of the raising and lowering operators $K^{\pm}=K_1\pm K_2$ is $\Delta l=0$, $\Delta m=0$, and $\Delta n=\pm 1$, and is shown on the weight diagram in Fig. 1. The states $|nlm\rangle$, for fixed l and m, form therefore a basis for an irreducible representation D_+^{ϕ} of this $O(2,1)$, the transition group,[4] characterized by the lowest eigenvalue of $K_3=N$ which is clearly $n=l+1$. The matrix elements for D_+ of the finite $O(2,1)$ transformations are given by[6]

$$e^{-i\vartheta K_1}|n'lm\rangle=|nlm\rangle\mathcal{V}_{nn'}{}^{l+1}(\vartheta)\,, \tag{A10}$$

where $\mathcal{V}_{nn'}{}^{l+1}$ is precisely the function introduced in (A7). Equation (A10) has been proved in Ref. 4. Therefore (A9) becomes

$$\langle n'lm|D_{(n\pm 1)/n}|n\pm 1, lm\rangle\frac{(n\pm 1)^2}{n} =\mp\frac{1}{\sinh\vartheta_{n'n}}\langle n'lm|e^{-i\vartheta_{n'n}K_1}|n\pm 1, lm\rangle. \tag{A11}$$

This equation inserted into (A1) gives finally Eq. (1.5). Q.E.D.

[6] V. Bargmann, Ann. Math. 48, 568 (1947).

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Correlation Theory of Quantized Electromagnetic Fields. I. Dynamical Equations and Conservation Laws*

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Dynamical equations are derived, which describe the space-time development of second-order coherence tensors of a quantized electromagnetic field *in vacuo*. With the help of these equations, laws for the conservation of correlations are obtained. Some new non-negative-definiteness conditions which the coherence tensors obey are also established.

I. INTRODUCTION

In recent years many investigations have been carried out concerning the coherence properties of the electromagnetic field.[1] Most of these investigations have been concerned with the quantum-mechanical definition of coherence of arbitrary order, with the properties of the so-called coherent states, with the differences in the statistical features of laser light and thermal light, and with the relation between the classical and the quantum-mechanical description of coherence. However, practically no studies have been made as to the space-time development of the coherence tensors that describe the correlation effects in a quantized electromagnetic field. The main part of this paper is concerned with this question.

In Secs. II and III dynamical equations are derived which describe the space-time development *in vacuo* of the second-order coherence tensors. In Sec. IV the associated conservation laws are deduced and in Sec. V some new non-negative-definiteness conditions, which the coherence tensors obey, are established.

In Paper II of this investigation, these results will be specialized to the case of the main practical importance, namely, when the statistical behavior of the field is describable by a stationary ensemble, and various properties of such fields will be discussed.

II. DYNAMICAL EQUATIONS FOR THE CORRELATION TENSORS OF THE ELECTROMAGNETIC FIELD

We begin with the operator form of Maxwell's equations for the electromagnetic field *in vacuo*. If $\hat{E}(\mathbf{r},t)$, $\hat{H}(\mathbf{r},t)$ denote the electric and the magnetic field operators,[2] respectively, at the point $\mathbf{r}$, at time t, and if subscripts i, j, k denote Cartesian components, the equations may be expressed in the form

$$\epsilon_{ijk}\frac{\partial\hat{E}_j}{\partial r_i}=-\frac{1}{c}\frac{\partial\hat{H}_k}{\partial t}\,, \tag{2.1}$$

$$\epsilon_{ijk}\frac{\partial\hat{H}_j}{\partial r_i}=\frac{1}{c}\frac{\partial\hat{E}_k}{\partial t}\,, \tag{2.2}$$

$$\frac{\partial\hat{E}_i}{\partial r_i}=0\,, \tag{2.3}$$

$$\frac{\partial\hat{H}_i}{\partial r_i}=0\,. \tag{2.4}$$

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[1] For a review of these researches see, for example, L. Mandel and E. Wolf, Rev. Mod. Phys. 37, 231 (1965).

