Taking partial waves, we deduce that

$$\tan\delta_l(E)=\langle k,l|V[1-H_r(E)V]^{-1}|k,l\rangle\,, \quad (14)$$

where $|k,l\rangle$ has the wave function

$$\phi_{k,l}(x)=\left(\frac{k}{2E}\right)^{1/2}j_l(kr)Y_l^0(\hat{r})\,, \quad (15)$$

and momentum k corresponds to energy E.

From this point we may use word for word the analysis of Gailitis,[3] who shows that if we find a trial potential $V_{\rm tr}$ such that

$$\langle\psi|V|\psi\rangle>\langle\psi|V_{\rm tr}|\psi\rangle \quad \text{for all} \quad |\psi\rangle\,, \quad (16)$$

then at a given energy

$$\delta_l>(\delta_l)_{\rm tr}\,. \quad (17)$$

The derivation is based on the fact that V and H_r are Hermitian operators. (Note that the V used above and in SZ corresponds to the negative of the potential in nonrelativistic theory.)

To construct a useful trial potential we may apply the technique of Sugar and Blankenbecler[4] (see also Löwdin[8]), who point out that if V is a positive operator, then

$$V\geq V^{(N)}\,, \quad (18)$$

where

$$V^{(N)}=\sum_{i,j=1}^{N}V|i\rangle A_{ij}\langle j|V\,. \quad (19)$$

The matrix A_{ij} is the inverse of $\langle i|V|j\rangle$, and the set $|i\rangle$, $i=1,\cdots,N$ is arbitrary. If we add another state to the set, we obtain a new trial potential $V^{(N+1)}$ with the property

$$V^{(N+1)}\geq V^{(N)}\,. \quad (20)$$

[8] P.-O. Löwdin, Phys. Rev. **139**, A357 (1965).

Thus, as N increases, $\delta_l^{(N)}$ form an increasing sequence bounded above by the exact phase shift δ_l.

To relate these results to the variational procedure used by SZ, we first point out that the phase shift $\delta_l^{(N)}$ calculated with the use of $V^{(N)}$ may be written as[4]

$$\tan\delta_l^{(N)}=\sum_{i,j=1}^{N}\langle k,l|V|i\rangle B_{ij}\langle j|V|k,l\rangle\,, \qquad (21)$$

where B_{ij} is the inverse of the matrix whose elements are $\langle i|(V-VH_rV)|j\rangle$. The principle used by SZ states that the exact phase shift is determined by finding the stationary value of

$$[\tan\delta_l]=2\langle k,l|U\rangle-\langle U|V^{-1}|U\rangle+\langle U|H_r|U\rangle\,, \qquad (22)$$

where $|U\rangle$ is any real wave function. At the stationary point, $|U\rangle=V|\phi_{k,l}\rangle$, where $|\phi_{k,l}\rangle$ is the partial-wave part of $|\phi_r\rangle$.

SZ write $|U\rangle$ as a linear combination of a number of basis states $|n\rangle$,

$$|U\rangle=\sum a_n|n\rangle\,, \qquad (23)$$

and vary the coefficients a_n to make (22) stationary. Define a new set of states $|\bar{n}\rangle$ by

$$|n\rangle=V|\bar{n}\rangle\,, \qquad (24)$$

and substitute (23) into (22). We must look for stationary values of the quadratic form

$$\sum_{n,m}a_na_m[\langle\bar{n}|VH_rV|\bar{m}\rangle-\langle\bar{n}|V|\bar{m}\rangle]+2\sum_n a_n\langle k,l|V|\bar{n}\rangle. \qquad (25)$$

The result is a value of $\tan\delta_l$ identical with that given by (21) if the states $|\bar{n}\rangle$ are identified with the set $|i\rangle$ used above. We are therefore able to explain the behavior of the numerical results commented on by SZ.

