Renormalized Born Approximation

W. F. FORD
National Aeronautics and Space Administration–Lewis Research Center, Cleveland, Ohio
(Received 23 December 1966)

A well-known defect of the Born approximation is that it often yields cross sections correct in shape, but orders of magnitude too large. A method is suggested for renormalizing approximate wave functions so that their amplitude is correct, not asymptotically but at some other more appropriately chosen point. This leads to an expression for the T matrix which is independent of the over-all amplitude of the approximate function. The method is applied to a plane wave Born approximation and a simple distorted-wave Born-approximation calculation of scattering from a square well. The renormalized results are in excellent agreement with exact calculations.

MOST problems in quantum scattering theory involve evaluation of the T matrix

$$T=\langle\phi_f|V|\psi\rangle. \qquad (1)$$

Here ϕ_f is the final-state wave function and V is the interaction responsible for the transition, both of which are usually assumed known. The total wave function ψ satisfies an integral equation

$$\psi(x)=\phi(x)+\int K(x,y)\psi(y)dy$$
$$\equiv\phi(x)+\langle K(x)\psi\rangle, \qquad (2)$$

which is often difficult or impossible to solve; thus approximations for ψ must be used in evaluating T.

If we assume that the dominant contribution to T comes from a region about the point $x=x_0$, and that this point is known, then there are two obvious requirements for a good approximation to ψ: (1) It must have the correct amplitude at $x=x_0$, and (2) it must vary properly in the region about $x=x_0$. Usually physical arguments are invoked to suggest trial functions which hopefully satisfy the latter requirement, but the first requirement seems to be universally ignored. Instead, the trial function is made to have the proper asymptotic normalization, which is quite ineffective since V vanishes asymptotically (i.e., $x_0\neq\infty$). In this paper we propose a method of "renormalizing" the trial functions which has proved remarkably successful.

Suppose A is the amplitude of ψ at $x=x_0$. Then we may define a new function $\hat{\psi}$ with unit amplitude at x_0,

$$\psi(x)=A\hat{\psi}(x), \quad \hat{\psi}(x_0)=1, \qquad (3)$$

in terms of which

$$T=A\langle\phi_f|V|\hat{\psi}\rangle. \qquad (4)$$

The advantage of this formulation is that any approximation to $\hat{\psi}$ will automatically have the proper amplitude just where the contribution to T is greatest.

To find A, we simply use the definition of $\hat{\psi}$ in Eq. (2):

$$A=\psi(x_0)=\phi(x_0)+A\langle K(x_0)\hat{\psi}\rangle$$
$$=\phi(x_0)/[1-\langle K(x_0)\hat{\psi}\rangle]; \qquad (5)$$

if now we write $\hat{\psi}(x)=\psi(x)/\psi(x_0)$, the final expression for T is[1]

$$T=\phi(x_0)\frac{\langle\phi_f|V|\psi\rangle}{\psi(x_0)-\langle K(x_0)\psi\rangle}. \qquad (6)$$

This is obviously exact if ψ is exact, and in fact can be derived in one step from Eqs. (1) and (2). If exact wave functions are used, T must be independent of x_0; we have given the above derivation to indicate one criterion[2] for choosing x_0 when approximate wave functions are used. In this regard, note that Eq. (6) is independent of the over-all amplitude of ψ, as opposed to Eq. (1); the price we pay for this advantage is having to estimate x_0 and having to calculate the additional factor, which in many cases is not much more difficult to evaluate than $\langle\phi_f|V|\psi\rangle$.

As an example, we calculate the total cross section for scattering from a square well of strength $U_0=2mV_0/\hbar^2$ and range R. The plane-wave Born approximation based on Eq. (1) is

$$T_{\mathrm{Born}}=U_0R^2j_1(QR)/Q, \quad Q=2k\sin\tfrac{1}{2}\theta; \qquad (7)$$

the renormalized approximation based on Eq. (6) is

$$T_{\mathrm{renorm}}=T_{\mathrm{Born}}/[1+U_0R^2(e^{ix}-1-ix)/x^2], \qquad (8)$$

where $x\equiv 2kR$ and we have used $x_0=0$. The total cross sections obtained from these expressions are shown in Figs. 1–3, plotted against U_0R^2 for various values of kR. In the resonance region $U_0R^2>\pi^2$, the Born approximation fails badly, whereas the renormalized result gives an adequate account of the nonresonant or average cross section. Note the flattening of the

[1] In partial-wave analysis Eq. (6) (or a variation thereof) has a venerable history. The renormalization was first introduced in order to obtain an integral equation for $\hat{\psi}_l$ with an absolutely convergent iterative solution [R. Jost, Helv. Phys. Acta **20**, 256 (1947); G. F. Drukarev, Zh. Eksperim. i Teor. Fiz. **25**, 139 (1953)]. Later various approximation theories made use of it, but without emphasizing the flexibility of the trial function [e.g., P. Swan, Nucl. Phys. **18**, 245 (1960); Henry Brysk, Phys. Rev. **133**, B1625 (1964)]. Most recently it has appeared in strong-interaction calculations [H. Pierre Noyes, Phys. Rev. Letters **15**, 538 (1965); K. L. Kowalski, *ibid.* **15**, 798 (1965)].

