Theory of Electromagnetic Properties of Strong-Coupling and Impure Superconductors. II*

SANG BOO NAM†
Department of Physics, University of Illinois, Urbana, Illinois

(Received 4 November 1966)

The effects of strong coupling and of spin-flip scattering by magnetic impurities on the electromagnetic properties of superconductors are discussed. We have found that the surface resistance (R_s/R_N) of lead, a strong-coupling superconductor, reflects the phonon spectrum and has characteristic peaks ($\sim$1.02) near frequencies $\sim 8\times10^{-3}$ and 11×10^{-3} eV, and has a small dip ($\sim$0.95) near frequency $\sim 22\times10^{-3}$ eV. Extensive calculations of the conductivity and surface impedance for superconductors containing magnetic impurities at finite temperatures, especially in the gapless region, are described.

1. INTRODUCTION

A GENERAL theory of electromagnetic properties of superconductors based on a generalized pairing scheme has been discussed by the author in the previous paper of this series.[1]

The main purpose of this paper is to discuss applications of the theory to a strong-coupling superconductor Pb and to weak-coupling superconductors containing magnetic impurities.

For strong-coupling superconductors such as Pb and Hg, it is necessary to take into account the retarded nature of electron-phonon interaction, as has been discussed by Schrieffer *et al.*,[2] and others.[3] Schrieffer *et al.* calculated the density of states for Pb assuming that the phonon spectrum is Lorentzian, and obtained excellent agreement with data from tunneling experiments.

We have here applied a general expression for the electromagnetic response function given in I to calculate the conductivity and surface resistance of Pb. In these calculations, we have used the results of Schrieffer *et al.* for the frequency-dependent gap parameter $\Delta(\omega)$ for Pb. We have found that the surface resistance (R_s/R_N) has characteristic peaks near frequencies $\sim 8\times10^{-3}$ and $\sim 11\times10^{-3}$ eV, and has a small dip ($\sim$0.95) near frequency $\sim 22\times10^{-3}$ eV.

The effect of spin-flip scattering by impurities with localized magnetic moments on the superconducting properties has been discussed by Abrikosov and Gorkov.[4] They have found that at a certain concentration of magnetic impurities the effective energy gap vanishes while the system still remains superconducting, so-called gapless superconductivity. Reif and Woolf,[5] on the basis of tunneling experiments, have experimentally demonstrated that there is a difference between the slopes in the transition temperature and in the effective energy gap as functions of the concentration of magnetic impurities, in accord with the prediction of the theory. A similar phenomena also occurs in a superconductor containing nonmagnetic impurities but in a high static magnetic field.[6] Weiss *et al.*[7] have carried out more detailed calculations of the properties of superconductors with magnetic impurities to determine the transition temperature $T_c(\Gamma_s)$ and gap parameter $\Delta(\Gamma_s,T)$ as functions of the parameter Γ_s, where $2\Gamma_s$ is the inverse relaxation time resulting from spin-flip scattering by magnetic impurities. They have calculated also the electrical conductivity at zero temperature. The thermal conductivity of magnetic impure superconductors has been discussed by Ambegaokar and Griffin.[8]

In this paper we describe results of similar calculations for $T_c(\Gamma_s)$ and $\Delta(\Gamma_s,T)$, and in addition calculations of the conductivity and surface impedance at finite temperatures. We have found that in the nonzero gap region conductivities and surface impedances have a behavior similar to those based on the Mattis and Bardeen theory. On the other hand, in the gapless region the real part of the conductivity has a finite value near zero frequency, and approaches the normal-state value at high frequencies, as one expects.

For convenience, we repeat from I Eqs. (5.7) and (5.10) which are useful limiting forms for the response

* This work was supported by the U. S. Army Research Office (Durham) under Contract No. DA-31-124-ARO(D)-114. Parts of the paper are based on a thesis submitted by the author in partial fulfillment of the requirements for a Ph.D. degree in physics, University of Illinois, 1966.

† Present address: Department of Physics, Rutgers, The State University, New Brunswick, New Jersey.

[1] S. B. Nam, preceding paper, Phys. Rev. **155**, 470 (1967). We refer to this as I. More detailed references can be found here. *Note added in proof.* The upper limit of the first integral in Eq. (4.18) of I should be ω_g instead of $-\omega_g$.

[2] J. R. Schrieffer, D. J. Scalapino, and J. W. Wilkins, Phys. Rev. Letters **10**, 336 (1963); **148**, 263 (1966); J. M. Rowell, P. W. Anderson, and D. E. Thomas, **10**, 334 (1963).