PADÉ APPROXIMANTS

To connect Padé approximants with the previous discussion, we shall show that the states $|i\rangle$ may be chosen in such a way that $\tan\delta_l^{(N)}$ given by (21) is identical with the $[N,N]$ Padé approximant to $\tan\delta_l$, which we write as $t_P^{(N)}$. To calculate $t_P^{(N)}$ (see Baker[6]) we write $t_P^{(N)}$ in the form of a ratio of two Nth-order polynomials in the interaction strength λ. The coefficients in the polynomials are found by comparing the first $2N$ coefficients in the formal power-series expansion of $t_P^{(N)}$ with those in the series for $\tan\delta_l$,

$$\tan\delta_l=\sum_{i=1}^{\infty}\alpha_i\lambda^i. \qquad (26)$$

A compact expression for the $[N,N]$ Padé approximant may be obtained by slightly rearranging the general formula quoted by Baker, and we find

$$t_P^{(N)}=\mathbf{m}^TM^{-1}\mathbf{m}\,, \qquad (27)$$

where M is the $N\times N$ matrix

$$M_{ij}=\alpha_{i+j-1}-\lambda\alpha_{i+j}\,,\quad i=1,\cdots N \qquad (28)$$

and $\mathbf{m}$ is the column vector

$$m_i=\alpha_i\,,\quad i=1,\cdots N. \qquad (29)$$

Since (14) shows that, using $V=\lambda\mathcal{V}$,

$$\alpha_i=\langle k,l|\mathcal{V}(H_r\mathcal{V})^{i-1}|k,l\rangle\,, \qquad (30)$$

it is not hard to see that (26) and (21) will coincide if we choose the states $|i\rangle$ to be

$$|i\rangle=(H_r\mathcal{V})^{i-1}|k,l\rangle\,,\quad i=1,\cdots,N. \qquad (31)$$

Thus we have shown that to obtain the $[N,N]$ Padé approximant we solve exactly the problem with potential $V^{(N)}$ given by (19) using states $|i\rangle$ defined in (31). It may be shown after a certain amount of analysis that this potential is identical with one given by Tani[9] in his discussion of Padé approximants in potential theory.

Having shown that the $[N,N]$ Padé approximants may be obtained from a set of Sugar-Blankenbecler trial potentials, we may now apply the results of the previous section. It follows that, for an attractive potential, the Padé phase shifts $\delta_{lP}^{(N)}$, for fixed l and E, form an increasing sequence as N increases, bounded above by the exact phase shift. Thus, the Padé phase shifts approach a limit which we now proceed to show to be the exact phase shift.

The result is that the limit of the Padé phase shifts is the exact phase shift and our results about Padé approximants may probably be proved with the help of techniques for studying the series of Stieltjes described in Baker's article. That the series (26) is related to a series of Stieltjes follows from the fact that we may rewrite (30) to read

$$\alpha_i=\langle\chi|K^{i-1}|\chi\rangle\,, \qquad (32)$$

with

$$|\chi\rangle=\mathcal{V}^{1/2}|k,l\rangle \qquad (33)$$

and

$$K=\mathcal{V}^{1/2}H_r\mathcal{V}^{1/2}. \qquad (34)$$

For positive potentials, K is a self-adjoint operator with a spectral resolution

$$K=\int_{-\infty}^{\infty}\lambda dE(\lambda)\,, \qquad (35)$$

where $E(\lambda)$ is a family of projections. In (32) we now obtain

$$\alpha_i=\int_{-\infty}^{\infty}\lambda^{i-1}d\langle\chi|E(\lambda)|\chi\rangle\,, \qquad (36)$$

showing that we have a series of Stieltjes except that the integral in (36) runs from $-\infty$ to ∞.

[9] S. Tani, Phys. Rev. 139, B1011 (1965).