[2] All operators are indicated by a caret.

Here $\partial/\partial r_i$ ($i=1, 2, 3$) are the Cartesian components of the operator ∇ and ϵ_{ijk} denotes the completely antisymmetric unit tensor of Levi-Civita, i.e., $\epsilon_{ijk}=+1$ or -1, according as the subscripts (i,j,k) are an even or an odd permutation of (1,2,3) and $\epsilon_{ijk}=0$ when any two suffixes are equal. Unless stated otherwise we employ the usual summation convention, according to which summation is implied over repeated dummy indices.

Let us represent $\hat{E}$ as a Fourier integral with respect to the time variable:

$$\hat{E}(\mathbf{r},t)=\int_{-\infty}^{\infty}\hat{e}(\mathbf{r},\nu)e^{-2\pi i\nu t}d\nu\,, \tag{2.5}$$

and let $\hat{E}^{(+)}$ and $\hat{E}^{(-)}$ denote its positive- and negative-frequency parts, respectively:

$$\hat{E}^{(+)}(\mathbf{r},t)=\int_{0}^{\infty}\hat{e}^{(+)}(\mathbf{r},\nu)e^{-2\pi i\nu t}d\nu\,, \tag{2.6}$$

$$\hat{E}^{(-)}(\mathbf{r},t)=\int_{0}^{\infty}\hat{e}^{(-)}(\mathbf{r},\nu)e^{+2\pi i\nu t}d\nu\,, \tag{2.7}$$

where

$$\hat{e}^{(+)}(\mathbf{r},\nu)=\hat{e}(\mathbf{r},\nu)\,,\quad(\nu\geqslant 0)\,, \tag{2.8}$$

$$\hat{e}^{(-)}(\mathbf{r},\nu)=\hat{e}(\mathbf{r},-\nu)\,,\quad(\nu\geqslant 0)\,. \tag{2.9}$$

Since the operator $\hat{E}$ is Hermitian, $\hat{e}(\mathbf{r},\nu)=\hat{e}^{\dagger}(\mathbf{r},-\nu)$ and hence

$$\hat{E}^{(-)}(\mathbf{r},t)=\{\hat{E}^{(+)}(\mathbf{r},t)\}^{\dagger}. \tag{2.10}$$

Similar notation will also be used for the corresponding decomposition of the magnetic field operator.

It is readily seen that the pair of operators $\hat{E}^{(+)}$, $\hat{H}^{(+)}$ and also the pair of the operators $\hat{E}^{(-)}$, $\hat{H}^{(-)}$ are coupled by equations of the form (2.1)–(2.4), i.e.,

$$\epsilon_{ijk}\frac{\partial\hat{E}_j^{(\pm)}}{\partial r_i}=-\frac{1}{c}\frac{\partial\hat{H}_k^{(\pm)}}{\partial t}\,, \tag{2.11}$$

$$\epsilon_{ijk}\frac{\partial\hat{H}_j^{(\pm)}}{\partial r_i}=\frac{1}{c}\frac{\partial\hat{E}_k^{(\pm)}}{\partial t}\,, \tag{2.12}$$

$$\frac{\partial\hat{E}_i^{(\pm)}}{\partial r_i}=0\,, \tag{2.13}$$

$$\frac{\partial\hat{H}_i^{(\pm)}}{\partial r_i}=0\,, \tag{2.14}$$

where it is understood that the equations hold separately when either the upper or the lower signs are taken. Equations (2.11)–(2.14) can be derived either by taking the Fourier transform of Eqs. (2.1)–(2.4) and then integrating the resulting equations over the positive- or the negative-frequency range, or by applying to Eqs. (2.1)–(2.4) the well-known integral transform relations[3]

$$\hat{E}^{(\pm)}(\mathbf{r},t)=\lim_{\eta\to+0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\hat{E}(\mathbf{r},t-t')}{\pm t'-i\eta}dt' \tag{2.15}$$

and the corresponding relations for $\mathbf{H}^{(\pm)}$.

