Analysis of π^--Mesonic Atoms*

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A phenomenological analysis of the π^--mesonic atom in light nuclei has been made by use of a new approximation technique for the exterior wave function outside of the region of strong interaction. This approximation is a series expansion of the Coulomb wave function in terms of the energy-level-shift ratio $\Delta E_{1S}/E_{1S}$. From the new measurements of Jenkins, Kunselman, Simmons, and Yamazaki of the shifts and spreads of the $2P \rightarrow 1S$ transition energy in π^--mesonic atoms, we have calculated the complex π^--nucleus potential and the exact π^--nucleus scattering length. The π^--nucleus scattering lengths are found not to satisfy the hypothesis that they are the sums of π^--nucleon scattering lengths. Also the scattering lengths do not exhibit any apparent isotopic-spin dependence. It is also found that level shift depends only on the volume integral of the potential and not on its shape. Using this potential, the π^--mesonic wave function inside the nucleus is obtained in analytic form, and it is found that the probability that the π^--meon is inside the nucleus has only about half the value given by a first-order perturbation calculation.

I. INTRODUCTION

IN the last twelve years, several models and phenomenological calculations of the π^--mesonic atom have been made. Deser *et al.*,[1] Brueckner,[2] and Ivanenko and Pustovalov[3] tried to connect the energy-level shifts and spreads with the low-energy π^--nucleon scattering data using first-order perturbation theory. In all cases the energy-level shifts predicted by these calculations have a stronger atomic-number dependence than the experimental shift and spread data.[4–7] However, first-order perturbation theory is not valid, even for the lightest nuclei, where the energy-level shifts are very small ($\sim$3%). This can be understood as follows: The π^--nucleus potential which perturbs the pure Coulomb interaction is large, but it has a small effect on the binding energy because it occupies a volume which is small compared with the volume occupied by the meson wave function. This wave function is expected to deviate greatly from the unperturbed wave function only in the vicinity of the nuclear volume. However, it is just the wave function inside the nucleus which is needed to calculate the energy shift. In Sec. V, we demonstrate quantitatively that perturbation theory is not valid in the π^--mesonic-atom problem.

Several attempts have been made to do a phenomenological analysis of the problem and to calculate the energy shift more carefully. Ericson,[8] Pustovalov,[9] and Byers[10] have solved the Schrödinger equation in various approximations with a phenomenological π^--nucleus potential added to the Coulomb potential. The problem is hard because it is difficult to find a good approximation for the Coulomb wave function outside the nucleus. The methods used by these authors all involve expansions which make error estimates very difficult.

In Sec. II, we shall present a very simple approximation for the external Coulomb wave function which makes use of the fact that it deviates only slightly from the unperturbed wave function. It is an expansion in the energy shift and is valid for all values of r. In Secs. III and IV, this approximation is used to calculate the π^--nucleus potential for several nuclei by fitting the observed energy shifts and spreads. The internal wave functions were found numerically and simple analytic forms for these are given in Sec. V, which may be of use in π^--capture calculations. In Sec. VI we discuss some of the implications of our results.

II. FORMULATION

We shall consider the π^--mesonic atom only in light nuclei ($6 \leq A \leq 24$), because there exists a series of new experiments for atoms in this region.[7] Furthermore, the effect of the strong interaction on the $2P$ state for these nuclei was experimentally found to be negligible,[4,5,11] so that only the shift and spread in the $1S$ state need be considered.[12,13]

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[3] D. D. Ivanenko and G. E. Pustovalov, Usp. Fiz. Nauk **61**, 27 (1957) [English transl.: Soviet Phys.—Usp. **63**, 1043 (1961)].

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[9] G. E. Pustovalov, Zh. Eksperim. i Teor. Fiz. **36**, 1806 (1959) [English transl.: Soviet Phys.—JETP **9**, 1288 (1959)].

[10] N. Byers, Phys. Rev. **107**, 843 (1957).

[11] M. Carmac, M. L. Halbert, and J. B. Platt, Phys. Rev. **99**, 505 (1955).

[12] This simplifies the analysis considerably because it is known that the π^--nucleus interaction has a strong momentum dependence at low energy associated with the 3-3 resonance of the π-nucleon interaction. This momentum dependence requires the determination of an additional complex parameter, referred to as a nonlocal interaction parameter by Ericson (Ref. 13). It can be shown that the energy-level shift and spread of the $1S$ state is hardly affected by this momentum-dependent interaction, provided the radial wave function is nearly constant inside the nucleus, while for the analysis of the $2P$ state its inclusion is essential, since even the unperturbed wave function varies rapidly inside the nucleus.