In place of this approach, we now present a direct method of proof which is based on the fact that K is also an L^2 operator. This property is proved in the Appendix. Let us define a set of states $|\chi_i\rangle$ given by

$$|\chi_i\rangle = K^{i-1}|\chi\rangle = \mathcal{V}^{1/2}|i\rangle. \tag{37}$$

Suppose δ_N is the space spanned by the states $|\chi_i\rangle$, $i=1, \cdots, N$, and that P_N is the orthogonal projector onto this space (the linear hull of all the $|\chi_i\rangle$ is denoted by S_∞). Using (27) it is easy to see that the $[N,N]$ Padé approximant may be written

$$t_P{}^{(N)} = \lambda\langle\chi|\omega_N\rangle, \tag{38}$$

where $|\omega_N\rangle \in S_N$ and satisfies

$$|\omega_N\rangle = |\chi\rangle + \lambda P_N K|\omega_N\rangle. \tag{39}$$

If we are not at a resonance, there will be a solution $|\omega\rangle$ of the equation

$$|\omega\rangle = |\chi\rangle + \lambda K|\omega\rangle, \tag{40}$$

and it may be shown (see below) that $|\omega\rangle \in S_\infty$. The exact phase shift is given by

$$\tan\delta_l = \lambda\langle\chi|\omega\rangle = \lambda\langle\omega|\chi\rangle. \tag{41}$$

By taking the scalar product of (39) and (40) with $|\omega\rangle$ and $|\omega_N\rangle$, respectively, and subtracting, we deduce that

$$\langle\omega|\chi\rangle - \langle\omega_N|\chi\rangle = \lambda\langle\omega|P_N K|\omega_N\rangle - \lambda\langle\omega|K|\omega_N\rangle. \tag{42}$$

Now let us suppose that there is an infinite sequence of N for which $|\omega_N\rangle$ are uniformly bounded. Since K is a compact operator,[10] there must be an infinite subsequence for which $K|\omega_N\rangle$ converges strongly to some state $|z\rangle \in S_\infty$. Moreover, it follows that

$$P_N K|\omega_N\rangle \to |z\rangle. \tag{43}$$

Consequently, the right-hand side of (43) approaches zero, and for this subsequence,

$$\langle\chi|\omega_N\rangle - \langle\chi|\omega\rangle \to 0, \tag{44}$$

and so

$$t_P{}^{(N)} \to \tan\delta_l. \tag{45}$$

However, we have already shown that all the $t_P{}^{(N)}$ converge to a limit, which must therefore be $\tan\delta_l$.

To complete the proof, we must consider the possibility that the $|\omega_N\rangle$ are not uniformly bounded as $N\to\infty$. In this case, the sequence of states $|\chi_N\rangle = (\langle\omega_N|\omega_N\rangle)^{-1/2}|\omega_N\rangle$ will have the following property:

$$|x_N\rangle - P_N K|x_N\rangle \to 0. \tag{46}$$

Since $|x_N\rangle$ has norm 1, there must be a subsequence of $|x_N\rangle$ for which $K|x_N\rangle$ converges to a limit $|x\rangle$, say, and as before we shall have

$$\begin{aligned} |x_N\rangle &\to |x\rangle \in S_\infty, \\ P_N K|x_N\rangle &\to |x\rangle. \end{aligned} \tag{47}$$

[10] L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces* (The Macmillan Company, New York, 1964).

Taking scalar products as before and subtracting, we shall find

$$\langle x_N|\chi\rangle = \lambda\langle\omega|K|x_N\rangle - \lambda\langle\omega|P_N K|x_N\rangle, \tag{48}$$

and (47) allows us to deduce that

$$\langle x|\chi\rangle = 0. \tag{49}$$

However, in (40) we could replace $|\chi\rangle$ by any state in S_∞ and obtain a corresponding $|\omega\rangle$. Repeating the above argument leads to the conclusion that $|x\rangle$ is orthogonal to S_∞, a contradiction, since $|x\rangle \in S_\infty$, and cannot be the zero state since all the $|x_N\rangle$ have unit norm.