[3] D. J. Scalapino and P. W. Anderson, Phys. Rev. **133**, A921 (1964); D. J. Scalapino, Y. Wada, and J. C. Swihart, Phys. Rev. Letters **14**, 102 (1965); **14**, 106 (1965); V. Ambegaokar and L. Tewordt, Phys. Rev. **134**, A805 (1964); J. Bardeen and M. Stephen, *ibid.* **136**, A1485 (1964); M. Fibich, Phys. Rev. Letters **14**, 561 (1965).

[4] A. A. Abrikosov and L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. **39**, 1781 (1960) [English transl.: Soviet Phys.—JETP **12**, 1243 (1961)].

[5] F. Reif and M. A. Woolf, Phys. Rev. Letters **9**, 315 (1962); Phys. Rev. **137**, A557 (1965).

[6] A. I. Larkin, Zh. Eksperim. i Teor. Fiz. **48**, 232 (1965) [English transl.: Soviet Phys.—JETP **21**, 153 (1965)]; K. Maki, Progr. Theoret. Phys. (Kyoto) **29**, 603 (1963); **31**, 731 (1964).

[7] S. Skalski, O. Betbeder-Matibet, and P. R. Weiss, Phys. Rev. **136**, A1500 (1964).

[8] V. Ambegaokar and A. Griffin, Phys. Rev. **137**, A1151 (1965).

function and the conductivity in the Pippard ($q\rightarrow\infty$) and London ($q\rightarrow 0, \Gamma\rightarrow\infty$) limits. These limiting expressions both may be written in the form

$$K(q,\omega)\approx\sigma^N 4\pi i\omega\{\sigma_1(\omega)-i\sigma_2(\omega)\}\,, \qquad (1.1)$$

$$(\sigma^s/\sigma^N)(\omega)\approx\sigma_1(\omega)-i\sigma_2(\omega)\,, \qquad (1.2)$$

where the normal-state conductivity σ^N is

$$\sigma^N\approx 3\pi/4qv_0\Lambda \qquad (1.3a)$$

for the Pippard limit, and

$$\sigma^N\approx\frac{1}{\Lambda}\frac{1}{2\Gamma_{\text{eff}}} \qquad (1.3b)$$

for the London limit. The latter corresponds to the case of an effective mean free path $L\ll\xi_0$, where ξ_0 is the coherence distance. The London parameter Λ is defined by

$$1/\Lambda=ne^2/m=\tfrac{2}{3}N(0)e^2v_0^2\,, \qquad (1.4)$$

where $N(0)$ and v_0 are the density of states and the velocity on the Fermi surface in the normal state. The real and imaginary parts of the ratio of conductivities $\sigma_1(\omega)$ and $\sigma_2(\omega)$ are the same in both limits;

$$\sigma_1(\omega)=\frac{1}{\omega}\int_{\omega_g-\omega}^{-\omega_g}d\omega'\,g_1(\omega',\omega+\omega')\tanh\tfrac{1}{2}\beta(\omega+\omega')$$
$$+\frac{1}{\omega}\int_{\omega_g}^{\infty}d\omega'\,g_1(\omega',\omega+\omega')$$
$$\times[\tanh\tfrac{1}{2}\beta(\omega+\omega')-\tanh\tfrac{1}{2}\beta\omega'], \qquad (1.5a)$$

$$\sigma_2(\omega)=\frac{1}{\omega}\int_{[\omega_g-\omega,-\omega_g]}^{\omega_g}d\omega'\,g_2(\omega',\omega+\omega')\tanh\tfrac{1}{2}\beta(\omega+\omega')$$
$$+\frac{1}{\omega}\int_{\omega_g}^{\infty}d\omega'\,[g_2(\omega',\omega'+\omega)\tanh\tfrac{1}{2}\beta(\omega+\omega')$$
$$+g_2(\omega+\omega',\omega')\tanh\tfrac{1}{2}\beta\omega'], \qquad (1.5b)$$

where $[\omega_g-\omega, -\omega_g]$ denotes that the algebraically larger of the two numbers is to be used. Here the functions g_1 and g_2 are coherence factors defined by

$$g_1(\omega',\omega+\omega')=n(\omega')n(\omega+\omega')+p(\omega')p(\omega+\omega')\,, \qquad (1.6a)$$

$$g_2(\omega',\omega+\omega')=\tilde{n}(\omega')n(\omega+\omega')+\tilde{p}(\omega')p(\omega+\omega')\,, \qquad (1.6b)$$

where $n(\omega)$, $\tilde{n}(\omega)$, $p(\omega)$, and $\tilde{p}(\omega)$ are defined by

$$n(\omega)+i\tilde{n}(\omega)=\omega/[\omega^2-\Delta^2(\omega)]^{1/2}\,, \qquad (1.7a)$$

$$p(\omega)+i\tilde{p}(\omega)=\Delta(\omega)/[\omega^2-\Delta^2(\omega)]^{1/2}. \qquad (1.7b)$$

The effective energy gap ω_g is defined as the frequency at which the density of states $n(\omega)$ first begins to have finite value;

$$n(\omega<\omega_g)=0. \qquad (1.8)$$

The minimum energy required to create a pair of excitations from the ground state is then $2\omega_g$. It may be seen from Eq. (1.5a) that at zero temperature the real part of the conductivity vanishes for frequency $\omega<2\omega_g$, as one expects.

