# Correlation Theory of Quantized Electromagnetic Fields. II. Stationary Fields and Their Spectral Properties*

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In Paper I of this investigation, dynamical laws which describe the space-time development of second-order coherence tensors of a quantized electromagnetic field *in vacuo*, and the associated conservation laws, were derived. In the first part of the present paper these results are specialized to stationary fields. Second-order cross-spectral tensors for such fields are then introduced, and some of their properties are discussed. A relation analogous to the Wiener-Khintchine theorem of the theory of stationary random processes is derived. Various non-negative-definiteness conditions obeyed by the cross-spectral tensors are established, and equations which govern the spatial variation of these tensors are deduced. Certain analytic properties of the correlation tensors are derived, and some of their consequences are examined. It is also shown that in the limiting case when the two space-time arguments of the coherence tensors coincide, two of our conservation laws reduce to the averaged form of the energy and the momentum conservation laws of the electromagnetic field.

## I. INTRODUCTION

In Paper I of this investigation,[1] equations were derived, which govern the space-time development of the second-order coherence tensors of a quantized electromagnetic field. In the present paper some consequences of these equations are deduced for the case which is of particular importance in practice, namely, the case when the field is describable by a stationary ensemble. Some spectral properties of such fields are also discussed.

We begin in Sec. II with specializing the field equations and the conservation laws derived in Paper I to stationary fields. It is shown, in Appendix II, that in the limiting case, when the two space-time points coincide, the real parts of two of the conservation laws reduce to the average form of the usual laws for the conservation of energy and momentum.

In Sec. III we introduce the concept of the (second-order) cross-spectral tensors of a quantized stationary electromagnetic field and discuss some of their properties. In particular, we establish the Wiener-Khintchine theorem for such a field. Our formulation is free of the assumption of homogeneity of the field, made implicitly in an earlier published version of this theorem. Various non-negative-definiteness conditions, obeyed by the cross-spectral tensors, are established and the equations which govern their spatial variation are also given.

## II. DYNAMICAL EQUATIONS AND ASSOCIATED CONSERVATION LAWS

Since the most commonly occurring fields encountered in nature are *stationary*, we begin by specializing the basic dynamical equations derived in Paper I to fields of this type. For a stationary field, the density operator commutes with the Hamiltonian operator and all the second-order correlation tensors depend on the two time arguments $t_1$ and $t_2$ only through the difference[2]

$$\tau = t_2 - t_1. \tag{2.1}$$

We will therefore write $\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)$ in place of $\mathcal{E}_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)$ etc. In the differential equations (2.27a)–(2.34a) and (2.27b)–(2.34b) derived in Paper I we may then replace $\partial/\partial t_1$ by $-\partial/\partial\tau$ and $\partial/\partial t_2$ by $\partial/\partial\tau$, and we obtain the following *dynamical equations relating to second-order coherence tensors of a stationary field in vacuo:*

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}\mathcal{E}_{jl} = -\frac{1}{c}\frac{\partial}{\partial\tau}\mathfrak{N}_{kl}, \tag{2.2a}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}\mathfrak{N}_{jl} = \frac{1}{c}\frac{\partial}{\partial\tau}\mathcal{E}_{kl}, \tag{2.3a}$$

$$\frac{\partial}{\partial r_{1i}}\mathcal{E}_{il} = 0, \tag{2.4a}$$

$$\frac{\partial}{\partial r_{1i}}\mathfrak{N}_{il} = 0; \tag{2.5a}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}\mathcal{H}_{jl} = \frac{1}{c}\frac{\partial}{\partial\tau}\mathfrak{M}_{kl}, \tag{2.6a}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}\mathfrak{M}_{jl} = -\frac{1}{c}\frac{\partial}{\partial\tau}\mathcal{H}_{kl}, \tag{2.7a}$$

$$\frac{\partial}{\partial r_{1i}}\mathcal{H}_{il} = 0, \tag{2.8a}$$

$$\frac{\partial}{\partial r_{1i}}\mathfrak{M}_{il} = 0; \tag{2.9a}$$

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[1] C. L. Mehta and E. Wolf, preceding paper, Phys. Rev. **157**, 1183 (1967).
[2] Y. Kano, Ann. Phys. (N. Y.), **30**, 127 (1964).

$$\epsilon_{ijk}\frac{\partial}{\partial r_{2i}}\mathcal{E}_{lj}=-\frac{1}{c}\frac{\partial}{\partial\tau}\mathcal{M}_{lk}\,, \tag{2.2b}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{2i}}\mathcal{M}_{lj}=\frac{1}{c}\frac{\partial}{\partial\tau}\mathcal{E}_{lk}\,, \tag{2.3b}$$

$$\frac{\partial}{\partial r_{2i}}\mathcal{E}_{li}=0\,, \tag{2.4b}$$

$$\frac{\partial}{\partial r_{2i}}\mathcal{M}_{li}=0; \tag{2.5b}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{2i}}\mathcal{H}_{lj}=\frac{1}{c}\frac{\partial}{\partial\tau}\mathcal{N}_{lk}\,, \tag{2.6b}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{2i}}\mathcal{N}_{lj}=-\frac{1}{c}\frac{\partial}{\partial\tau}\mathcal{H}_{lk}\,, \tag{2.7b}$$

$$\frac{\partial}{\partial r_{2i}}\mathcal{H}_{li}=0\,, \tag{2.8b}$$

$$\frac{\partial}{\partial r_{2i}}\mathcal{N}_{li}=0\,. \tag{2.9b}$$

As in Paper I, summation over repeated dummy indices is implied.

The differential equations (2.2a)–(2.9a) and (2.2b)–(2.9b) are identical with the differential equations which couple the second-order coherence tensors of the classical field.[3]

For the sake of completeness we also write down the second-order differential equations which follow from Eqs. (2.2)–(2.9) or more briefly from the second-order differential equations of Sec. III of Paper I of this investigation. Thus from[4] I(3.3) and I(3.4) we have

$$\nabla_\alpha^2\mathcal{E}_{kl}=\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\mathcal{E}_{kl},\quad(\alpha=1,\,2)\,. \tag{2.10}$$

Each of the other three correlation tensors also obeys, of course, such a wave equation.

The remaining second-order equations are immediately obtained from Eqs. I(3.5)–I(3.8) and are

$$\epsilon_{ijk}\epsilon_{mnl}\frac{\partial^2\mathcal{H}_{jn}}{\partial r_{1i}\partial r_{2m}}=-\frac{1}{c^2}\frac{\partial^2\mathcal{E}_{kl}}{\partial\tau^2}\,, \tag{2.11}$$

$$\epsilon_{ijk}\epsilon_{mnl}\frac{\partial^2\mathcal{E}_{jn}}{\partial r_{1i}\partial r_{2m}}=-\frac{1}{c^2}\frac{\partial^2\mathcal{H}_{kl}}{\partial\tau^2}\,, \tag{2.12}$$

$$\epsilon_{ijk}\epsilon_{mnl}\frac{\partial^2\mathcal{M}_{jn}}{\partial r_{1i}\partial r_{2m}}=\frac{1}{c^2}\frac{\partial^2\mathcal{N}_{kl}}{\partial\tau^2}\,, \tag{2.13}$$

$$\epsilon_{ijk}\epsilon_{mnl}\frac{\partial^2\mathcal{N}_{jn}}{\partial r_{1i}\partial r_{2m}}=\frac{1}{c^2}\frac{\partial^2\mathcal{M}_{kl}}{\partial\tau^2}\,. \tag{2.14}$$

From Eqs. (2.2)–(2.9) one may derive a number of conservation laws. Alternatively, and more simply, one may obtain these laws immediately from the conservation laws of Sec. IV in Paper I by specializing to stationary fields. For this purpose we introduce the tensors

$$U_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)+\mathcal{H}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)\,, \tag{2.15}$$

$$S_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\mathcal{M}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)-\mathcal{N}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)\,, \tag{2.16}$$

and the associated scalars, vectors, and tensors

$$U(\mathbf{r}_1,\mathbf{r}_2,\tau)=U_{kk}(\mathbf{r}_1,\mathbf{r}_2,\tau)\,, \tag{2.17}$$

$$S(\mathbf{r}_1,\mathbf{r}_2,\tau)=S_{kk}(\mathbf{r}_1,\mathbf{r}_2,\tau)\,, \tag{2.18}$$

$$U_i(\mathbf{r}_1,\mathbf{r}_2,\tau)=\epsilon_{ijk}U_{jk}(\mathbf{r}_1,\mathbf{r}_2,\tau)\,, \tag{2.19}$$

$$S_i(\mathbf{r}_1,\mathbf{r}_2,\tau)=\epsilon_{ijk}S_{jk}(\mathbf{r}_1,\mathbf{r}_2,\tau)\,, \tag{2.20}$$

$$T_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=U_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)+U_{ji}(\mathbf{r}_1,\mathbf{r}_2,\tau)-\delta_{ij}U_{kk}(\mathbf{r}_1,\mathbf{r}_2,\tau)\,, \tag{2.21}$$

$$Q_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=S_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)+S_{ji}(\mathbf{r}_1,\mathbf{r}_2,\tau)-\delta_{ij}S_{kk}(\mathbf{r}_1,\mathbf{r}_2,\tau)\,. \tag{2.22}$$

Some of these quantities are generalizations of quantities which enter the usual conservation laws of the electromagnetic field. In fact, as is shown in Appendix I [Eq. (A1.19), (A1.21), and (A1.22)], the quantities $(1/4\pi)U(\mathbf{r},\mathbf{r},0)$, $(c/4\pi)\mathbf{S}(\mathbf{r},\mathbf{r},0)$, $(1/4\pi c)\mathbf{S}(\mathbf{r},\mathbf{r},0)$, and $(1/4\pi)T_{ij}(\mathbf{r},\mathbf{r},0)$ represent the expectation values of the electromagnetic energy density, the Poynting vector, the field momentum density, and the Maxwell stress tensor, respectively, provided that contributions of the vacuum field are neglected.