[13] T. E. O. Ericson, in International Conference on High-Energy Physics and Nuclear Structure, CERN Report No. 63-78, 1963, p. 47 (unpublished).

To describe the strong π^--nucleus interaction, we use a simple square-well potential, since, as will appear later (Sec. IV), the level shift is not sensitive to the shape of the potential. Thus, determination of the shape is quite difficult, as it is in the case of low-energy nucleon-nucleon scattering.

The Schrödinger equation for the radial wave function $U(r)/r$ of the π^--mesonic atom is then

$$-(\hbar^2/2\mu)U''(r)-(Ze^2/r)U(r)+[\hbar^2l(l+1)/2\mu r^2]U(r)=EU(r), \quad r\geq R \tag{1a}$$

$$-(\hbar^2/2\mu)U''(r)+\{V_0+(Ze^2/2R)[(r/R)^2-3]\}U(r)+[\hbar^2l(l+1)/2\mu r^2]U(r)=EU(r), \quad R>r\geq 0 \tag{1b}$$

where $(Ze^2/2R)[(r/R)^2-3]$ is the Coulomb potential inside the nucleus due to a uniform charge distribution of radius R, V_0 is the height of the π^--nucleus potential, also of radius R, and μ is the reduced mass of the π^--nucleus system.

In terms of the Bohr radius $a_0=\hbar^2/(Ze^2\mu)=1/\alpha$ and the unperturbed ground-state energy $E_0=-Ze^2/2a_0=-\hbar^2\alpha^2/2\mu$, we introduce the following dimensionless variables:

$$\rho=\alpha r, \quad \rho_0=\alpha R,$$
$$\epsilon=(E-E_0)/|E_0|=\Delta E/|E_0|,$$
$$\mathcal{V}_0=V_0/|E_0|=2\mu V_0/(\alpha^2\hbar^2).$$

As defined, the real parts of ϵ and $\mathcal{V}_0$ are positive and the imaginary parts are negative. Note that the complex parameter ϵ describes the energy-level shift and spread caused not only by the strong interaction, but also by the finite charge distribution inside the nucleus. In terms of these variables the equation for the $1S$ state is

$$U''(\rho)+2U(\rho)/\rho-(1-\epsilon)U(\rho)=0, \quad \rho\geq\rho_0 \tag{2a}$$

$$U''(\rho)-(\mathcal{V}_0+1-\epsilon)U(\rho)-[(\rho/\rho_0)^2-3]U(\rho)/\rho_0=0, \quad \rho_0>\rho\geq 0. \tag{2b}$$

Our problem is to find the complex parameter $\mathcal{V}_0$, given the experimental value of ϵ.

Since the perturbed exterior wave function U_{ex} is only slightly deviated from the unperturbed wave function U_0 outside the nucleus, it is convenient to write U_{ex} in terms of U_0:

$$U_{ex}(\rho)=U_0(\rho)\varphi(\rho). \tag{3}$$

Here, $\varphi(\rho)$ is a function which depends on the energy shift ϵ and has the properties that $\varphi(\rho)|_{\epsilon=0}=1$ and $\lim_{\rho\to\infty}\rho^{-N}\varphi(\rho)=0$ for some N. The unperturbed wave wave function $U_0(\rho)$ satisfies the equation

$$U_0''(\rho)+2U_0(\rho)/\rho-U_0(\rho)=0, \tag{4}$$

and has the form

$$U_0(\rho)=C\rho e^{-\rho}. \tag{5}$$

Substituting Eqs. (3) and (5) into Eq. (2a), we obtain the equation for $\varphi(\rho)$:

$$\varphi''(\rho)+2(1/\rho-1)\varphi'(\rho)+\epsilon\varphi(\rho)=0. \tag{6}$$

$\varphi(\rho)$ can be written in a series in terms of the small complex number ϵ:

$$\varphi(\rho)=1+\epsilon\xi_1(\rho)+\epsilon^2\xi_2(\rho)+\epsilon^2\xi_3(\rho)+\cdots, \tag{7}$$

where each ξ_n satisfies successive differential equations

$$\xi_n''(\rho)+2(1/\rho-1)\xi_n'(\rho)+\xi_{n-1}(\rho)=0, \tag{8}$$

with $\xi_0=1$.