DISCUSSION

Our work has brought to light an interesting connection between Padé approximants to the solution of integral equations and variational principles. In two-particle scattering, both nonrelativistic and, as shown here, relativistic, an on-energy-shell R-matrix element may be written in the following form:

$$\langle\phi'|R|\phi\rangle = \lambda\langle\chi'|(1-\lambda K)^{-1}|\chi\rangle, \tag{50}$$

where

$$|\chi\rangle = \mathcal{V}^{1/2}|\phi\rangle, \quad \langle\chi'| = \langle\phi'|\mathcal{V}^{1/2}, \tag{51}$$

and K is the symmetrized kernel (34), or the corresponding nonrelativistic expression. We know that K is an L^2 operator, even when the potential $\mathcal{V}$ is not a positive operator.

The Schwinger variational principle states that $\langle\phi'|R|\phi\rangle$ is the stationary value of the following expression when $|\chi\rangle$ and $|\chi'\rangle$ are varied throughout the Hilbert space.

$$[R] = \lambda[\langle\chi'|\psi\rangle + \langle\psi'|\chi\rangle - \langle\psi'|\psi\rangle + \lambda\langle\psi'|K|\psi\rangle]. \tag{52}$$

$[R]$ is stationary when $|\psi\rangle$, $|\psi'\rangle$ satisfy

$$|\psi\rangle = |\chi\rangle + \lambda K|\psi\rangle, \tag{53a}$$

$$\langle\psi'| = \langle\chi'| + \lambda\langle\psi'|K. \tag{53b}$$

We define the space S_N spanned by

$$|\chi\rangle,\, K|\chi\rangle,\, \cdots K^{N-1}|\chi\rangle$$

and S_N' spanned by

$$\langle\chi'|,\, \langle\chi'K,\, \cdots\langle\chi'|K^{N-1}.$$

The linear hull of all $K^N|\chi\rangle$ is S_∞ and similarly S_∞'. Since S_∞ is an invariant subspace, it is easy to see that $|\chi\rangle \in S_\infty$ and similarly $\langle\psi'| \in S_\infty'$. With P_N defined as before, the unique solution $|\psi_S\rangle \in S_\infty$ of the equation

$$|\psi_S\rangle = |\chi\rangle + \lambda P_\infty K P_\infty|\psi_S\rangle \tag{54}$$

will satisfy (53a), for $P_\infty K P_\infty|\psi_S\rangle = K|\psi_S\rangle$.

Since we know the stationary value of $|\psi\rangle$ lies in S_∞, it seems reasonable to restrict the variation of $|\psi\rangle$ to this space, and this is the idea behind the Padé approximant method. We obtain an approximate stationary value of $[R]$ by restricting the variation of $|\psi\rangle$, $|\psi'\rangle$ to S_N, S_N', respectively, and our previous analysis is easily modified to show that this stationary value is just the $[N,N]$ Padé approximant to $\langle\phi'|R|\phi\rangle$.

The author is not aware of any rigorous proof of the convergence of this scheme unless $|\chi\rangle=|\chi'\rangle$ and K is self-adjoint, which is the situation studied earlier in this paper.

It must be pointed out that the operator $H_r(x,x')$ is not self-adjoint unless the scattered particles have equal mass. If this is not the case, the Gailitis argument does not apply and we cannot find a Sugar-Blankenbecler bound. Also the proof of the convergence of the Padé approximants is not valid, for K is not then self-adjoint.