The conservation laws (4.11), (4.12), (4.13), and (4.15) of Paper I and the corresponding laws involving differentiation with respect to the second space-time point are then given by the following set of equations:

$$\nabla_\alpha\cdot\mathbf{U}=\mp\frac{1}{c}\frac{\partial}{\partial\tau}S\,, \tag{2.23}$$

$$\nabla_\alpha\cdot\mathbf{S}=\pm\frac{1}{c}\frac{\partial}{\partial\tau}U\,, \tag{2.24}$$

$$\frac{\partial}{\partial r_{\alpha l}}T_{ml}=\mp\frac{1}{c}\frac{\partial}{\partial\tau}S_m\,, \tag{2.25}$$

$$\frac{\partial}{\partial r_{\alpha l}}Q_{ml}=\pm\frac{1}{c}\frac{\partial}{\partial\tau}U_m\,. \tag{2.26}$$

Here the upper or lower signs are taken on the right-hand sides according as $\alpha$ takes on the value 1 or 2, respectively.

[3] P. Roman and E. Wolf, Nuovo Cimento **20**, 462 (1961). There is a misprint in Eq. (3.12b) of this reference: $\mathcal{G}_{mj}$ must be replaced by $\tilde{\mathcal{G}}_{mj}$.

[4] All equations preceded by "I" refer to equations of Paper I (Ref. 1).

The conservation laws (2.23)–(2.26) are identical in form with the conservation laws derived for the classical field[5]. It was shown there that the real part of the equations of the form (2.24) and (2.25) may be regarded as the generalization of the usual laws of conservation of energy and momentum, respectively, in their averaged form and reduces to them in the limit $\mathbf{r}_2 \to \mathbf{r}_1$ and $\tau \to 0$. We show in Appendix II that the same is true for Eqs. (2.24) and (2.25) relating to the quantized field, provided that the contributions from the vacuum field are omitted.

Finally, we note a number of relations that will be needed later. We have, from Eqs. I(2.20)–I(2.23), specialized to a stationary field,

$$\mathcal{E}_{ji}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\mathcal{E}_{ij}^*(\mathbf{r}_2,\,\mathbf{r}_1,\,-\tau)\,, \tag{2.27}$$

$$\mathcal{H}_{ji}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\mathcal{H}_{ij}^*(\mathbf{r}_2,\,\mathbf{r}_1,\,-\tau)\,, \tag{2.28}$$

$$\mathfrak{M}_{ji}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\mathfrak{N}_{ij}^*(\mathbf{r}_2,\,\mathbf{r}_1,\,-\tau)\,. \tag{2.29}$$

Making use of these relations, we see that the tensors $U_{ij}$, $S_{ij}$; $T_{ij}$, and $Q_{ij}$ which are defined by (2.15), (2.16), (2.21), and (2.22), obey the relations

$$U_{ji}(\mathbf{r}_1,\mathbf{r}_2,\tau)=U_{ij}^*(\mathbf{r}_2,\,\mathbf{r}_1,\,-\tau)\,, \tag{2.30}$$

$$S_{ji}(\mathbf{r}_1,\mathbf{r}_2,\tau)=-S_{ij}^*(\mathbf{r}_2,\,\mathbf{r}_1,\,-\tau)\,, \tag{2.31}$$

$$T_{ji}(\mathbf{r}_1,\mathbf{r}_2,\tau)=T_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=T_{ij}^*(\mathbf{r}_2,\,\mathbf{r}_1,\,-\tau)\,, \tag{2.32}$$

$$Q_{ji}(\mathbf{r}_1,\mathbf{r}_2,\tau)=Q_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=-Q_{ij}^*(\mathbf{r}_2,\,\mathbf{r}_1,\,-\tau)\,. \tag{2.33}$$

From (2.17) and (2.30) it readily follows that

$$\mathrm{Im}U(\mathbf{r},\mathbf{r},0)=0\,. \tag{2.34}$$

From (2.20) and (2.31) one has

$$\mathrm{Im}S_i(\mathbf{r},\mathbf{r},0)=0\,, \tag{2.35}$$

and from (2.32)

$$\mathrm{Im}T_{ij}(\mathbf{r},\mathbf{r},0)=0\,. \tag{2.36}$$

In a similar way one can show that

$$\mathrm{Re}S(\mathbf{r},\mathbf{r},0)=0\,, \tag{2.37}$$

$$\mathrm{Re}U_i(\mathbf{r},\mathbf{r},0)=0\,, \tag{2.38}$$

$$\mathrm{Re}Q_{ij}(\mathbf{r},\mathbf{r},0)=0\,. \tag{2.39}$$

## III. THE CROSS-SPECTRAL TENSORS: A QUANTUM-MECHANICAL ANALOG OF THE WIENER-KHINTCHINE THEOREM FOR THE ELECTROMAGNETIC FIELD

In order to understand some of the spectral properties of the field it is desirable to introduce another set of second-rank tensors. These tensors appear naturally if one considers the correlation amongst the spectral components of the positive and the negative parts of the field operators.

Consider the Fourier transforms $\hat{e}^{(+)}(\mathbf{r},\nu)$ and $\hat{e}^{(-)}(\mathbf{r},\nu)$ of $\hat{E}^{(+)}(\mathbf{r},t)$ and $\hat{E}^{(-)}(\mathbf{r},t)$, respectively, [cf. I(2.6) and I(2.7)], viz.,

$$\hat{e}^{(+)}(\mathbf{r},\nu)=\int_{-\infty}^{\infty}\hat{E}^{(+)}(\mathbf{r},t)e^{2\pi i\nu t}dt\,, \tag{3.1}$$

$$\hat{e}^{(-)}(\mathbf{r},\nu)=\int_{-\infty}^{\infty}\hat{E}^{(-)}(\mathbf{r},t)e^{-2\pi i\nu t}dt\,. \tag{3.2}$$

The "spectral correlation" may be defined as the expectation value of the normally ordered product of the two operators $\hat{e}^{(-)}(\mathbf{r}_1,\nu)$ and $\hat{e}^{(+)}(\mathbf{r}_2,\nu)$. If we use (3.1) and (3.2), this correlation is given by

$$\begin{aligned}\mathrm{tr}\{\hat{\rho}\hat{e}_i^{(-)}(\mathbf{r}_1,\nu)\hat{e}_j^{(+)}(\mathbf{r}_2,\nu')\}&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{tr}\{\hat{\rho}\hat{E}_i^{(-)}(\mathbf{r}_1,t_1)\hat{E}_j^{(+)}(\mathbf{r}_2,t_2)\}\exp\{2\pi i(\nu't_2-\nu t_1)\}dt_1dt_2\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{E}_{ij}(\mathbf{r}_1,t_1;\,\mathbf{r}_2,t_2)\exp\{2\pi i\nu(t_2-t_1)\}\exp\{2\pi i(\nu'-\nu)t_2\}dt_1dt_2.\end{aligned}$$

Since the field is stationary, $\mathcal{E}_{ij}(\mathbf{r}_1,\,\mathbf{r}_2,\,t_2-t_1)$ may be written in place of $\mathcal{E}_{ij}(\mathbf{r}_1,t_1;\,\mathbf{r}_2,t_2)$ and we have, if we introduce new variables $\tau=t_2-t_1$, $\theta=t_2$,

$$\begin{aligned}\mathrm{tr}\{\hat{\rho}\hat{e}_i^{(-)}(\mathbf{r}_1,\nu)\hat{e}_j^{(+)}(\mathbf{r}_2,\nu')\}&=\int_{-\infty}^{\infty}\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)e^{2\pi i\nu\tau}d\tau\int_{-\infty}^{\infty}e^{2\pi i\theta(\nu'-\nu)}d\theta\\&=W_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)\delta(\nu-\nu')\,,\end{aligned} \tag{3.3}$$

where[6]

$$W_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)=\int_{-\infty}^{\infty}\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)e^{2\pi i\nu\tau}d\tau\,. \tag{3.4}$$

[5] P. Roman and E. Wolf, Nuovo Cimento **20**, 477 (1961). Equation (2.26) of this reference is incorrect. It should read

$$\partial_k\equiv\partial_k{}^1\to\partial_k{}^2,\quad \mathcal{S}\to\mathcal{S}^T,\quad \mathcal{W}\to\mathcal{W}^T,$$

where superscript $T$ denotes the transpose.