Now the basic correlation tensors (second-order coherence tensors) of the quantized field are defined by the following formulas:

$$\mathcal{E}_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\mathrm{tr}\{\hat{\rho}\hat{E}_i^{(-)}(\mathbf{r}_1,t_1)\hat{E}_j^{(+)}(\mathbf{r}_2,t_2)\}\,, \tag{2.16}$$

$$\mathcal{H}_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\mathrm{tr}\{\hat{\rho}\hat{H}_i^{(-)}(\mathbf{r}_1,t_1)\hat{H}_j^{(+)}(\mathbf{r}_2,t_2)\}\,, \tag{2.17}$$

$$\mathfrak{M}_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\mathrm{tr}\{\hat{\rho}\hat{E}_i^{(-)}(\mathbf{r}_1,t_1)\hat{H}_j^{(+)}(\mathbf{r}_2,t_2)\}\,, \tag{2.18}$$

$$\mathfrak{N}_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\mathrm{tr}\{\hat{\rho}\hat{H}_i^{(-)}(\mathbf{r}_1,t_1)\hat{E}_j^{(+)}(\mathbf{r}_2,t_2)\}\,, \tag{2.19}$$

where $\hat{\rho}$ is the density operator of the field. The correlation tensor $\mathcal{E}_{ij}$ was introduced by Glauber[4] and corresponds to a tensor introduced previously (for stationary fields) by Wolf[5] on the basis of classical theory. The tensors $\mathcal{H}$, $\mathfrak{M}$, and $\mathfrak{N}$ correspond to the other three tensors of the classical theory ($\mathfrak{M}$ and $\mathfrak{N}$ correspond to $\mathcal{G}$ and $\tilde{\mathcal{G}}$, respectively, of Refs. 6 and 7). The behavior of these tensors for blackbody radiation was discussed by Mehta and Wolf.[7]

If we make use of Eq. (2.10) and the corresponding relation involving $\hat{H}^{(+)}$ and $\hat{H}^{(-)}$ and use also the fact that $\hat{\rho}$ is a Hermitian operator, we obtain the relations

$$\mathcal{E}_{ji}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\mathcal{E}^*_{ij}(\mathbf{r}_2,t_2;\mathbf{r}_1,t_1)\,, \tag{2.20}$$

$$\mathcal{H}_{ji}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\mathcal{H}^*_{ij}(\mathbf{r}_2,t_2;\mathbf{r}_1,t_1)\,, \tag{2.21}$$

$$\mathfrak{M}_{ji}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\mathfrak{N}^*_{ij}(\mathbf{r}_2,t_2;\mathbf{r}_1,t_1)\,. \tag{2.22}$$

We will now show that the four correlation tensors $\mathcal{E}$, $\mathcal{H}$, $\mathfrak{M}$, and $\mathfrak{N}$ are coupled by a set of first-order partial differential equations. To derive these equations, let us first rewrite Eqs. (2.11)–(2.14) for the negative-frequency parts in the form

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}\hat{E}_j^{(-)}(\mathbf{r}_1,t_1)=-\frac{1}{c}\frac{\partial}{\partial t_1}\hat{H}_k^{(-)}(\mathbf{r}_1,t_1)\,, \tag{2.23}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}\hat{H}_j^{(-)}(\mathbf{r}_1,t_1)=\frac{1}{c}\frac{\partial}{\partial t_1}\hat{E}_k^{(-)}(\mathbf{r}_1,t_1)\,, \tag{2.24}$$

[3] Equation (2.15) follows immediately from the Fourier convolution theorem and the relation

$$\lim_{\eta\to+0}(1/2\pi i)\{1/(\pm t-i\eta)\}=\delta_{\pm}(t).$$

Cf. W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, England, 1954), 3rd ed., pp. 69–70.

[4] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).

[5] E. Wolf, Nuovo Cimento **12**, 884 (1954).