These differential equations can be easily integrated, successively subject to the above boundary condition on $\varphi(\rho)$. The wave function comes out to be[14]

$$U_{ex}(\rho)=C\rho e^{-\rho}\varphi(\rho),$$

$$\varphi(\rho)=1+\tfrac{1}{2}\epsilon(\rho+\ln\rho-1/2\rho)+(\epsilon^2/16)\int_\infty^\rho \rho^{-2}[4\rho^3+6\rho^2+6\rho+5+2(2\rho^2+2\rho+1)\ln\rho+2e^{+2\rho}E_1(2\rho)]d\rho+O(\epsilon^3), \tag{9}$$

and its logarithmic derivative is

$$\begin{aligned}d(\ln U_{ex})/d\rho &= 1/\rho-1+\epsilon\xi_1'(\rho)+\epsilon^2[\xi_2'(\rho)-\xi_1'(\rho)\xi_1(\rho)]\\ &+\epsilon^3\left[\xi_1'(\rho)\xi_1^2(\rho)-2\xi_1(\rho)\xi_2'(\rho)-e^{2\rho}\rho^{-2}\int_\infty^\rho \xi_1^2(\rho')e^{-2\rho'}\rho'^2d\rho'\right]+O(\epsilon^4)\\ &=1/\rho-1+\tfrac{1}{2}\epsilon(1+1/\rho+1/2\rho^2)+(\epsilon^2/16)[2+6/\rho+7/\rho^2+1/\rho^3+2e^{+2\rho}E_1(2\rho)/\rho^2]\\ &+(\epsilon^3/32)\Big\{2+10/\rho+11/\rho^2+6/\rho^3+1/2\rho^4-2(2\rho+2+1/\rho^2)\ln^2\rho\\ &+e^{+2\rho}\Big[(5+1/\rho-2/\rho-2\ln\rho)E_1(2\rho)+\pi^2/6+(\gamma+\ln 2)^2-2\int_0^\rho \ln^2\rho e^{-2\rho}d\rho\Big]\Big/16\Big\}+O(\epsilon^4),\end{aligned} \tag{10}$$

[14] $E_1(z)$ is the exponential integral defined in Ref. 15, p. 228. Note that $\exp(2\rho)E_1(2\rho)\to 1/2\rho$ as $\rho\to\infty$ so that φ as given in Eq. (9) does not increase exponentially in this limit.

TABLE I. The potentials and scattering lengths which fit the measured energy shift and spread using the radii obtained from electron scattering. ΔE_{sh} is the difference in the $1S$ level shift between the experimental value and the theoretical value for a point Coulomb potential, corrected for the vacuum-polarization effect. That is, ΔE_{sh} includes the strong-interaction effect and the finite-volume effect. ΔE_{sp} is the measured energy spread of the $1S$ level. These data are from the measurements of Jenkins *et al.* (Ref. 7).

	R[a] (F)	ΔE_{sh} (keV)	ΔE_{sp} (keV)	V_R (MeV)	V_I (MeV)	R^3V_R (F^3 MeV)	R^3V_I (F^3 MeV)	a_R (F)	a_I (F)
Li^6	2.97	0.7±0.2	0.39±0.36	$6.6^{+0.9}_{-1.9}$	$-5.2^{+6.0}_{-4.8}$	170^{+20}_{-50}	-140^{+160}_{-130}	$0.37^{+0.10}_{-0.09}$	$-0.23^{+0.21}_{-0.25}$
Li^7	2.97	0.9±0.2	0.57±0.30	$8.5^{+0.8}_{-1.7}$	$-8.4^{+5.7}_{-4.8}$	220^{+20}_{-50}	-220^{+160}_{-130}	$0.49^{+0.10}_{-0.08}$	$-0.34^{+0.19}_{-0.21}$
Be^9	2.84	2.0±0.2	0.85±0.28	9.9±0.9	−6.3±2.4	230±20	−140±60	0.46±0.05	−0.22±0.08
B^{10}	3.07	4.3±0.2	1.4 ±0.5	9.3±0.2	−4.6±1.9	270±10	−130±60	0.53±0.03	−0.20±0.08
B^{11}	3.29	4.7±0.2	2.3 ±0.5	7.7±0.2	−6.3±1.4	270±10	−270±50	0.58±0.07	−0.34±0.07
C^{12}	3.12	6.7±0.5	2.6 ±0.5	7.5±0.7	−4.6±1.0	230±20	−140±30	0.47±0.04	−0.22±0.04
N^{14}	3.16	12.6±0.5	4.1 ±0.4	9.5±0.7	−5.2±0.6	300±20	−160±20	0.59±0.04	−0.24±0.02
O^{16}	3.41	17.7±0.5	4.5 ±1.0	7.2±0.2	−3.0±0.6	290±10	−120±20	0.56±0.02	−0.18±0.03
F^{19}	3.57	30.6±0.5	4.6 ±1.0	8.8±0.2	−2.3±0.6	400±10	−110±30	0.73±0.02	−0.14±0.04
Na^{23}	3.70	63.9±1.0	4.6 ±1.0	10.3±0.2	−1.5±0.2	520±10	−80±10	0.92±0.01	−0.09±0.01
Mg^{24}	3.75	76.6±1.0		8.6±0.2		450±10		0.85±0.02	