[6] We implicitly assume here that the Fourier transform (3.4) of $\mathcal{E}_{ij}$ exists, as will be the case in most situations of practical interest. Whether or not the Fourier transform exists in the ordinary sense, one may define a frequency distribution tensor $F_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)$ by means of a Fourier-Stieltjes integral. In the case when $W_{ij}^{(e)}$ exists

$$\partial F_{ij}^{(e)}/\partial\nu=W_{ij}^{(e)}.$$

However, we will not consider here this refinement which is well known in the theory of stationary random processes [see for example, A. M. Yaglom, *An Introduction to the Theory of Stationary Random Functions* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1962), Secs. 10 and 15].

*Note added in proof.* The tensor analogous to $W^{(e)}$ was discussed in the framework of the classical theory of electromagnetic fields in a recent paper by A. D. Jacobson [IEE Trans. Antennas Propagation **15**, 24 (1967)]. He defined this tensor as the Fourier transform of the corresponding second-order correlation tensor $\mathcal{E}$ rather than as a spectral correlation.

Since $\delta(\nu-\nu')=0$ if $\nu\neq\nu'$, we see that *the different frequency components of the two operators* $\hat{e}^{(+)}$ *and* $\hat{e}^{(-)}$ *at any two (distinct or coincident) points are uncorrelated.* Correlation only exists for spectral components of the same frequency and may then be expressed in the form

$$\lim_{\Delta\nu\to 0}\int_{\nu-\Delta\nu/2}^{\nu+\Delta\nu/2}\mathrm{tr}\{\hat{\rho}\hat{e}_i^{(-)}(\mathbf{r}_1,\nu)\hat{e}_j^{(+)}(\mathbf{r}_2,\nu')\}d\nu' = W_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)\,, \quad (3.5)$$

*where the tensor* $W_{ij}^{(e)}$ *is the Fourier transform of the second-order coherence tensor* $\mathcal{E}_{ij}$. It is evident that in the language of the theory of random processes $W_{ij}(\mathbf{r}_1,\mathbf{r}_2,\nu)$ is the *(second-order) electric cross-spectral tensor of the quantized electric field.*

Let us now set

$$W^{(e)}(\mathbf{r},\mathbf{r},\nu)=W_{ii}^{(e)}(\mathbf{r},\mathbf{r},\nu) = \lim_{\Delta\nu\to 0}\int_{\nu-\Delta\nu/2}^{\nu+\Delta\nu/2}\mathrm{tr}\{\hat{\rho}\hat{e}^{(-)}(\mathbf{r},\nu)\cdot\hat{e}^{(+)}(\mathbf{r},\nu')\}d\nu'\,, \quad (3.6)$$

$$\mathcal{E}(\mathbf{r},\mathbf{r},\tau)=\mathcal{E}_{ii}(\mathbf{r},\mathbf{r},\tau) = \mathrm{tr}\{\hat{\rho}\hat{E}^{(-)}(\mathbf{r},t)\cdot\hat{E}^{(+)}(\mathbf{r},t+\tau)\}\,. \quad (3.7)$$

From (3.4) it then follows that

$$W^{(e)}(\mathbf{r},\mathbf{r},\nu)=\int_{-\infty}^{\infty}\mathcal{E}(\mathbf{r},\mathbf{r},\tau)e^{2\pi i\nu\tau}d\tau\,, \quad (3.8a)$$

$$\mathcal{E}(\mathbf{r},\mathbf{r},\tau)=\int_{0}^{\infty}W^{(e)}(\mathbf{r},\mathbf{r},\nu)e^{-2\pi i\nu\tau}d\nu\,. \quad (3.8b)$$

By writing the lower limit of integration in (3.8b) as zero rather than $-\infty$, we imply that $\mathcal{E}$ does not contain any negative-frequency components. That this is so, follows from the analytic behavior of $\mathcal{E}$, as shown in Appendix I, Eq. (A1.5).

According to (3.7), $\mathcal{E}(\mathbf{r},\mathbf{r},\tau)$ represents the trace of the electric correlation tensor for the special case when the two points $\mathbf{r}_1$ and $\mathbf{r}_2$ coincide ($\mathbf{r}_1=\mathbf{r}_2=\mathbf{r}$). Moreover, it follows from (3.8b) and Eq. (A1.17) of Appendix I that $(1/4\pi)W^{(e)}(\mathbf{r},\mathbf{r},\nu)d\nu$ represents the contribution from the frequency range $\nu$, $\nu+d\nu$ to the expectation value of the electric energy at the point $\mathbf{r}$, provided that the contribution from the vacuum field is neglected. The Fourier transform relation (3.8b), which connects $\mathcal{E}(\mathbf{r},\mathbf{r},\tau)$ and $W^{(e)}(\mathbf{r},\mathbf{r},\nu)$ is thus evidently the quantum-mechanical analog of the *Wiener-Khintchine theorem*[7] of the theory of stationary random processes.

[7] A restricted formulation of this quantum-mechanical analog was given by R. J. Glauber [Phys. Rev. **131**, 2786 (1963)]. Though not explicitly stated, his formulation applies only to the narrow class of stationary fields which are spatially homogeneous. This restriction is evident from the fact that Glauber's expression (10.15) for the energy spectrum is independent of position; this result is a consequence of the incorrect assumption, made in his Eq. (10.13), that for any stationary fields his $P(\{\alpha_k\})$ function is independent of the phases of the $\alpha_k$'s. It has been shown by Kano (Ref. 2) that for a radiation field with a finite or countable infinite number of modes $N(>1)$, the phase independence implies not only stationarity, but also homogeneity of the field.

Returning to Eq. (3.5), we note that the electric cross-spectral tensor satisfies the relation

$$W_{ji}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)=W_{ij}^{(e)*}(\mathbf{r}_2,\mathbf{r}_1,\nu)\,. \quad (3.9)$$

Moreover, this tensor is non-negative-definite in the sense that for any arbitrary set of functions $f_i(\mathbf{r})$, $(i=1, 2, 3)$, for which the integral on the left-hand side of (3.10) below is defined,

$$\int d^3r_1\int d^3r_2\, f_i^*(r_1)W_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)f_j(\mathbf{r}_2)\geqslant 0\,. \quad (3.10)$$

The proof of this inequality as well as its relationship to the non-negative-definiteness condition I(5.4) which the electric coherence tensor obeys, is discussed in Appendix III. Here we only note that with the special choice $f_i(\mathbf{r}_1)=\delta_{ik}\delta^{(3)}(\mathbf{r}-\mathbf{r}_1)$, (3.10) implies that each diagonal component of the tensor $W_{ij}$ for $\mathbf{r}_1=\mathbf{r}_2$ is non-negative, so that

$$W^{(e)}(\mathbf{r},\mathbf{r},\nu)=W_{ii}^{(e)}(\mathbf{r},\mathbf{r},\nu)\geqslant 0\,, \quad (3.11)$$

a result that was to be expected from the physical significance of $W^{(e)}$.

By analogy with the electric cross-spectral tensor, we may introduce three other cross-spectral tensors, which involve the magnetic field. Let $\hat{h}^{(+)}$ and $\hat{h}^{(-)}$ be the operators which bear the same relationship to the magnetic field operator as $\hat{e}^{(+)}$ and $\hat{e}^{(-)}$ bear to the electric field operator. Then [cf. (3.1) and (3.2)]

$$\hat{h}^{(+)}(\mathbf{r},\nu)=\int_{-\infty}^{\infty}\hat{H}^{(+)}(\mathbf{r},t)e^{2\pi i\nu t}dt\,, \quad (3.12)$$

$$\hat{h}^{(-)}(\mathbf{r},\nu)=\int_{-\infty}^{\infty}\hat{H}^{(-)}(\mathbf{r},t)e^{-2\pi i\nu t}dt\,, \quad (3.13)$$

where $\hat{H}^{(+)}$ and $\hat{H}^{(-)}$ are the positive- and negative-frequency parts of the magnetic field operator $\hat{H}$. We then readily find, by analogy with (3.3), that

$$\mathrm{tr}\{\hat{\rho}\hat{h}_i^{(-)}(\mathbf{r}_1,\nu)\hat{h}_j^{(+)}(\mathbf{r}_2,\nu')\} = \delta(\nu-\nu')W_{ij}^{(h)}(\mathbf{r}_1,\mathbf{r}_2,\nu)\,, \quad (3.14)$$

$$\mathrm{tr}\{\hat{\rho}\hat{e}_i^{(-)}(\mathbf{r}_1,\nu)\hat{h}_j^{(+)}(\mathbf{r}_2,\nu')\} = \delta(\nu-\nu')W_{ij}^{(m)}(\mathbf{r}_1,\mathbf{r}_2,\nu)\,, \quad (3.15)$$

$$\mathrm{tr}\{\hat{\rho}\hat{h}_i^{(-)}(\mathbf{r}_1,\nu)\hat{e}_j^{(+)}(\mathbf{r}_2,\nu')\} = \delta(\nu-\nu')W_{ij}^{(n)}(\mathbf{r}_1,\mathbf{r}_2,\nu)\,, \quad (3.16)$$

where

$$W_{ij}^{(h)}(\mathbf{r}_1,\mathbf{r}_2,\nu)=\int_{-\infty}^{\infty}\mathcal{H}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)e^{2\pi i\nu\tau}d\tau\,, \quad (3.17)$$

$$W_{ij}^{(m)}(\mathbf{r}_1,\mathbf{r}_2,\nu)=\int_{-\infty}^{\infty}\mathcal{M}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)e^{2\pi i\nu\tau}d\tau\,, \quad (3.18)$$

$$W_{ij}^{(n)}(\mathbf{r}_1,\mathbf{r}_2.\nu)=\int_{-\infty}^{\infty}\mathcal{N}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)e^{2\pi i\nu\tau}d\tau\,. \quad (3.19)$$