[6] E. Wolf, in *Proceedings of the Symposium on Astronomical Optics and Related Subjects*, edited by Z. Kopal (North-Holland Publishing Company, Amsterdam, 1956), p. 177; P. Roman and E. Wolf, Nuovo Cimento **17**, 462 (1960).

[7] C. L. Mehta and E. Wolf, Phys. Rev. **134**, A1143 (1964); **134**, A1149 (1964).

$$\frac{\partial}{\partial r_{1i}}\hat{E}_i^{(-)}(\mathbf{r}_1,t_1)=0, \tag{2.25}$$

$$\frac{\partial}{\partial r_{1i}}\hat{H}_i^{(-)}(\mathbf{r}_1,t_1)=0, \tag{2.26}$$

where $\partial/\partial r_{1i}$, $(i=1, 2, 3)$ are the Cartesian components of the operator ∇_1 (differentiation with respect to the coordinates of $\mathbf{r}_1$). If we multiply each of these equations by $\hat{\rho}$ from the left and by $\hat{E}_l^{(+)}(\mathbf{r}_2,t_2)$ from the right and take the trace we obtain the equations

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}\mathcal{E}_{jl}=-\frac{1}{c}\frac{\partial}{\partial t_1}\mathfrak{M}_{kl}, \tag{2.27a}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}\mathfrak{N}_{jl}=\frac{1}{c}\frac{\partial}{\partial t_1}\mathcal{E}_{kl}, \tag{2.28a}$$

$$\frac{\partial}{\partial r_{1i}}\mathcal{E}_{il}=0, \tag{2.29a}$$

$$\frac{\partial}{\partial r_{1i}}\mathfrak{N}_{il}=0. \tag{2.30a}$$

Similarly, if we multiply each of the Eqs. (2.23)–(2.26) by $\hat{\rho}$ from the left and by $\hat{H}_l^{(+)}(\mathbf{r}_2,t_2)$ from the right and take the trace, we obtain the equations

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}\mathfrak{IC}_{jl}=\frac{1}{c}\frac{\partial}{\partial t_1}\mathfrak{M}_{kl}, \tag{2.31a}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}\mathfrak{M}_{jl}=-\frac{1}{c}\frac{\partial}{\partial t_1}\mathfrak{IC}_{kl}, \tag{2.32a}$$

$$\frac{\partial}{\partial r_{1i}}\mathfrak{IC}_{il}=0, \tag{2.33a}$$

$$\frac{\partial}{\partial r_{1i}}\mathfrak{M}_{il}=0. \tag{2.34a}$$

In addition to the basic set of equations [(2.27a)–(2.34a)] which we just derived, there is another set of similar equations which involve derivatives with respect to the second space-time point $(\mathbf{r}_2,t_2)$ rather than with respect to the first one $(\mathbf{r}_1,t_1)$:

$$\epsilon_{ijk}\frac{\partial}{\partial r_{2i}}\mathcal{E}_{lj}=-\frac{1}{c}\frac{\partial}{\partial t_2}\mathfrak{M}_{lk}, \tag{2.27b}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{2i}}\mathfrak{M}_{lj}=\frac{1}{c}\frac{\partial}{\partial t_2}\mathcal{E}_{lk}, \tag{2.28b}$$

$$\frac{\partial}{\partial r_{2i}}\mathcal{E}_{li}=0, \tag{2.29b}$$

$$\frac{\partial}{\partial r_{2i}}\mathfrak{M}_{li}=0, \tag{2.30b}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{2i}}\mathfrak{IC}_{lj}=\frac{1}{c}\frac{\partial}{\partial t_2}\mathfrak{N}_{lk}, \tag{2.31b}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{2i}}\mathfrak{N}_{lj}=-\frac{1}{c}\frac{\partial}{\partial t_2}\mathfrak{IC}_{lk}, \tag{2.32b}$$

$$\frac{\partial}{\partial r_{2i}}\mathfrak{IC}_{li}=0, \tag{2.33b}$$

$$\frac{\partial}{\partial r_{2i}}\mathfrak{N}_{li}=0. \tag{2.34b}$$

This second set of equations [(2.27b)–(2.34b)] may be derived in a strictly similar manner as the set (2.27a)–(2.34a), by starting from Maxwell's equations for the positive-frequency parts of the field operators, or alternatively by applying to Eqs. (2.27a)–(2.34a) the relations (2.20)–(2.22).