[a] Radii are taken from R. Herman and R. Hofstadter [*High-Energy Electron Scattering Table* (Stanford University Press, Stanford, California, 1960)] except for lithium radii, which are from P. Sood *et al.*, Nucl. Phys. 4, 274 (1957).

where γ is Euler's constant, $\gamma=0.5772$. This expression for the logarithmic derivative of the external wave function is valid for all values of $\rho \geqq \rho_0$, and from the rapid convergence of this series it appears that the contribution of the ϵ^4 term is much less than the errors on the experimental data under consideration.

Equation (2b) for U_{in}, the interior solution, is easily solved numerically, and the complex parameter $\mathcal{V}_0$ adjusted so that $d(\ln U_{in})/d\rho$ calculated from Eq. (2b) is equal to $d(\ln U_{ex})/d\rho$ from Eq. (10) at $\rho=\rho_0$.

An independent check of our approximation was done by expanding the Whittaker function[15] for the exterior wave function to first order in ϵ and comparing with Eq. (9). Also, a calculation was made of the μ^--atomic energy shift and compared with numerical solutions of the Dirac equation given by Ford and Wills.[16] Our results agree within several percent for nuclei with $Z \leq 40$, the difference arising from our use of a square well and the nonrelativistic Schrödinger equation. Application of this approximation method to the Dirac equation is in process and will be compared with numerical analyses of the μ^- atom.

III. CALCULATIONS AND RESULTS

Comprehensive experimental data of the π^--mesonic-atom $1S \rightarrow 2P$ transition energies and their spreads in light nuclei were recently reported by Jenkins *et al.*[7] These data were measured with germanium solid-state detectors in contrast to earlier experiments in which NaI spectrometers were used.[4–6]

Assuming no shifts and spreads of the $2P$ energy levels, we fit these new data using complex square-well $V=V_R+iV_I$ to describe the π^--nucleus strong interaction, with the equivalent square-well radius R obtained from the electron-nucleus scattering measurements by the Stanford group.[17] (This will be referred to as the Stanford radius.) The volume integrals of the potentials and the exact π^--nucleus scattering lentghs,[18]

$$a=R-\tanh(\beta R)/\beta,$$
$$\beta=(2\mu V/\hbar^2)^{1/2},$$

were calculated from these potentials and were found to be independent of R. This will be discussed more fully in Sec. IV.

In Table I, we list the energy-level-shift and spread data, the best-fit potentials and their volume integrals, and the π^--nucleus scattering lengths. Errors shown in this table (and also in Table II) are the experimental errors in the γ-ray energy measurement. The higher-order terms which are neglected in our approximation

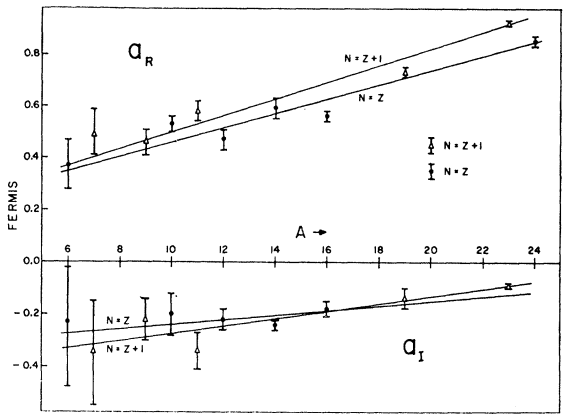

FIG. 1. The π^--nucleus scattering lengths and their linear curve fits, calculated from the best complex potential in Table I. Indicated errors are the experimental ones. The errors caused by neglecting the ϵ^4 term were estimated to be much less than these.

[15] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., 1964), Appl. Math. Ser. 55.

[16] K. W. Ford and J. G. Wills, Nucl. Phys. **35**, 295 (1962).

[17] R. Herman and R. Hofstadter, *High-Energy Electron Scattering Table* (Stanford University Press, Stanford, California, 1960).