The physical interpretation of Eqs. (3.14)–(3.19) is, of course, strictly analogous to that given in connection with Eqs. (3.3) and (3.8a). We may call $W_{ij}^{(h)}$ the (second-order) magnetic cross-spectral tensors and $W_{ij}^{(m)}$ and $W_{ij}^{(n)}$ the (second-order) mixed cross-spectral tensors of the quantized electromagnetic field. These tensors evidently satisfy the following relations which correspond to (3.9):

$$W_{ji}^{(h)}(\mathbf{r}_1,\mathbf{r}_2,\nu)=W_{ij}^{(h)*}(\mathbf{r}_2,\mathbf{r}_1,\nu)\,, \qquad (3.20)$$

$$W_{ji}^{(m)}(\mathbf{r}_1,\mathbf{r}_2,\nu)=W_{ij}^{(n)*}(\mathbf{r}_2,\mathbf{r}_1,\nu)\,. \qquad (3.21)$$

The Fourier inverse of the relation (3.17) is essentially Wiener-Khintchine theorem for the magnetic field. If we add this Fourier inverse relation and the relation (3.8a) involving the electric correlation and recall the definition (2.17) of $U$, we obtain the relation

$$U(\mathbf{r},\mathbf{r},\tau)=\int_0^\infty \{W^{(e)}(\mathbf{r},\mathbf{r},\nu)+W^{(h)}(\mathbf{r},\mathbf{r},\nu)\}e^{-2\pi i\nu\tau}d\nu. \quad (3.22)$$

This relation expresses the Wiener-Khintchine theorem for the total (electromagnetic) field.

There are corresponding relations involving the vector $\mathbf{S}$ and the tensor $T_{ij}$ defined by Eqs. (2.20) and (2.21). From Eqs. (2.20) and (2.16) and from the Fourier inverses of Eqs. (3.18) and (3.19), it follows that

$$S_i(\mathbf{r},\mathbf{r},\tau)=\epsilon_{ijk}\int_0^\infty \{W_{jk}^{(m)}(\mathbf{r},\mathbf{r},\nu) - W_{jk}^{(n)}(\mathbf{r},\mathbf{r},\nu)\}e^{-2\pi i\nu\tau}d\nu. \quad (3.23)$$

From the physical interpretation of the vector $\mathbf{S}(\mathbf{r},\mathbf{r},0)$ given in Appendix I, Eq. (A1.20), and from (3.21) it then follows that the quantity

$$\frac{1}{4\pi c}\epsilon_{ijk}\{W_{jk}^{(m)}(\mathbf{r},\mathbf{r},\nu)-W_{jk}^{(n)}(\mathbf{r},\mathbf{r},\nu)\}d\nu, \qquad (i=1,\,2,\,3), \quad (3.24)$$

represents the contribution, from the frequency range $\nu$, $\nu+d\nu$, to the expectation value of the field momentum, provided that the contribution from the vacuum field is neglected.

From Eqs. (2.21), (2.15), (3.8b), and the Fourier inverse of Eq. (3.17), it follows that

$$T_{ij}(\mathbf{r},\mathbf{r},\tau)=\int_0^\infty \{W_{ij}^{(e)}(\mathbf{r},\mathbf{r},\nu)+W_{ji}^{(e)}(\mathbf{r},\mathbf{r},\nu) + W_{ij}^{(h)}(\mathbf{r},\mathbf{r},\nu)+W_{ji}^{(h)}(\mathbf{r},\mathbf{r},\nu) - \delta_{ij}[W_{kk}^{(e)}(\mathbf{r},\mathbf{r},\nu)+W_{kk}^{(h)}(\mathbf{r},\mathbf{r},\nu)]\}e^{-2\pi i\nu\tau}d\nu. \quad (3.25)$$

From the physical interpretation of the tensor $T_{ij}(\mathbf{r},\mathbf{r},0)$ given in Appendix I, Eq. (A1.22), and from Eqs. (3.9) and (3.20), it then follows that the quantity

$$\frac{1}{2\pi}\{W_{ij}^{(e)}(\mathbf{r},\mathbf{r},\nu)+W_{ij}^{(h)}(\mathbf{r},\mathbf{r},\nu) - \tfrac{1}{2}\delta_{ij}[W_{kk}^{(e)}(\mathbf{r},\mathbf{r},\nu)+W_{kk}^{(h)}(\mathbf{r},\mathbf{r},\nu)]\}d\nu \quad (3.26)$$

represents the contribution from the frequency range $\nu$, $\nu+d\nu$ to the expectation value of the Maxwell's electromagnetic stress tensor, provided that the contributions from the vacuum field are neglected.

Representations analogous to (3.22), (3.23), and (3.25) also exist, of course, for the scalar $S(\mathbf{r},\mathbf{r},\tau)$, the vector $U_i(\mathbf{r},\mathbf{r},\tau)$, and the tensor $Q_{ij}(\mathbf{r},\mathbf{r},\tau)$.

The following non-negative-definiteness condition involving the four cross-spectral tensors holds:

$$\int d^3r_1\int d^3r_2\{f_i^*(\mathbf{r}_1)W_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)f_j(\mathbf{r}_2) + g_i^*(\mathbf{r}_1)W_{ij}^{(h)}(\mathbf{r}_1,\mathbf{r}_2,\nu)g_j(\mathbf{r}_2) + f_i^*(\mathbf{r}_1)W_{ij}^{(m)}(\mathbf{r}_1,\mathbf{r}_2,\nu)g_j(\mathbf{r}_2) + g_i^*(\mathbf{r}_1)W_{ij}^{(n)}(\mathbf{r}_1,\mathbf{r}_2,\nu)f_j(\mathbf{r}_2)\}\geqslant 0. \quad (3.27)$$

Here $f_i(\mathbf{r})$ and $g_i(\mathbf{r})$, ($i=1,\,2,\,3$), are arbitrary sets of functions for which the integral on the left-hand side of (3.27) is defined. This inequality is established in Appendix III. Here we only note that if we choose $g_i(\mathbf{r})\equiv 0$, ($i=1,\,2,\,3$), we recover the non-negative-definiteness condition (3.10) on the electric cross-spectral tensor. If we choose $f_i(\mathbf{r})\equiv 0$, ($i=1,\,2,\,3$), we obtain a similar condition on the magnetic cross-spectral tensor.

For the sake of completeness, we also write down differential equations which couple the four cross-spectral tensors. If we take the Fourier transforms of Eqs. (2.2a)–(2.9a) and use (3.4) and (3.17)–(3.19), we obtain the set of equations

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}W_{jl}^{(e)}=-\frac{2\pi i\nu}{c}W_{kl}^{(n)}\,, \qquad (3.28a)$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}W_{jl}^{(h)}=\frac{2\pi i\nu}{c}W_{kl}^{(m)}\,, \qquad (3.29a)$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}W_{jl}^{(n)}=\frac{2\pi i\nu}{c}W_{kl}^{(e)}\,, \qquad (3.30a)$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}W_{jl}^{(m)}=-\frac{2\pi i\nu}{c}W_{kl}^{(h)}\,, \qquad (3.31a)$$

$$\frac{\partial}{\partial r_{1i}}W_{il}^{(\mu)}=0, \quad (\mu=e,\,h,\,m,\text{ or }n)\,, \qquad (3.32a)$$

the arguments of all the $W$'s being $\mathbf{r}_1,\mathbf{r}_2,\nu$.