The equations (2.27)–(2.34) may be regarded as the basic equations of the second-order coherence theory of the quantized electromagnetic field.

III. SECOND-ORDER EQUATIONS

From the basic first-order differential equations which we just derived, one may derive a number of second-order equations, which are also of interest. For example, if we apply the operator $(1/c)\partial/\partial t_1$ to both sides of Eq. (2.28a) and use (2.27a), we obtain the equation

$$-\epsilon_{ijk}\epsilon_{mnj}\frac{\partial^2\mathcal{E}_{nl}}{\partial r_{1i}\partial r_{1m}}=\frac{1}{c^2}\frac{\partial^2\mathcal{E}_{kl}}{\partial t_1^2}. \tag{3.1}$$

If we now use the well-known identity[8]

$$\epsilon_{ijk}\epsilon_{mnj}=\delta_{km}\delta_{ni}-\delta_{kn}\delta_{mi}, \tag{3.2}$$

where δ is the Kronecker symbol and use also Eq. (2.29a), we obtain a wave equation for $\mathcal{E}$:

$$\nabla_1^2\mathcal{E}_{kl}=\frac{1}{c^2}\frac{\partial^2}{\partial t_1^2}\mathcal{E}_{kl}. \tag{3.3}$$

Here ∇_1^2 is the Laplacian operator with respect to the coordinates of the point $\mathbf{r}_1$. In a strictly similar manner one may also derive a second wave equation for $\mathcal{E}$ which contains second-order derivatives with respect to the second space-time point:

$$\nabla_2^2\mathcal{E}_{kl}=\frac{1}{c^2}\frac{\partial^2}{\partial t_2^2}\mathcal{E}_{kl}. \tag{3.4}$$

[8] H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics* (Cambridge University Press, New York, 1950), 2nd ed., p. 73.

Moreover, one can also show, in an analogous manner, that each of the other three tensors $\mathcal{H}$, $\mathfrak{M}$, and $\mathfrak{N}$ satisfy such a pair of wave equations.

From our basic set of first-order equations, one may also derive a number of second-order equations which couple some of the correlation tensors. For example, if we operate on both sides of (2.28a) by $(1/c)\partial/\partial t_2$ and if we use (2.31b) we find that

$$\epsilon_{ijk}\epsilon_{mnl}\frac{\partial^2\mathcal{H}_{jn}}{\partial r_{1i}\partial r_{2m}}=\frac{1}{c^2}\frac{\partial^2\mathcal{E}_{kl}}{\partial t_1\partial t_2}. \tag{3.5}$$

In a similar way it follows from (2.32a) and (2.27b) that

$$\epsilon_{ijk}\epsilon_{mnl}\frac{\partial^2\mathcal{E}_{jn}}{\partial r_{1i}\partial r_{2m}}=\frac{1}{c^2}\frac{\partial^2\mathcal{H}_{kl}}{\partial t_1\partial t_2}. \tag{3.6}$$

Further, if we operate on both sides of (2.27a) by $(1/c)\partial/\partial t_2$ and use (2.28b), we obtain the equation

$$\epsilon_{ijk}\epsilon_{mnl}\frac{\partial^2\mathfrak{M}_{jn}}{\partial r_{1i}\partial r_{2m}}=-\frac{1}{c^2}\frac{\partial^2\mathfrak{N}_{kl}}{\partial t_1\partial t_2}. \tag{3.7}$$