[18] T. Y. Wu and T. Ohmura, *Quantum Theory of Scattering* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1962), p. 73.

are estimated to contribute less than the experimental error. The scattering lengths are also plotted in Fig. 1 as a function of the atomic number A. From these results, we see that the $1S$-state π^--nucleus interaction is repulsive as obtained independently from measurements of the low-energy π^--nucleus scattering.[19] Furthermore, we recognize two important features of the interaction from the quantities listed in Table I and shown in Fig. 1:

1. The depths and heights of the potentials do not show any obvious atomic-number dependence, but the values of their volume integrals do increase slowly with A as do the scattering lengths. While an isotopic-spin dependence of the π^--nucleus interaction has been used by Ericson[8,13] in previous analyses, no such isotopic-spin dependence is apparent here.

2. The π^--nucleus scattering lengths calculated exactly from these potentials do not satisfy the additivity hypothesis that they are the coherent sum of the π^--nucleon scattering lengths. On the basis of this hypothesis, the real part of the π^--nucleus scattering length is given by

$$a_R=\bar{a}A$$

for nuclei with $N=Z$, and by

$$a_R=b+\bar{a}A$$

for nuclei with $N=Z+1$. Here $\bar{a}=\frac{1}{2}(a_n+a_p)$ and $b=\frac{1}{2}(a_n-a_p)$, where a_n and a_p are the π^--neutron and π^--proton scattering lengths, respectively. The most recent evaluation of these quantities gives[20] $\bar{a}=0.012\pm0.004$ F and $b=0.097\pm0.006$ F. The actual scattering lengths exhibit no simple A dependence, but we have nevertheless calculated the best-fit linear curves to them. These fits are shown in Fig. 1. The results are as follows: For the real part of the scattering lengths we get for $N=Z$ nuclei

$$a_R=0.18\pm0.04+(0.028\pm0.002)A \quad (\chi^2=19.5) \tag{11a}$$

and for $N=Z+1$ nuclei

$$a_R=0.18\pm0.05+(0.032\pm0.003)A \quad (\chi^2=12.1). \tag{11b}$$

The fits are not good and the coefficients are in serious disagreement with the values calculated on the basis of additivity. Furthermore, there is no evidence for an isotopic-spin dependence since the two curves are identical within the statistics. For the imaginary part of the scattering length we have for the $N=Z$ nuclei

$$a_I=-0.31\pm0.14+(0.006\pm0.010)A \quad (\chi^2=1.7) \tag{12a}$$

and for the $N=Z+1$ nuclei

$$a_I=-0.41\pm0.14+(0.014\pm0.004)A \quad (\chi^2=2.0). \tag{12b}$$

Here we see that both have little A dependence and are again identical within statistics.

IV. SUPPLEMENTAL CALCULATIONS

Just before the data of Jenkins *et al.* became avilable, we had done a series of calculations using the old Stearns and Stearns data.[4] In this section, we give some of the results of these calculations which demonstrate a couple of general features of our optical-model calculation.

In their data only the energy spread of Be^9 is given. Their energy-level-shift and spread data for Be^9, $\Delta E_{1S}=[(1.98\pm0.10)-i(0.6\pm0.1)]$ keV, was used to examine how the real and imaginary parts of ΔE_{1S} interfere in the determination of V_R and V_I, assuming no shift and spread of the $2P$ energy level. This ΔE_{1S} was fit using a square-well π^--nucleus potential V_R+iV_I for two different nuclear radii, the Stanford radius and the radius given by the $A^{1/3}$ law, $R=r_0A^{1/3}$ with $r_0=1.3$ F. The results are, for the Stanford radius $R=2.84$ F,

$$\begin{aligned} V_R&=10.3\pm0.7 \text{ MeV},\\ V_I&=-4.8\pm1.0 \text{ MeV}, \end{aligned} \tag{13a}$$

and for $R=r_0A^{1/3}=2.70$ F,

$$\begin{aligned} V_R&=12.1\pm0.8 \text{ MeV},\\ V_I&=-5.5\pm1.0 \text{ MeV}. \end{aligned} \tag{13b}$$

Then, omitting the energy spread, real potentials were found which fit the energy shift $\Delta E_{1S}=1.98\pm0.10$ keV. The results are, for the Stanford radius $R=2.84$ F,

$$V_R=10.6\pm0.6 \text{ MeV} \tag{14a}$$

and for $R=r_0A^{1/3}=2.70$ F,

$$V_R=12.4\pm0.8 \text{ MeV}. \tag{14b}$$

Comparing the two results, (13) and (14), we see that inclusion of the energy-level spread has only a negligible effect on the determination of V_R. It should be noted that this does not contradict Brueckner's theory[2] of the π^--mesonic atom in which he estimated that the π^--nucleus scattering contributes to the energy shift by a half of its value and another half is a result of the pure absorption of the π^- meson. The real part of ΔE_{1S} used here is the observed total energy-level shift and such a contribution from the π^--meson absorption would be already included in the observed value.