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APPENDIX

In this Appendix we show that $K=V^{1/2}H_rV^{1/2}$ is an L^2 operator, and therefore compact. Our proof applies to the case of scattered particles of equal mass, but no doubt the result is true in general. We must verify the convergence of $\mathrm{Tr}(KK^\dagger)$, which may be written, in the equal mass case,

$$\mathrm{Tr}(KK^\dagger)=\mathrm{Tr}(K^2)=\int d^4x d^4x' V(x)H_r^2(x',0)V(x+x'). \quad \text{(A1)}$$

The main point at issue is the behavior of the integrand for large values of x and x'. We know $V(x)$ explicitly [SZ (2.35)] and its asymptotic form is

$$V(x)\sim \mathrm{const}\, R^{-3/2}e^{-MR}, \quad \text{(A2)}$$

with

$$R=(x^2)^{1/2}=(\tau^2+\mathbf{r}^2)^{1/2}. \quad \text{(A3)}$$

Our function $H_r(x',0)$ is just the real part of SZ's $H(x',0)$, and we could attempt to make use of the form given by SZ (2.40) to obtain its asymptotic behavior. However, this is not adequate for our purposes, and we go back to an earlier expression [SZ (2.14)], from which we deduce that

$$H(x,0)=\mathrm{const}\int_{-\omega}^{\omega} d\beta\, e^{\beta\tau}K_0(QR). \quad \text{(A4)}$$

Here $Q=(\beta^2-\mathbf{k}^2)^{1/2}$ and $\omega=(\mathbf{k}^2+m^2)^{1/2}$. The contour is chosen so that Q lies in the fourth quadrant.

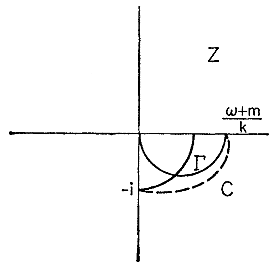

FIG. 1. Contour of integration C in the Z plane.

For large R, $K_0(QR)$ has the form

$$K_0(QR)\sim \mathrm{const}(QR)^{-1/2}e^{-QR}. \quad \text{(A5)}$$

The behavior of $H(x,0)$ will be dominated by the factor $e^{\beta\tau-QR}$ in the integrand, and for $\tau>0$, we need only consider the region $\mathrm{Re}\beta>0$. In this region we have

$$\mathrm{Re}(\beta\tau-QR)\leq R\,\mathrm{Re}(\beta-Q). \quad \text{(A6)}$$

We shall now show that the integration contour running from $\beta=0$ to $\beta=\omega$ may be chosen so that

$$\mathrm{Re}(\beta-Q)\leq\omega-m. \quad \text{(A7)}$$

To do this we change the variable to

$$Z=\rho e^{i\theta}=k^{-1}(\beta+Q),$$

so that

$$\beta-Q=kZ^{-1}=(k/\rho)e^{-i\theta}.$$

In the Z plane, the contour must run from $Z=-i$ to $Z=k^{-1}(\omega+m)$. Both Q and β will lie in the fourth quadrant if $-\frac{1}{2}\pi\leq Q\leq 0$ and $\rho\geq 1$, so the contour must lie outside the circle $\rho=1$. The points satisfying $\mathrm{Re}(\beta-Q)=\omega-m$ are given by

$$\rho=[(\omega+m)/k]\cos\theta$$

and lie on the circle Γ of Fig. 1. Those points lying below Γ satisfy $\mathrm{Re}(\beta-Q)<\omega-m$, and so the contour C sketched in Fig. 1 fulfills all our requirements.

This argument shows that $H(x,0)$ is dominated by $e^{R(\omega-m)}$ for large R, whatever the size of the ratio τ/R. The same result must apply to the real part of $H(x,0)$, $H_r(x,0)$. Using (A2) it is quite easy to see that the integral (A1) converges for large x and x', so long as $\omega<m+\frac{1}{2}M$, for the integrand decreases exponentially if one or both of x and x' are large.

The functions $V(x)$ and $H_r(x,0)$ are singular at $R=0$, and we must check that this does not destroy the convergence of (A1). The most singular parts of $V(x)$ and $H_r(x,0)$ are R^{-2} and $\ln R$, respectively, and factors obtained after doing angular integrations in (A1) will nullify this behavior, so that we deduce that $\mathrm{Tr}(KK^\dagger)<\infty$. The argument can no doubt be generalized to show that $V^{1/2}HV^{1/2}$ is compact even for unequal-mass particles.