Finally, from (2.28a) and (2.27b) we obtain, in a similar manner, the equation

$$\epsilon_{ijk}\epsilon_{mnl}\frac{\partial^2\mathfrak{N}_{jn}}{\partial r_{1i}\partial r_{2m}}=-\frac{1}{c^2}\frac{\partial^2\mathfrak{M}_{kl}}{\partial t_1\partial t_2}. \tag{3.8}$$

IV. CONSERVATION LAWS

In order to formulate various conservation laws which involve second-order correlations of the electromagnetic field, we introduce two tensors $U_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)$ and $S_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)$ defined as follows:

$$U_{ij}=\mathcal{E}_{ij}+\mathcal{H}_{ij}, \tag{4.1}$$

$$S_{ij}=\mathfrak{M}_{ij}-\mathfrak{N}_{ij}. \tag{4.2}$$

On adding Eqs. (2.27a) and (2.31a) and on subtracting (2.28a) from (2.32a), it follows that

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}U_{jl}=\frac{1}{c}\frac{\partial}{\partial t_1}S_{kl}, \tag{4.3}$$

$$\epsilon_{ijk}\frac{\partial}{\partial r_{1i}}S_{jl}=-\frac{1}{c}\frac{\partial}{\partial t_1}U_{kl}. \tag{4.4}$$

Also, on adding Eqs. (2.29a) and (2.33a) and on subtracting Eq. (2.30a) from (2.34a) it follows that

$$\frac{\partial}{\partial r_{1i}}U_{ij}=0, \tag{4.5}$$

$$\frac{\partial}{\partial r_{1i}}S_{ij}=0. \tag{4.6}$$

Let us now associate with the tensor U_{ij} a scalar U and a vector $\mathbf{U}(U_1,U_2,U_3)$ defined as follows:

$$\begin{aligned}U(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)&=U_{kk}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)\\&=\langle\hat{E}^{(-)}(\mathbf{r}_1,t_1)\cdot\hat{E}^{(+)}(\mathbf{r}_2,t_2)\rangle\\&\quad+\langle\hat{H}^{(-)}(\mathbf{r}_1,t_1)\cdot\hat{H}^{(+)}(\mathbf{r}_2,t_2)\rangle,\end{aligned} \tag{4.7}$$

$$\begin{aligned}U_i(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)&=\epsilon_{ijk}U_{jk}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)\\&=\{\langle\hat{E}^{(-)}(\mathbf{r}_1,t_1)\times\hat{E}^{(+)}(\mathbf{r}_2,t_2)\rangle_i\\&\quad+\langle\hat{H}^{(-)}(\mathbf{r}_1,t_1)\times\hat{H}^{(+)}(\mathbf{r}_2,t_2)\rangle_i\}.\end{aligned} \tag{4.8}$$

In Eqs. (4.7) and (4.8) angular brackets denote the quantum-mechanical expectation value.

We also associate a scalar and a vector with the tensor S_{ij}:

$$\begin{aligned}S(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)&=S_{kk}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)\\&=\langle\hat{E}^{(-)}(\mathbf{r}_1,t_1)\cdot\hat{H}^{(+)}(\mathbf{r}_2,t_2)\rangle\\&\quad-\langle\hat{H}^{(-)}(\mathbf{r}_1,t_1)\cdot\hat{E}^{(+)}(\mathbf{r}_2,t_2)\rangle,\end{aligned} \tag{4.9}$$

$$\begin{aligned}S_i(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)&=\epsilon_{ijk}S_{jk}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2),\\&=\{\langle\hat{E}^{(-)}(\mathbf{r}_1,t_1)\times\hat{H}^{(+)}(\mathbf{r}_2,t_2)\rangle_i\\&\quad-\langle\hat{H}^{(-)}(\mathbf{r}_1,t_1)\times\hat{E}^{(+)}(\mathbf{r}_2,t_2)\rangle_i\}.\end{aligned} \tag{4.10}$$