[2] One might instead require that the approximate T be stationary with respect to x_0.

renormalized curves at U_0R^2 grows large; this occurs because Eq. (8) is independent of V_0 to first order in $1/V_0$ when $\exp(i\mathbf{k}\cdot\mathbf{r})$ is used for ψ.

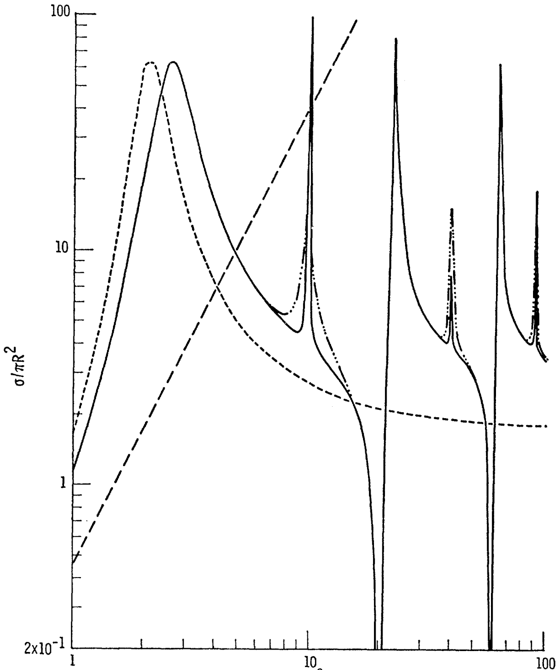

FIG. 1. Total cross section for scattering from a square well for $kR=0.25$. The curves represent the exact value (solid line), the ordinary Born approximation (long dash), the renormalized Born approximation (short dash), and the renormalized approximation based on Eq. (10) (dot-dash).

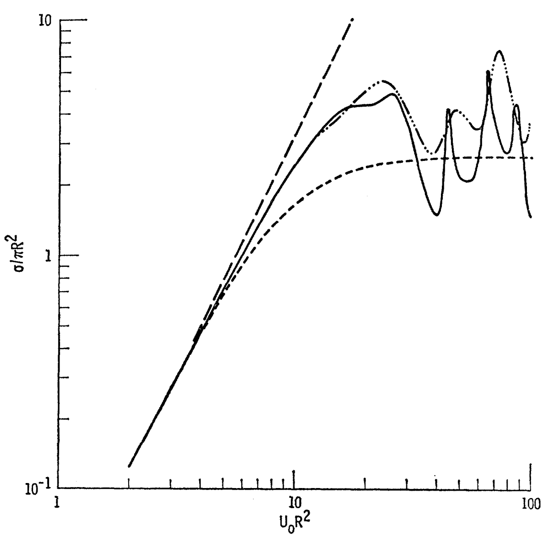

FIG. 2. Same as Fig. 1, except for $kR=4$.

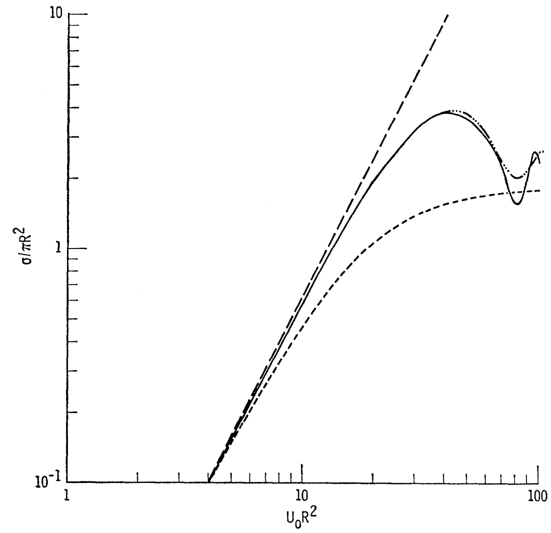

FIG. 3. Same as Fig. 1, except for $kR=9$.

To capitalize on the amplitude-independent feature of this approach, one should use trial functions whose behavior near x_0 is more reasonable. In the present case we may use a plane wave with the correct wavelength

$$\psi=e^{i\mathbf{K}\cdot\mathbf{r}},\quad K^2=2m(E+V_0)/\hbar^2, \tag{9}$$

which can be still further improved by adding a wave reflected off the back surface of the well

$$\psi=e^{i\mathbf{K}\cdot\mathbf{r}}+\beta e^{-i\mathbf{K}\cdot\mathbf{r}},\quad \beta=(k-K)/(k+K)e^{2iKR}. \tag{10}$$

Calculation of T with these wave functions is no harder than before, and the results using Eq. (10) are also presented in Figs. 1–3.

The success of Eq. (10) as a trial function for $kR\ll1$ and $kR\gg1$ is easily explained. One can show[3] that in a partial wave expansion for ψ, the terms for $l<kR$ are given quite well by Eq. (10), and in fact the $l=0$ term is exact. The terms for $l>kR$ are incorrect but make negligible contribution anyway. Hence Eq. (10) should give good results, provided that the terms for $l\sim kR$ do not contribute a large fraction of the total cross section.

Equation (6) may also be used for phase-shift calculations, and for direct reactions if the denominator can be reasonably approximated. Encouraging results have been obtained in both cases, and will be reported in the near future.

[3] A. L. Latter, Phys. Rev. **83**, 1056 (1951).