In a similar way, the Fourier inversion of Eqs. (2.2b)–(2.9b) gives the set of equations

$$\epsilon_{ijk}\frac{\partial}{\partial r_{2i}}W_{lj}^{(e)}=\frac{2\pi i\nu}{c}W_{lk}^{(m)}\,, \qquad (3.28b)$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{2i}}W_{lj}^{(h)}=-\frac{2\pi i\nu}{c}W_{lk}^{(n)}\,, \qquad (3.29b)$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{2i}}W_{lj}^{(n)}=\frac{2\pi i\nu}{c}W_{lk}^{(h)}, \tag{3.30b}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{2i}}W_{lj}^{(m)}=-\frac{2\pi i\nu}{c}W_{lk}^{(e)}, \tag{3.31b}$$

$$\frac{\partial}{\partial r_{2i}}W_{li}^{(\mu)}=0,\quad (\mu=e,\,h,\,m,\text{ or }n). \tag{3.32b}$$

From these two sets of equations one may derive, in a way similar to that of the derivation of the equations of Sec. III in Paper I, the following second-order equations:

$$\left(\nabla_\alpha^2+\frac{4\pi^2\nu^2}{c^2}\right)W_{ij}^{(\mu)}=0, \tag{3.33}$$

($\alpha=1$ or 2, $\mu=e,\,h,\,m$, or $n$);

$$\epsilon_{ijk}\epsilon_{mnl}\frac{\partial^2}{\partial r_{1i}\partial r_{2m}}W_{jn}^{(e)}=\frac{4\pi^2\nu^2}{c^2}W_{kl}^{(h)}, \tag{3.34}$$

$$\epsilon_{ijk}\epsilon_{mnl}\frac{\partial^2}{\partial r_{1i}\partial r_{2m}}W_{jn}^{(h)}=\frac{4\pi^2\nu^2}{c^2}W_{kl}^{(e)}, \tag{3.35}$$

$$\epsilon_{ijk}\epsilon_{mnl}\frac{\partial^2}{\partial r_{1i}\partial r_{2m}}W_{jn}^{(m)}=-\frac{4\pi^2\nu^2}{c^2}W_{kl}^{(n)}, \tag{3.36}$$

$$\epsilon_{ijk}\epsilon_{mnl}\frac{\partial^2}{\partial r_{1i}\partial r_{2m}}W_{jn}^{(n)}=-\frac{4\pi^2\nu^2}{c^2}W_{kl}^{(m)}. \tag{3.37}$$

This completes our discussion of the mathematical framework of the second-order coherence theory of the quantized field. An application of some of our results to the study of the cross-spectral properties of blackbody radiation will be described elsewhere.

## APPENDIX I: ANALYTIC PROPERTIES OF THE CORRELATION TENSORS AND SOME OF THEIR CONSEQUENCES

Let $\hat{F}(t)$ be an arbitrary operator and let the superscript (+) and (−) denote its positive- and negative-frequency parts, respectively, defined as in I(2.6) and I(2.7). The positive- and the negative-frequency parts may be expressed in the form (cf. footnote 3 of Paper I)

$$\hat{F}^{(\pm)}(t)=\lim_{\eta\to+0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\hat{F}(t-\tau)}{\pm\tau-i\eta}d\tau. \tag{A1.1}$$

Let $\hat{G}^{(+)}$ and $\hat{G}^{(-)}$ be the positive- and the negative-frequency parts, respectively, of another operator $\hat{G}$, and let us consider the correlation function

$$C(t_2-t_1)=\langle\hat{F}^{(-)}(t_1)\hat{G}^{(+)}(t_2)\rangle, \tag{A1.2}$$

where the quantum-mechanical expectation value is to be taken with respect to a density operator of a *stationary* field. We have, if we use (A1.1) and a similar relation for $\hat{G}^{(+)}$,

$$C(\tau)=\lim_{\eta_1\to+0}\lim_{\eta_2\to+0}\frac{1}{4\pi^2}\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\tau_1d\tau_2\frac{\langle\hat{F}(t-\tau_1)\hat{G}(t+\tau-\tau_2)\rangle}{(\tau_1+i\eta_1)(\tau_2-i\eta_2)}. \tag{A1.3}$$

Since the field is assumed to be stationary

$$\langle\hat{F}(t-\tau_1)\hat{G}(t+\tau-\tau_2)\rangle\equiv\langle\hat{F}(t)\hat{G}(t+\tau+\tau_1-\tau_2)\rangle,$$

and using this result the integral on the right-hand side of (A1.3) may be simplified by introducing new variables $\alpha=\tau_2-\tau_1$, $\beta=\tau_2+\tau_1$. The integration over $\beta$ can then immediately be carried out and we obtain

$$C(\tau)=\lim_{\eta\to+0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\alpha\frac{\langle\hat{F}(t)\hat{G}(t+\tau-\alpha)\rangle}{\alpha-i\eta}. \tag{A1.4}$$

The quantity on the right-hand side is nothing but [cf. (A1.1)] the positive-frequency part of the correlation function $\langle\hat{F}(t)\hat{G}(t+\tau)\rangle$, so that we have the result

$$C(\tau)\equiv\langle\hat{F}^{(-)}(t)\hat{G}^{(+)}(t+\tau)\rangle=\langle\hat{F}(t)\hat{G}(t+\tau)\rangle^{(+)}. \tag{A1.5}$$

In a similar manner one may show that

$$\langle\hat{F}^{(+)}(t)\hat{G}^{(-)}(t+\tau)\rangle=\langle\hat{F}(t)\hat{G}(t+\tau)\rangle^{(-)}. \tag{A1.6}$$

The equation (A1.6) may be rewritten in the form

$$\langle\hat{G}^{(-)}(t+\tau)\hat{F}^{(+)}(t)\rangle=\langle\hat{F}(t)\hat{G}(t+\tau)\rangle^{(-)}-\langle[\hat{F}^{(+)}(t),\hat{G}^{(-)}(t+\tau)]\rangle, \tag{A1.7}$$

where $[\hat{A},\hat{B}]$ denotes the commutator $\hat{A}\hat{B}-\hat{B}\hat{A}$.

It is seen from (A1.5) that the correlation function $C(\tau)$ contains only positive-frequency components. Hence under very general conditions assumed here to be satisfied,[8] $C(\tau)$ is the boundary value on the real $\tau$ axis of a function which is analytic and regular in the lower-half of the complex $\tau$ plane. This statement is equivalent to saying that the real and imaginary parts of $C(\tau)$ are Hilbert transforms of each other,[8] i.e.,

$$\begin{aligned}\mathrm{Re}C(\tau)&=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Im}C(\tau')}{\tau'-\tau}d\tau',\\ \mathrm{Im}C(\tau)&=+\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Re}C(\tau')}{\tau'-\tau}d\tau',\end{aligned} \tag{A1.8}$$

where $P$ denotes the Cauchy principal value of the integral at $\tau'=\tau$.

In particular, if we choose $\hat{F}$ and $\hat{G}$ to be the Cartesian components of the electric or of the magnetic field operators at the space-time points $\mathbf{r}_1$, $t$ and $\mathbf{r}_2$, $t+\tau$, respectively, we may conclude that all the four correlation tensors $\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)$, $\mathcal{H}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)$, $\mathcal{M}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)$, and

[8] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon Press, Oxford, England, 1948), 2nd ed., Chap. V.

$\mathfrak{N}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)$ are analytic signals.[9] Further, it follows from I(2.20), specialized to stationary fields, that

$$\mathrm{Re}\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\tfrac{1}{2}\langle\hat{E}_i^{(-)}(\mathbf{r}_1,t)\hat{E}_j^{(+)}(\mathbf{r}_2,t+\tau)\rangle+\tfrac{1}{2}\langle\hat{E}_j^{(-)}(\mathbf{r}_2,t+\tau)\hat{E}_i^{(+)}(\mathbf{r}_1,t)\rangle.$$

If on the right-hand side we make use of the identities (A1.5) and (A1.7), we find that

$$\mathrm{Re}\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\tfrac{1}{2}\langle\hat{E}_i(\mathbf{r}_1,t)\hat{E}_j(\mathbf{r}_2,t+\tau)\rangle-\tfrac{1}{2}\langle[\hat{E}_i^{(+)}(\mathbf{r}_1,t),\hat{E}_j^{(-)}(\mathbf{r}_2,t+\tau)]\rangle. \quad (A1.9)$$

Now the term on the right-hand side, involving the commutator, has a simple interpretation. To see this let $|0\rangle$ represent the vacuum state, so that

$$\hat{E}^{(+)}|0\rangle=0 \quad\text{and}\quad \langle 0|\hat{E}^{(-)}=0. \quad (A1.10)$$

The expectation value in the vacuum state of the operator $\hat{E}_i(\mathbf{r}_1,t)\hat{E}_j(\mathbf{r}_2,t+\tau)$, which for short we denote by $\hat{E}_i(1)\hat{E}_j(2)$ is

$$\langle 0|\hat{E}_i(1)\hat{E}_j(2)|0\rangle=\langle 0|\{\hat{E}_i^{(-)}(1)+\hat{E}_i^{(+)}(1)\}\times\{\hat{E}_j^{(-)}(2)+\hat{E}_j^{(+)}(2)\}|0\rangle=\langle 0|[\hat{E}_i^{(+)}(1),\hat{E}_j^{(-)}(2)]|0\rangle, \quad (A1.11)$$

where (A1.10) was used. Now since the commutator is a $c$-number, the expectation value of the commutator on the right-hand side of (A1.11) may be replaced by its expectation value with respect to *any* state of the field. Hence the average of the commutator on the right-hand side of (A1.9) is the vacuum expectation value (denoted by subscript "vac") of the operator $\hat{E}_i(\mathbf{r}_1,t)\times\hat{E}_j(\mathbf{r}_2,t+\tau)$, i.e.,

$$\langle[\hat{E}_i^{(+)}(\mathbf{r}_1,t),\hat{E}_j^{(-)}(\mathbf{r}_2,t+\tau)]\rangle=\langle\hat{E}_i(\mathbf{r}_1,t)\hat{E}_j(\mathbf{r}_2.\ t+\tau)\rangle_{\mathrm{vac}}. \quad (A1.12)$$