Since the interference between the real part and imaginary part of our optical-model calculation is negligible, we calculated V_R from the energy-shift data for the rest of the atoms measured by Stearns and Stearns to examine how our calculation depends on the radius R. We calculated V_R, its volume integral, and

[19] V. S. Demidov, V. G. Kirillov-Ugryumov, A. K. Ponosov, V. P. Protasov, and F. M. Sergeev, Zh. Eksperim. i Teor. Fiz. **42**, 1687 (1962) [English transl.: Soviet Phys.—JETP **15**, 1172 (1962)].

[20] V. K. Samaranayake and W. S. Woolcook, Phys. Rev. Letters **15**, 936 (1965). See also J. Hamilton [Phys. Letters **20**, 687 (1966)], where $\bar{a}$ is found to be only 0.001 ± 0.004 F. We use the opposite sign convention for the scattering lengths, in order to conform with the convention used in potential calculation (Ref. 18).

the scattering length for the two kinds of radii R as before. The results are listed in Table II. From this table we see that the calculations using different radii give the same value for the volume integral, even though perturbation theory is not valid. It is the volume integral of the potential rather than its shape which determines the energy shift [and also the spread according to Eq. (13)]. Given the fact that the volume integral is the same for different radii, it follows from Born approximation (see Sec. V) that the scattering lengths will also be the same.

V. THE π^--MESONIC WAVE FUNCTION AND THE π^--CAPTURE PROBLEM

As mentioned before, although the exterior wave function differs only slightly from the bound Coulomb wave function, the interior wave function is completely different from it. Nevertheless, the bound Coulomb wave function has been commonly used in the π^--capture problem as the interior wave function. Usage of the new interior wave function obtained from the π^--mesonic-atom data corresponds to treating the initial state by the distorted-wave Born approximation (DWBA). The only difference is that in the bound π^--nucleus interaction problem, the initial incident particle is described as the bound Coulomb wave function instead of the plane wave, and this "incident wave" is deformed from the bound Coulomb wave function through the π^--nuclear interaction and the finite charge distribution inside the nucleus. In calculations of the π^--capture rates, the DWBA has been applied to the final-state interaction to calculate the reaction probabilities for one- and two-nucleon ejections.[21] Since the initial pion wave function is greatly perturbed by the nucleus, as will be shown numerically at the end of this section, the new interior wave function obtained from the π^--mesonic-atom data should also be used in π^--capture calculations.

The interior wave function, a solution of Eq. (1b) with $l=0$, has an explicit analytic form[22]

$$U_{\text{in}}(r) \propto r[\exp(-\tfrac{1}{2}\eta r^2)]M\{\tfrac{1}{4}[3-2(m/\eta) \times(E+3Ze^2/2R-V_0)],\tfrac{3}{2},\eta r^2\}, \quad (15)$$

where $\eta=(MZe^2/\hbar^2R^3)^{1/2}$. It is, however, difficult to manipulate this function in this form since the major part of the optical and the electrostatic potentials appears as a parameter in the confluent hypergeometric function. Although in the phenomenological calculation in Sec. III we numerically integrated Eq. (1b) to avoid any possible errors, we present here an approximate simple analytical form of the interior wave function for future usage in the π^--capture problem.

Since the major part of the potential in Eq. (1b) is its constant part $\bar{V}=V_0-3Ze^2/2R$, we may write the interior wave function $U_{\text{in}}(r)$ as

$$U_{\text{in}}(r)=U_{\text{sq}}(r)\varphi(r) =2C\sinh\{[2m(\bar{V}-E)/\hbar^2]^{1/2}r\}\varphi(r), \quad (16)$$

where $U_{\text{sq}}(r)$ is the solution of Eq. (1b) neglecting the term $Ze^2r^2/2R^3$ in the potential, and $\varphi(r)$ is a smoothly varying modifying function to correct for this. Inserting Eq. (16) into Eq. (1b), we obtain an equation for $\varphi(r)$. Neglecting the $d^2\varphi/dr^2$ term in the equation, we get

$$\varphi(r)\propto\exp\left[(\tfrac{1}{2}\eta^2/\gamma)\int_0^r r^2\tanh(\gamma r)dr\right] =\exp[\eta^2r^4/8+O(r^6)].$$