If in Eqs. (4.3) and (4.4) we put $k=l$ and sum over l, the resulting equations may be expressed in the form of two *scalar conservation laws*:

$$\nabla_1\cdot\mathbf{U}=\frac{1}{c}\frac{\partial}{\partial t_1}S, \tag{4.11}$$

$$\nabla_1\cdot\mathbf{S}=-\frac{1}{c}\frac{\partial}{\partial t_1}U. \tag{4.12}$$

Finally, if we multiply (4.3) by ϵ_{mkl} and use the identity (3.2) we readily find that

$$\frac{\partial}{\partial r_{1l}}[U_{ml}-\delta_{ml}U_{kk}]=\frac{1}{c}\frac{\partial}{\partial t_1}S_m.$$

If we also make use of Eq. (4.5), the last equation may be expressed in the form of a *vectorial conservation law*

$$\frac{\partial}{\partial r_{1l}}T_{ml}=\frac{1}{c}\frac{\partial}{\partial t_1}S_m, \tag{4.13}$$

where the tensor $T_{ml}\equiv T_{ml}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)$ is defined as

$$T_{ml}=U_{ml}+U_{lm}-\delta_{ml}U_{kk}. \tag{4.14}$$

In a similar way, if we start from Eq. (4.4) we arrive at the *vectorial conservation law*

$$\frac{\partial}{\partial r_{1l}}Q_{ml}=-\frac{1}{c}\frac{\partial}{\partial t_1}U_m, \tag{4.15}$$

where $Q_{ml}\equiv Q_{ml}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)$ is defined as

$$Q_{ml}=S_{ml}+S_{lm}-\delta_{ml}S_{kk}. \tag{4.16}$$

The four conservation laws which we just derived involve derivatives with respect to the space-time point $(\mathbf{r}_1,t_1)$. By starting from the appropriate equations of the "b" set rather than of the "a" set in Sec. II, one may derive, in a strictly similar manner, conservation laws that involve the derivatives with respect to the space-time point $(\mathbf{r}_2,t_2)$. These conservation laws are similar in form with those derived above. They are given by Eqs. (4.11), (4.12), (4.13), and (4.15), with ∇_1, $\partial/\partial r_{1l}$, and $\partial/\partial t_1$ replaced by ∇_2, $\partial/\partial r_{2l}$, and $\partial/\partial t_2$, respectively.

Some implications of these conservation laws will become apparent in Paper II of this investigation, which will deal with stationary electromagnetic fields.

V. NON-NEGATIVE-DEFINITENESS CONDITIONS

In Sec. II we have noted some simple consequences [Eqs. (2.20)–(2.22)] of the Hermiticity of the density operator $\hat{\rho}$. We will now briefly consider some consequences of the fact that $\hat{\rho}$ is also non-negative-definite, i.e., that for an arbitrary operator $\hat{A}$, for which $\mathrm{tr}(\hat{\rho}\hat{A}^\dagger\hat{A})$ is defined, the inequality

$$\mathrm{tr}(\hat{\rho}\hat{A}^\dagger\hat{A})\geqslant 0 \tag{5.1}$$

holds.

Let us choose

$$\hat{A}=\int d^3r dt\{f_i(\mathbf{r},t)\hat{E}_i^{(+)}(\mathbf{r},t)+g_i(\mathbf{r},t)\hat{H}_i^{(+)}(\mathbf{r},t)\}\,, \tag{5.2}$$

where $f_i(\mathbf{r},t)$ and $g_i(\mathbf{r},t)$, $i=1, 2, 3$, are arbitrary functions of the space-time point $\mathbf{r}$, t and the summation convention is again implied. We then obtain from (5.1) and the defining equations (2.16)–(2.19)

$$\begin{aligned}\int d^3r_1dt_1d^3r_2dt_2\{&f_i^*(\mathbf{r}_1,t_1)\mathcal{E}_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)f_j(\mathbf{r}_2,t_2)\\&+g_i^*(\mathbf{r}_1,t_1)\mathcal{H}_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)g_j(\mathbf{r}_2,t_2)\\&+f_i^*(\mathbf{r}_1,t_1)\mathfrak{M}_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)g_j(\mathbf{r}_2,t_2)\\&+g_i^*(\mathbf{r}_1,t_1)\mathfrak{N}_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)f_j(\mathbf{r}_2,t_2)\}\geqslant 0\,.\end{aligned} \tag{5.3}$$

It is, of course, understood that the arbitrary functions f_i and g_i are restricted to a class of functions for which the integral on the left-hand side of (5.3) is well defined.