The surface impedance, neglecting the small displacement current correction which is one of the order of $(\omega\tau)^2(v/c)^2$, is given by[9]

$$R(\omega)+iX(\omega)=4\pi i\omega I(\omega)\,, \qquad (1.9)$$

where

$$I(\omega)=\frac{2}{\pi}\int_0^{\infty}\frac{dq}{q^2+K(q,\omega)} \qquad (1.10a)$$

for specular reflection, and

$$I(\omega)=\pi\left\{\int_0^{\infty}dq\,\ln[1+K(q,\omega)/q^2]\right\}^{-1} \qquad (1.10b)$$

for random scattering.

One can in general calculate the surface impedance by using the general expression for the response function $K(q,\omega)$ given in I. For simplicity we use the limiting form of the response function, Eq. (1.1). Inserting Eq. (1.1) into Eqs. (1.10), we obtain a single form for both cases:

$$\frac{R_s(\omega)+iX_s(\omega)}{R_N(\omega)+iX_N(\omega)}\approx\{\sigma_1(\omega)-i\sigma_2(\omega)\}^n\,, \qquad (1.11)$$

where $n=-\frac{1}{3}$ for the Pippard limit, and $n=-\frac{1}{2}$ for the London limit. The surface resistance $R(\omega)$ and reactance $X(\omega)$ may be rewritten as

$$R_s/R_N=\text{Re}(\sigma_1-i\sigma_2)^n-(X_N/R_N)\,\text{Im}(\sigma_1-i\sigma_2)^n\,, \qquad (1.12a)$$

$$X_s/X_N=\text{Re}(\sigma_1-i\sigma_2)^n+(R_N/X_N)\,\text{Im}(\sigma_1-i\sigma_2)^n. \qquad (1.12b)$$

The penetration depth λ is given by

$$\lambda=I(\omega=0) \qquad (1.13)$$

from Eqs. (1.10). This can be calculated directly by using the expression for $K(q,0)$ given in I. For simplicity, we give here only the limiting forms valid in the Pippard and London limits. In both cases, this may be expressed in the form

$$K(q,0)\approx 4\pi^2\sigma^N\Delta^p(0)S\,, \qquad (1.14)$$

where $\Delta^p(0)=\Delta_{\text{BCS}}(0)$, and σ^N is given by Eqs. (1.3). The parameter S is given by

$$S=\frac{2}{\pi}\frac{2\pi T}{\Delta^P(0)}\,\text{Re}\sum_{n>0}\frac{\Delta_n^2}{\omega_n^2+\Delta_n^2}\,, \qquad (1.15)$$

[9] G. E. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1948).

where $\omega_n = (2n+1)\pi T$, with n an integer and $\Delta_n = \Delta(i\omega_n)$.

In the weak-coupling limit without magnetic impurities, Eq. (1.15) becomes identical with the corresponding BCS expression:

$$S = \frac{\Delta(T)}{\Delta(0)} \tanh \frac{\Delta(T)}{2T}. \quad (1.16)$$

Inserting Eq. (1.14) into Eqs. (1.10) and (1.13), we obtain in the Pippard limit

$$\frac{\lambda_L}{\lambda} = \left[\frac{3\pi^2}{4}\frac{\lambda_L}{\xi_0}S\right]^{1/3} \quad (1.17a)$$

for the random-scattering boundary condition, and

$$\frac{\lambda_L}{\lambda} = \frac{8}{9}\left[\frac{3\pi^2}{4}\frac{\lambda_L}{\xi_0}S\right]^{1/3} \quad (1.17b)$$

for specular reflection. In the London limit, both cases give the same result:

$$\frac{\lambda_L}{\lambda} = \left[\frac{L}{\xi_0}S\right]^{1/2}. \quad (1.18)$$

Here the London penetration depth λ_L, the effective mean free path L, and the coherence distance ξ_0 are defined by

$$\lambda_L = [\Lambda/4\pi]^{1/2},$$

$$L = v_0/2\Gamma_{\text{eff}},$$

and

$$\xi_0 = v_0/\pi\Delta^p(0).$$

In the London limit, $L \ll \xi_0$, S corresponds to the superfluid density of the two-fluid model.