Thus, the interior wave function can be written approximately,

$$U_{\text{in}}(r)=C(e^{+\gamma r}-e^{-\gamma r})e^{+\delta r^4}, \quad (17)$$

where

$$\gamma=\gamma_R+i\gamma_I=[2m(V_0-3Ze^2/2R-E)/\hbar^2]^{1/2},$$
$$\delta=\eta^2/8=(8a_0R^3)^{-2},$$

and a_0 is the Bohr radius. These parameters are listed in Table III.

Also in Table III, we compare the probability of the π^- being inside the nucleus,[23] $\int_0^R|U_{\text{in}}|^2dr=\Delta E_{\text{sp}}/V_I$ (where ΔE_{sp} is the measured $1S$ energy spread), with that obtained from the Coulomb wave function. The

TABLE II. Potentials and scattering lengths, which fit the measurements of Stearns and Stearns (Ref. 4) of the energy shifts, using the Stanford radii and $R=r_0A^{1/2}$ radii, with $r_0=1.3$ F. ΔE is the difference of the $1S$ level shift between the experimental value and the theoretical one corrected for the vacuum-polarization effect. That is, ΔE includes the strong-interaction effect and the finite-volume effect.

	ΔE (keV)	R (F) Stanford $R=r_0A^{1/3}$	V_R (MeV)	a (F)	R^3V_R (MeV F^3)
Li[a]	0.87±0.12	3.65	4.8±0.8	0.46±0.06	230±40
		2.49	17.1±3.2	0.48±0.07	260±50
Be^9	1.98±0.10	2.84	10.6±0.6	0.46±0.02	240±10
		2.70	12.4±0.8	0.46±0.02	240±20
B^{10}	3.88±0.2	3.07	8.4±0.5	0.47±0.02	240±10
		2.80	11.4±0.2	0.47±0.01	250±10
B^{11}	5.68±0.2	3.29	11.7±0.5	0.72±0.02	420±20
		2.89	17.5±0.5	0.71±0.01	420±10
C^{12}	7.27±0.4	3.12	9.2±0.5	0.53±0.02	280±20
		2.98	10.6±0.8	0.52±0.03	280±20
N^{14}	10.4 ±0.3	3.16	7.6±0.2	0.47+0.01	240±10
		3.13	7.8±0.3	0.47+0.02	240±10
O^{16}	23.5 ±0.8	3.41	11.5±0.4	0.78±0.02	460±20
		3.28	13.2±0.7	0.79±0.03	470±30
F^{19}	30.6 ±1.0	3.57	8.8±0.4	0.72+0.03	400±20
		3.47	9.7±0.5	0.72+0.02	400±20

[a] A mixture of 92% Li^7 and 8% Li^6.

[21] R. M. Spector, Phys. Rev. 134, B101 (1964).

[22] $M(a,b,z)$ is Kummer's function, one of the confluent hypergeometric functions defined in Ref. 15, p. 405.

[23] This simple result for the interior probability follows from the fact that $\Delta E_{\text{sp}}=\int_0^\infty|U|^2V_Idr=V_I\int_0^R|U_{\text{in}}|^2dr$.

TABLE III. Parameters of the interior wave functions and probabilities. The probabilities are for the π^- meson to be inside the nucleus. The unperturbed probability is calculated from the unperturbed Coulomb wave function and the perturbed probability is equal to E_{sp}/V_I. The parameters are to be used in Eq. (17).

	Unperturbed probability (%)	Perturbed probability (%)	γ_R (F^{-1})	γ_I (F^{-1})	δ ($10^{-3}F^{-4}$)	C ($10^{-1}F^{-1/2}$)
Li^6	0.0113	0.00750	0.198	0.0919	0.0721	0.0649
Li^7	0.0114	0.00679	0.243	0.122	0.0724	0.0488
Be^9	0.0235	0.0135	0.238	0.0933	0.111	0.0786
B^{10}	0.0565	0.0304	0.214	0.0758	0.110	0.118
B^{11}	0.0692	0.0365	0.206	0.108	0.0894	0.114
C^{12}	0.101	0.0565	0.177	0.0921	0.126	0.182
N^{14}	0.163	0.0789	0.202	0.0915	0.142	0.188
O^{16}	0.294	0.150	0.139	0.0764	0.129	0.328
F^{19}	0.465	0.200	0.158	0.0518	0.127	0.336
Na^{23}	0.890	0.307	0.161	0.0331	0.139	0.396

Coulomb wave function is found to give a probability about twice as large as that given by the exact wave function. Thus the π^--capture rate should be about half the value given by calculations using the bound Coulomb wave function.