It follows that (A1.9) may be expressed in the form

$$\mathrm{Re}\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\tfrac{1}{2}\langle\hat{E}_i(\mathbf{r}_1,t)\hat{E}_j(\mathbf{r}_2,t+\tau)\rangle-\tfrac{1}{2}\langle\hat{E}_i(\mathbf{r}_1,t)\hat{E}_j(\mathbf{r}_2,t+\tau)\rangle_{\mathrm{vac}}. \quad (A1.13)$$

In a strictly similar manner it may be shown that

$$\mathrm{Re}\mathcal{H}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\tfrac{1}{2}\langle\hat{H}_i(\mathbf{r}_1,t)\hat{H}_j(\mathbf{r}_2,t+\tau)\rangle-\tfrac{1}{2}\langle\hat{H}_i(\mathbf{r}_1,t)\hat{H}_j(\mathbf{r}_2,t+\tau)\rangle_{\mathrm{vac}}, \quad (A1.14)$$

$$\mathrm{Re}\mathfrak{M}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\tfrac{1}{2}\langle\hat{E}_i(\mathbf{r}_1,t)\hat{H}_j(\mathbf{r}_2,t+\tau)\rangle-\tfrac{1}{2}\langle\hat{E}_i(\mathbf{r}_1,t)\hat{H}_j(\mathbf{r}_2,t+\tau)\rangle_{\mathrm{vac}}, \quad (A1.15)$$

$$\mathrm{Re}\mathfrak{N}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\tfrac{1}{2}\langle\hat{H}_i(\mathbf{r}_1,t)\hat{E}_j(\mathbf{r}_2,t+\tau)\rangle-\tfrac{1}{2}\langle\hat{H}_i(\mathbf{r}_1,t)\hat{E}_j(\mathbf{r}_2,t+\tau)\rangle_{\mathrm{vac}}. \quad (A1.16)$$

With the help of the relations (A1.13)–(A1.16) we may readily write down simple expressions for quantities of special physical interest. We have from (3.7) and (A1.13) and from the fact that, because of (2.27), $\mathcal{E}(\mathbf{r},\mathbf{r},0)$ is real,

$$\mathcal{E}(\mathbf{r},\mathbf{r},0)\equiv\mathcal{E}_{ii}(\mathbf{r},\mathbf{r},0)=\tfrac{1}{2}\langle\{\hat{E}(\mathbf{r},t)\}^2\rangle-\tfrac{1}{2}\langle\{\hat{E}(\mathbf{r},t)\}^2\rangle_{\mathrm{vac}},$$

so that

*the expectation value of the electric energy density* is

$$(1/4\pi)\mathcal{E}(\mathbf{r},\mathbf{r},0)+\text{v.c.}, \quad (A1.17)$$

where v.c. denotes "vacuum contribution. Similarly, *the expectation value of the magnetic energy density* is

$$(1/4\pi)\mathcal{H}(\mathbf{r},\mathbf{r},0)+\text{v.c.} \quad (A1.18)$$

On combining (A1.17) and (A1.18) and on using (2.17) and (2.15), we see that *the expectation value of the electromagnetic energy density* is

$$(1/4\pi)U(\mathbf{r},\mathbf{r},0)+\text{v.c.} \quad (A1.19)$$

Further we have from (2.20), (2.16), (A1.15), and (A1.16), on taking the real part and using (2.35),

$$S_i(\mathbf{r},\mathbf{r},0)=\mathrm{Re}\epsilon_{ijk}S_{jk}(\mathbf{r},\mathbf{r},0)=\tfrac{1}{2}[\langle\{\hat{E}(\mathbf{r},t)\times\hat{H}(\mathbf{r},t)\}_i\rangle-\langle\{\hat{H}(\mathbf{r},t)\times\hat{E}(\mathbf{r},t)\}_i\rangle]+\text{v.c.},$$

so that

*the expectation value of the energy flux* (*Poynting vector*) is

$$(c/4\pi)\mathbf{S}(\mathbf{r},\mathbf{r},0)+\text{v.c.} \quad (A1.20)$$

and

*the expectation value of the electromagnetic momentum density* is

$$(1/4\pi c)\mathbf{S}(\mathbf{r},\mathbf{r},0)+\text{v.c.} \quad (A1.21)$$

Finally, we have from (2.21), (2.15), (A1.13), and (A1.14) on taking the real part and using (2.36),

$$T_{ij}(\mathbf{r},\mathbf{r},0)=\tfrac{1}{2}\langle\{\hat{E}_i(\mathbf{r},t)\hat{E}_j(\mathbf{r},t)\rangle+\langle\hat{E}_j(\mathbf{r},t)\hat{E}_i(\mathbf{r},t)\rangle-\delta_{ij}\langle\hat{E}_k(\mathbf{r},t)\hat{E}_k(\mathbf{r},t)\rangle+\langle\hat{H}_i(\mathbf{r},t)\hat{H}_j(\mathbf{r},t)\rangle+\langle\hat{H}_j(\mathbf{r},t)\hat{H}_i(\mathbf{r},t)\rangle-\delta_{ij}\langle\hat{H}_k(\mathbf{r},t)\hat{H}_k(\mathbf{r},t)\rangle\}+\text{v.c.}$$

so that

*the expectation value of the Maxwell's electromagnetic stress tensor* is

$$(1/4\pi)T_{ij}(\mathbf{r},\mathbf{r},0)+\text{v.c.} \quad (A1.22)$$

## APPENDIX II: LIMITING FORMS OF THE CONSERVATION LAWS (2.24) AND ((2.25)

### A. The Conservation Law (2.24)

Consider first the real part of the conservation law (2.24), with $\alpha=1$;

$$\mathrm{Re}\boldsymbol{\nabla}_1\cdot\mathbf{S}(\mathbf{r}_1,\mathbf{r}_2,\tau)=-\frac{1}{c}\mathrm{Re}\frac{\partial}{\partial\tau}U(\mathbf{r}_1,\mathbf{r}_2,\tau). \quad (A2.1)$$

We examine each side separately. We have, if we use (2.20), (2.16), (A1.15), and (A1.16),

$$\mathrm{Re}\boldsymbol{\nabla}_1\cdot\mathbf{S}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\tfrac{1}{2}\epsilon_{ijk}\left\langle\frac{\partial}{\partial r_{1i}}\{\hat{E}_j(1)\hat{H}_k(2)-\hat{H}_j(1)\hat{E}_k(2)\}\right\rangle+\text{v.c.} \quad (A2.2)$$

---

[9] It follows that each of the quantities $U_{ij}$, $S_{ij}$, $U$, $S$, $\mathbf{U}$, $\mathbf{S}$, $T_{ij}$, and $Q_{ij}$, defined by Eqs. (2.15)–(2.22) is also an analytic signal.

As before, the variables (1) and (2) in the various operators refer to space-time point $\mathbf{r}_1,t$ and $\mathbf{r}_2$, $t+\tau$, respectively, and v.c. denotes a vacuum contribution. In (A2.2) we proceed to the limit $\mathbf{r}_1\rightarrow\mathbf{r}_2=\mathbf{r}$, $\tau\rightarrow 0$ and obtain the formula

$$\mathrm{Re}\nabla_1\cdot\mathbf{S}(\mathbf{r}_1,\mathbf{r}_2,\tau)\,|_{\mathbf{r}_1=\mathbf{r}_2=\mathbf{r};\ \tau=0}=\tfrac{1}{2}\epsilon_{ijk}\left\langle\frac{\partial}{\partial r_i}(\hat{E}_j\hat{H}_k-\hat{H}_j\hat{E}_k)\right\rangle -\tfrac{1}{2}\epsilon_{ijk}\left\langle\hat{E}_j\frac{\partial\hat{H}_k}{\partial r_i}-\hat{H}_j\frac{\partial\hat{E}_k}{\partial r_i}\right\rangle+\text{v.c.}$$

Here and elsewhere in this Appendix, the variables not shown explicitly in the various operators are $\mathbf{r}$, $t$, e.g., $\hat{E}_j=\hat{E}_j(\mathbf{r},t)$, etc.