Let us consider some particular cases of the inequality (5.3). If we choose $g_i(\mathbf{r},t)\equiv 0$, $(i=1, 2, 3)$, we obtain from (5.3) the following non-negative-definiteness condition obeyed by the electric correlation tensor:

$$\int d^3r_1dt_1d^3r_2dt_2\{f_i^*(\mathbf{r}_1,t_1)\mathcal{E}_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)f_j(\mathbf{r}_2,t_2)\}\geqslant 0\,. \tag{5.4}$$

Similarly, if we put $f_i\equiv 0$ in (5.3), we obtain the corresponding non-negative-definiteness condition which the magnetic correlation tensor obeys:

$$\int d^3r_1dt_1d^3r_2dt_2\{g_i^*(\mathbf{r}_1,t_1)\mathcal{H}_{ij}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)g_j(\mathbf{r}_2,t_2)\}\geqslant 0\,. \tag{5.5}$$

The non-negative-definiteness conditions (5.3), (5.4), and (5.5) can easily be rewritten in a form that contains summation rather than integration over the space-time variables. For this purpose we choose

$$f_i(\mathbf{r},t)=\sum_{m=1}^{M}\sum_{k=1}^{N}\alpha_{mki}\delta^{(3)}(\mathbf{r}_m-\mathbf{r})\delta(t_k-t)\,, \tag{5.6}$$

$$g_i(\mathbf{r},t)=\sum_{m=1}^{M}\sum_{k=1}^{N}\beta_{mki}\delta^{(3)}(\mathbf{r}_m-\mathbf{r})\delta(t_k-t)\,, \tag{5.7}$$

where M and N are arbitrary positive integers, α_{mki}, β_{mki} $(m=1, 2, \cdots M;\ k=1, 2, \cdots N;\ i=1, 2, 3)$ are arbitrary sets of constants, $\mathbf{r}_m$ are arbitrary points in space, and t_k are arbitrary time instants. We then obtain from (5.3), (5.4), and (5.5) the following non-negative-definiteness conditions:

$$\begin{aligned}\alpha^*_{mki}\mathcal{E}_{ij}(\mathbf{r}_m,t_k;\mathbf{r}_n,t_l)\alpha_{nlj}&+\beta^*_{mki}\mathcal{H}_{ij}(\mathbf{r}_m,t_k;\mathbf{r}_n,t_l)\beta_{nlj}\\&+\alpha^*_{mki}\mathfrak{M}_{ij}(\mathbf{r}_m,t_k;\mathbf{r}_n,t_l)\beta_{nlj}\\&+\beta^*_{mki}\mathfrak{N}_{ij}(\mathbf{r}_m,t_k;\mathbf{r}_n,t_l)\alpha_{nlj}\geqslant 0\,,\end{aligned} \tag{5.8}$$

$$\alpha^*_{mki}\mathcal{E}_{ij}(\mathbf{r}_m,t_k;\mathbf{r}_n,t_l)\alpha_{nlj}\geqslant 0\,, \tag{5.9}$$

$$\beta^*_{mki}\mathcal{H}_{ij}(\mathbf{r}_m,t_k;\mathbf{r}_n,t_l)\beta_{nlj}\geqslant 0\,. \tag{5.10}$$

The summation convention is again implied here, i.e., summation over all the possible values of the dummy indices i, j, m, n, k, and l is understood. Some special cases of the inequality (5.9) have been found by Glauber.[4]