It should be noted that our criticism of perturbation approximation refers to its usage to obtain the potentials from the energy-level shifts and spreads. When the potential is once obtained exactly, Born approximation is valid to obtain the scattering length a from it, if and only if $a_B/R \ll 1$ or equivalently $a/R \ll 1$,[24] which is the case for the π^--nucleus interaction. Here a_B is the scattering length obtained from the exact potential in Born approximation. In our analysis, a is obtained exactly from the algebraic relation for the square-well potential. Either way, the major errors in previous analyses of π^--mesonic atoms occur in using perturbation theory to relate the level shifts to the potentials and not in using Born approximation to obtain the scattering lengths from these potentials.

VI. REMARKS AND CONCLUSIONS

The characteristic feature of the results of perturbation approximation and the additivity hypothesis of the scattering lengths is that they cannot explain the correct atomic-number dependence of the energy-level shift. The previous level-shift predictions[1–3] from the π^--nucleon scattering have a stronger dependence on A than the experimental values. As pointed out by Eisenberg and Kessler,[25] this cannot be corrected by a simple algebraic improvement within the first-order approximation, as long as additivity is assumed. We have shown that a more exact calculation of the π^--nucleus scattering length from the level shift does not give a simple A dependence for the π^--nucleus scattering length.

The additivity hypothesis is based on impulse approximation and the neglect of multiple scattering. Impulse approximation is used to relate the amplitude for scattering a π^- from a bound nucleon to the free π^--nucleon scattering amplitude. This approximation has been used in the multiple scattering treatment of Ericson and Ericson.[26] It appears to us, however, that significant errors may be involved in this approximation and this point is being investigated.

In fact, the quantity $\bar{a}=\frac{1}{2}(a_n+a_p)$ is very much smaller than either a_n or a_p. Samaranayake and Woolcock[20] find $a_p=-0.122\pm0.011$ F and $a_n=0.154\pm0.011$ F, which give $\bar{a}=0.017\pm0.006$ F. This value of $\bar{a}$ is much smaller than the coefficient of A which was found in Eq. (11) to fit the π^--nucleus scattering lengths. Thus, corrections to $\bar{a}$ coming from the fact that the nucleons are bound in the nucleus, which are proportional to $a_n^2+a_p^2$, may be comparable to $\bar{a}$ itself.

The low-energy π^--nucleus scattering could be a useful means to supplement or independently check the π^--mesonic data. Unfortunately, the experiments are extremely difficult and there are no systematic measurements available to compare with our scattering lengths. A single measurement that we are aware of is the experiment of Demidev *et al.*[19] on π^--C^{12} scattering. Our π^--C^{12} potential for the Stanford radius gives the values of $-0.084\pm0.006+i(0.041\pm0.009)$ and $-0.113\pm0.009+i(0.057\pm0.012)$ for the real and imaginary parts of the π^--C^{12} S-wave phase shifts at 5 and 10 MeV, respectively, which is in good agreement with the phases used by Demidev *et al.* to fit their data. This comparison is not complete because of a lack of measurements of the $2P$ energy-level shift and thus a determination of the P-wave π^--nucleus potential.

As a conclusion, we summarize our results as follows:

1. It is the volume integral of the potential rather than the detailed shape of the interaction which determines the energy-level shift.
2. The additivity of the scattering length and first-order perturbation approximation are invalid.
3. No simple A dependence of the potential is evident.

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[24] For a real-square barrier we have $a/R=1-\tanh(3a_B/R)^{1/2}/(3a_B/R)^{1/2}$ and for a real-square well, $a/R=1-\tan(3a_B/R)^{1/2}/(3a_B/R)^{1/2}$. Thus, $a_B \sim a$ occurs only if $|a_B/R| \ll 1$ (provided no bound states exist). This is the conclusion arrived at by F. von Hippel and J. H. Douglas [Phys. Rev. **146**, 1042 (1966)], though their derivation is inconsistent.

[25] Y. Eisenberg and D. Kessler, Phys. Rev. **130**, 2352 (1963).

[26] M. Ericson and T. E. O. Ericson, Ann. Phys. (N. Y.) **36**, 323 (1966).