If we use the operator form I(2.1) and I(2.2) of Maxwell's two main equations, the last equation may be rewritten in the form

$$\mathrm{Re}\nabla_1\cdot\mathbf{S}(\mathbf{r}_1,\mathbf{r}_2,\tau)\,|_{\mathbf{r}_1=\mathbf{r}_2=\mathbf{r};\ \tau=0}=\tfrac{1}{2}\langle\nabla\cdot(\hat{E}\times\hat{H}-\hat{H}\times\hat{E})\rangle +\frac{1}{2c}\left\langle\hat{E}\cdot\frac{\partial\hat{E}}{\partial t}+\hat{H}\cdot\frac{\partial\hat{H}}{\partial t}\right\rangle+\text{v.c.}\quad\text{(A2.3)}$$

Next let us consider the term on the right-hand side of (A2.1). We have, if we use (2.17), (2.15), (A1.13), and (A1.14),

$$\mathrm{Re}\frac{\partial}{\partial\tau}U(\mathbf{r}_1,\mathbf{r}_2,\tau) =\frac{1}{2}\frac{\partial}{\partial\tau}\{\langle\hat{E}_k(1)\hat{E}_k(2)\rangle+\langle\hat{H}_k(1)\hat{H}_k(2)\rangle\}+\text{v.c.}\quad\text{(A2.4)}$$

We again proceed to the limit $\mathbf{r}_1\rightarrow\mathbf{r}_2=\mathbf{r}$, $\tau\rightarrow 0$. If we make use of the fact that, because of stationarity, the two averages on the right of (A2.4) depend on the two time arguments only through their difference, we find that[10]

$$\mathrm{Re}\frac{\partial}{\partial\tau}U(\mathbf{r}_1,\mathbf{r}_2,\tau)\Big|_{\mathbf{r}_1=\mathbf{r}_2=\mathbf{r};\ \tau=0} =-\frac{1}{2}\left\{\left\langle\frac{\partial\hat{E}}{\partial t}\cdot\hat{E}\right\rangle+\left\langle\frac{\partial\hat{H}}{\partial t}\cdot\hat{H}\right\rangle\right\}+\text{v.c.}\quad\text{(A2.5)}$$

On substituting from (A2.5) and (A2.3) into (A2.1) (taken in the limit which we are now considering), we obtain the result

$$\langle\nabla\cdot(\hat{E}\times\hat{H}-\hat{H}\times\hat{E})\rangle=-\frac{1}{2c}\left\langle\frac{\partial}{\partial t}(\hat{E}^2+\hat{H}^2)\right\rangle+\text{v.c.}\quad\text{(A2.6)}$$

If we omit the vacuum contribution, (A2.6) is seen to be *the averaged form of the energy conservation law of the electromagnetic field.*

It is quite easy to see that had we started from the conservation law (2.24) with $\alpha=2$ rather than $\alpha=1$, we would have also been led to the conservation law (A2.6).

### B. The Conservation Law (2.25)

Next let us consider the real part of the conservation law (2.25), with $\alpha=1$:

$$\mathrm{Re}\frac{\partial}{\partial r_{1l}}T_{ml}(\mathbf{r}_1,\mathbf{r}_2,\tau)=-\frac{1}{c}\mathrm{Re}\frac{\partial}{\partial\tau}S_m(\mathbf{r}_1,\mathbf{r}_2,\tau).\quad\text{(A2.7)}$$

We again examine each side separately. We have, if we use (2.21), (2.15), (A1.13), and (A1.14),

$$\mathrm{Re}\frac{\partial}{\partial r_{1l}}T_{ml}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\frac{1}{2}\left\{\left\langle\frac{\partial\hat{E}_m(1)}{\partial r_{1l}}\hat{E}_l(2)\right\rangle+\left\langle\frac{\partial\hat{E}_l(1)}{\partial r_{1l}}\hat{E}_m(2)\right\rangle-\delta_{lm}\left\langle\frac{\partial\hat{E}_k(1)}{\partial r_{1l}}\hat{E}_k(2)\right\rangle\right\} +\frac{1}{2}\left\{\left\langle\frac{\partial\hat{H}_m(1)}{\partial r_{1l}}\hat{H}_l(2)\right\rangle+\left\langle\frac{\partial\hat{H}_l(1)}{\partial r_{1l}}\hat{H}_m(2)\right\rangle-\delta_{lm}\left\langle\frac{\partial\hat{H}_k(1)}{\partial r_{1l}}\hat{H}_k(2)\right\rangle\right\}+\text{v.c.}\quad\text{(A2.8)}$$

We next proceed to the limit $\mathbf{r}_1\rightarrow\mathbf{r}_2=\mathbf{r}$, $\tau\rightarrow 0$ and obtain

$$\mathrm{Re}\frac{\partial}{\partial r_{1l}}T_{ml}(\mathbf{r}_1,\mathbf{r}_2,\tau)\Big|_{\mathbf{r}_1=\mathbf{r}_2=\mathbf{r};\ \tau=0}=\frac{1}{2}\left\{\left\langle\frac{\partial}{\partial r_l}(\hat{E}_m\hat{E}_l+\hat{E}_l\hat{E}_m-\delta_{lm}\hat{E}_k\hat{E}_k)\right\rangle\right\}+\frac{1}{2}\left\{\left\langle\frac{\partial}{\partial r_l}(\hat{H}_m\hat{H}_l+\hat{H}_l\hat{H}_m-\delta_{lm}\hat{H}_k\hat{H}_k)\right\rangle\right\} -\frac{1}{2}\left\{\left\langle\hat{E}_m\frac{\partial\hat{E}_l}{\partial r_l}+\hat{E}_l\frac{\partial\hat{E}_m}{\partial r_l}\right\rangle-\left\langle\hat{E}_k\frac{\partial\hat{E}_k}{\partial r_m}\right\rangle\right\}-\frac{1}{2}\left\{\left\langle\hat{H}_m\frac{\partial\hat{H}_l}{\partial r_l}+\hat{H}_l\frac{\partial\hat{H}_m}{\partial r_l}\right\rangle-\left\langle\hat{H}_k\frac{\partial\hat{H}_k}{\partial r_m}\right\rangle\right\}+\text{v.c.}\quad\text{(A2.9)}$$

Let us now consider the term on the right-hand side of (A2.7). We have from (2.20), (2.16), (A1.15), and (A1.16),

$$\mathrm{Re}\frac{\partial}{\partial\tau}S_m(\mathbf{r}_1,\mathbf{r}_2,\tau)=\tfrac{1}{2}\epsilon_{mkl}\frac{\partial}{\partial\tau}\{\langle\hat{E}_k(1)\hat{H}_l(2)\rangle-\langle\hat{H}_k(1)\hat{E}_l(2)\rangle\}+\text{v.c.}\quad\text{(A2.10)}$$

Next we proceed to the limit $\mathbf{r}_1\rightarrow\mathbf{r}_2=\mathbf{r}$, $\tau\rightarrow 0$ and use an argument similar to that employed in connection

[10] Actually, for a stationary field, considered here, this term vanishes identically. This result follows from the relation $U(\mathbf{r},\mathbf{r},\tau)$ $U^*(\mathbf{r},\mathbf{r},-\tau)$ that can be deduced from (2.17) and (2.30).

with (A2.5). We then find that[11]

$$\mathrm{Re}\frac{\partial}{\partial\tau}S_m(\mathbf{r}_1,\mathbf{r}_2,\tau)\bigg|_{\mathbf{r}_1=\mathbf{r}_2=\mathbf{r};\ \tau=0}=-\tfrac{1}{2}\epsilon_{mkl}\left\{\left\langle\frac{\partial\hat{E}_k}{\partial t}\hat{H}_l\right\rangle-\left\langle\frac{\partial\hat{H}_k}{\partial t}\hat{E}_l\right\rangle\right\}+\text{v.c.}$$
$$=-\frac{1}{2}\left\langle\frac{\partial}{\partial t}\{(\hat{E}\times\hat{H})_m-(\hat{H}\times\hat{E})_m\}\right\rangle+\tfrac{1}{2}\epsilon_{mkl}\left\langle\left(\hat{E}_k\frac{\partial\hat{H}_l}{\partial t}-\hat{H}_k\frac{\partial\hat{E}_l}{\partial t}\right)\right\rangle+\text{v.c.}\quad(\text{A2.11})$$

Now with the help of the operator form of Maxwell's equations [I(2.1)–I(2.4)] the terms in the last line in (A2.11) may be rewritten in the same form as some of the terms in (A2.9). We have, if we use the first Maxwell equation, I(2.1),

$$\epsilon_{mkl}\left\langle\hat{E}_k\frac{\partial\hat{H}_l}{\partial t}\right\rangle=-c\epsilon_{mkl}\epsilon_{abl}\left\langle\hat{E}_k\frac{\partial\hat{E}_b}{\partial r_a}\right\rangle,$$

or, if we use the identity I(3.2),

$$\epsilon_{mkl}\left\langle\hat{E}_k\frac{\partial\hat{H}_l}{\partial t}\right\rangle=-c(\delta_{ma}\delta_{bk}-\delta_{mb}\delta_{ak})\left\langle\hat{E}_k\frac{\partial\hat{E}_b}{\partial r_a}\right\rangle$$
$$=-c\left\{\delta_{ma}\left\langle\hat{E}_k\frac{\partial\hat{E}_k}{\partial r_a}\right\rangle-\left\langle\hat{E}_k\frac{\partial\hat{E}_m}{\partial r_k}\right\rangle\right\}$$
$$=c\left\{\left\langle\hat{E}_m\frac{\partial\hat{E}_k}{\partial r_k}\right\rangle+\left\langle\hat{E}_k\frac{\partial\hat{E}_m}{\partial r_k}\right\rangle-\left\langle\hat{E}_k\frac{\partial\hat{E}_k}{\partial r_m}\right\rangle\right\}.\quad(\text{A2.12})$$

Here, in going to the last line we also used the Maxwell equation I(2.3).

In a strictly similar manner we obtain the identity

$$\epsilon_{mkl}\left\langle\hat{H}_k\frac{\partial\hat{E}_l}{\partial t}\right\rangle=-c\left\{\left\langle\hat{H}_m\frac{\partial\hat{H}_k}{\partial r_k}\right\rangle+\left\langle\hat{H}_k\frac{\partial\hat{H}_m}{\partial r_k}\right\rangle-\left\langle\hat{H}_k\frac{\partial\hat{H}_k}{\partial r_m}\right\rangle\right\}.\quad(\text{A2.13})$$

Finally on substituting from (A2.9) and (A2.11) into (A2.7) (taken in the limit $\mathbf{r}_1\to\mathbf{r}_2$, $\tau\to0$), and on using the identities (A2.12) and (A2.13), we obtain the result

$$\left\langle\frac{\partial}{\partial r_l}(\hat{E}_m\hat{E}_l+\hat{E}_l\hat{E}_m-\delta_{lm}\hat{E}_k\hat{E}_k+\hat{H}_m\hat{H}_l+\hat{H}_l\hat{H}_m-\delta_{lm}\hat{H}_k\hat{H}_k)\right\rangle=\frac{1}{c}\left\langle\frac{\partial}{\partial t}(\hat{E}\times\hat{H}-\hat{H}\times\hat{E})_m\right\rangle+\text{v.c.}\quad(\text{A2.14})$$

[11] A similar remark applies here as in connection with Eq. (A2.5): For a stationary field considered here, this term vanishes identically. This result follows immediately from the relation $S_m(\mathbf{r},\mathbf{r},\tau)=S_m^*(\mathbf{r},\mathbf{r},-\tau)$, that can be deduced from (2.20) and (2.31).

If we omit the vacuum contribution, (A2.14) is seen to be *the averaged form of the momentum conservation law of the electromagnetic field.*

It is not difficult to see that had we started from the conservation law (2.25) with $\alpha=2$ rather than $\alpha=1$, we would have also been led to the conservation law (A2.14).

We can summarize the result of the calculations carried in this Appendix by saying that we have shown that if vacuum contributions are neglected, the real parts of the conservation laws (2.24) and (2.25) reduce in the limit $\mathbf{r}_1\to\mathbf{r}_2$, $\tau\to0$ to the averaged form of the energy and the momentum conservation laws, respectively, of the electromagnetic field.

## APPENDIX III: NON-NEGATIVE-DEFINITENESS CONDITIONS FOR THE CROSS-SPECTRAL TENSORS

In this Appendix we will establish the non-negative-definiteness conditions (3.10) and (3.27), which the cross-spectral tensors satisfy and we will discuss their relationship to the non-negative-definiteness conditions which are obeyed by the field correlation tensors.

We again make use of the inequality

$$\mathrm{tr}(\hat{\rho}\hat{A}^\dagger\hat{A})\geqslant0,\quad(\text{A3.1})$$

which holds for any arbitrary operator $\hat{A}$ for which the left-hand side exists, because the density operator $\hat{\rho}$ is non-negative-definite. In particular, let us choose

$$\hat{A}=\int_{\nu-\Delta\nu/2}^{\nu+\Delta\nu/2}d\nu\int d^3r\ f_i(\mathbf{r})\hat{e}_i^{(+)}(\mathbf{r},\nu),\quad(\text{A3.2})$$

where $f_i(\mathbf{r})$, $(i=1,2,3)$, are arbitrary functions of the space point $\mathbf{r}$, for which the integrals in the equations which follow are well defined and $(\nu-\frac{1}{2}\Delta\nu,\nu+\frac{1}{2}\Delta\nu)$ is an arbitrarily small frequency interval. As before, summation over repeated dummy indices is implied. From (3.3), (A3.1), and (A3.2) it follows that

$$\int_{\nu-\Delta\nu/2}^{\nu+\Delta\nu/2}d\nu\int d^3r_1\int d^3r_2\ f_i^*(\mathbf{r}_1)\times W_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)f_j(\mathbf{r}_2)\geqslant0.\quad(\text{A3.3})$$

Since this inequality holds for integration over an arbitrarily small frequency range, it follows that the electric cross-spectral tensor $W_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)$ obeys the non-negative-definiteness condition

$$\int d^3r_1\int d^3r_2 f_i^*(\mathbf{r}_1)W_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)f_j(\mathbf{r}_2)\geqslant0,\quad(\text{A3.4})$$

which is (3.10).

If instead of the operator (A3.2) we started with the operator

$$\hat{A}=\int_{\nu-\Delta\nu/2}^{\nu+\Delta\nu/2} d\nu \int d^3r\{f_i(\mathbf{r})\hat{e}_i^{(+)}(\mathbf{r},\nu)+g_i(\mathbf{r})\hat{h}_i^{(+)}(\mathbf{r},\nu)\} \quad \text{(A3.5)}$$

and used also Eqs. (3.14)–(3.16), we would obtain in a similar manner the non-negative-definiteness condition (3.27), namely,

$$\int d^3r_1 \int d^3r_2\{f_i^*(\mathbf{r}_1)W_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)f_j(\mathbf{r}_2) +g_i^*(\mathbf{r}_1)W_{ij}^{(h)}(\mathbf{r}_1,\mathbf{r}_2,\nu)g_j(\mathbf{r}_2) +f_i^*(\mathbf{r}_1)W_{ij}^{(m)}(\mathbf{r}_1,\mathbf{r}_2,\nu)g_j(\mathbf{r}_2) +g_i^*(\mathbf{r}_1)W_{ij}^{(n)}(\mathbf{r}_1,\mathbf{r}_2,\nu)f_j(\mathbf{r}_2)\}\geqslant 0. \quad \text{(A3.6)}$$

The condition (A3.4) was derived here from first principles, but it may also be regarded to be a consequence of the non-negative-definiteness condition which the electric correlation tensor $\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)$ obeys, i.e., the condition I(5.4), specialized to a stationary field:

$$\int\cdots\int d^3r_1d^3r_2dt_1dt_2 \times\{f_i^*(\mathbf{r}_1,t_1)\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,t_2-t_1)f_j(\mathbf{r}_2,t_2)\}\geqslant 0. \quad \text{(A3.7)}$$

The proof of this statement is rather long and will not be given here. It will suffice to say that this result is essentially an obvious generalization of Bochner's theorem,[12] on the assumption, made implicitly throughout this paper, that the frequency distribution function is differentiable throughout the whole frequency range ($0\leqslant\nu\leqslant\infty$), i.e., that the Fourier-Stieltjes frequency representation of the electric coherence tensor $\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)$ may be replaced by the ordinary (Riemann) Fourier representation.

The converse, namely, that (A3.4) implies (A3.7), is also true and can be quite easily established: We have, if we use the Fourier inverse of (3.4), with the choice

$$f_i(\mathbf{r},t)=g_i(\mathbf{r})\delta(t), \quad \text{(A3.8)}$$

$$\int\cdots\int d^3r_1d^3r_2dt_1dt_2\{f_i^*(\mathbf{r}_1,t_1)\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,t_2-t_1)f_j(\mathbf{r}_2,t_2)\}$$
$$=\int d^3r_1\int d^3r_2\, g_i^*(\mathbf{r}_1)g_j(\mathbf{r}_2)\int d\nu\int dt_1\int dt_2\, \delta(t_1)\delta(t_2)W_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)e^{-2\pi i\nu(t_2-t_1)}$$
$$=\int_0^\infty d\nu\int d^3r_1\int d^3r_2\, g_i^*(\mathbf{r}_1)W_{ij}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\nu)g_j(\mathbf{r}_2). \quad \text{(A3.9)}$$

If now (A3.4) is assumed to hold, the integrand on the right-hand side of (A3.9) is non-negative and (A3.9) then shows that the non-negative-definiteness condition (A3.4) on the electric cross-power spectrum implies the non-negative-definiteness condition (A3.7) on the electric coherence tensor $\mathcal{E}_{ij}$ of a stationary field.

In a similar manner, it may be shown that the more general non-negative-definiteness condition (A3.6) on the cross-spectral tensors and the more general non-negative-definiteness condition I(5.3) on the coherence tensors, (specialized to a stationary field), are equivalent to each other, i.e., each of them may be derived from the other.

[12] S. Bochner, *Lectures on Fourier Integrals* (Princeton University Press, Princeton, New Jersey, 1959), p. 326; see also R. R. Goldberg, *Fourier Transforms* (Cambridge University Press, Cambridge, England, 1961), Chap. V.