Expressions for the surface impedance and for the penetration depth simplify in the Pippard limit if $\lambda \ll \xi_0$ and in the London limit if $L \ll \lambda$ or ξ_0.

2. CALCULATIONS FOR LEAD

In this section, we give the results of numerical calculations of the conductivity and surface resistance for lead.

In these calculations we have used the results of Schrieffer *et al.* for the frequency-dependent gap parameter $\Delta(\omega)$. The effective energy gap ω_g is

$$\omega_g = \Delta(\omega_g) = 1.34 \times 10^{-3} \text{ eV}.$$

Numerical calculations of the conductivity of Eq. (1.2) are presented in Fig. 1. For comparison, we have calculated also the conductivity assuming $\Delta(\omega) = \omega_g$, corresponding to the weak-coupling limit. The difference between the calculations made with $\Delta = \Delta(\omega)$ and $\Delta = \omega_g$ is illustrated in Fig. 2. It may be seen that taking $\Delta = \Delta(\omega)$ (strong coupling) yields a smaller conductivity

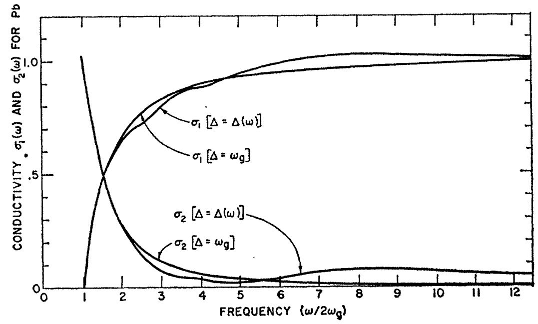


FIG. 1. The conductivity ratio, $(\sigma^s/\sigma^N)(\omega) = \sigma_1(\omega) - i\sigma_2(\omega)$, for lead. The curves of $\sigma_{1,2}[\Delta = \Delta(\omega)]$ and of $\sigma_{1,2}[\Delta = \omega_g]$ correspond to the conductivities calculated by using $\Delta = \Delta(\omega)$ and $\Delta = \omega_g$.

at low frequencies. Some indications of such an effect have been observed recently by Palmer[10]; his data are presented in Fig. 2(b). We see that his data are below the calculated curve based on $\Delta = \omega_g$, that is, the weak-coupling limit. It is noted that his value of $2\omega_g$ is 22.5 ± 0.5 cm^{-1}, which is equivalent to

$$\omega_g = (1.39 \pm 0.06) \times 10^{-3} \text{ eV}.$$

On the other hand, at high frequencies the conductivity is enhanced.

Using these results we have calculated the surface resistance (R_s/R_N) from Eq. (1.12). We have used $X_N/R_N = \sqrt{3}$ for the Pippard limit ($\lambda \ll \xi_0$) and $X_N/R_N = 1$ for the London limit ($\lambda \gg \xi_0$).[11] Actually for Pb, neither of these limits is applicable since $\xi_0/\lambda_L(0) = 2.2$.[12] In Fig. 3, we present the theoretical calculation of surface resistance in the Pippard limit, which should be more nearly valid.

We observe that R_s/R_N has characteristic peaks (~ 1.02) near frequencies $\sim 3.0(2\omega_g)$ and $\sim 4.3(2\omega_g)$, and has a small dip (~ 0.95) near frequency $\sim 8.5(2\omega_g)$. This dip, which is a reflection of the electron-phonon interaction, is perhaps large enough to be experimentally detectable even though it would be difficult to observe.

3. CALCULATIONS OF SUPERCONDUCTORS WITH MAGNETIC IMPURITIES

In this section we give some results of calculations of the gap parameter $\Delta(\Gamma_s, T)$ and the transition temperature $T_c(\Gamma_s)$, as well as a few typical results for the conductivity and surface impedance at finite temperatures, especially in the gapless region $[0 < \Delta(\Gamma_s, T) < 2\Gamma_s$

[10] L. H. Palmer, thesis, University of California, Berkeley, 1966 (unpublished). We thank Dr. Palmer for sending us his data. *Note added in proof.* The recent experiment by S. L. Norman and D. H. Douglass, Jr. [Phys. Rev. Letters **18**, 339 (1967)] also indicates that there is no infrared absorption at frequency below $2\omega_g$, as does that of Palmer, agreeing with the theory.

[11] A. B. Pippard, Proc. Roy. Soc. (London) **A216**, 547 (1953); Advan Electron. Electron Phys. **6**, 1 (1954).

[12] See, for example, J. Bardeen and J. R. Schrieffer, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1961), Vol. 3, p. 170.

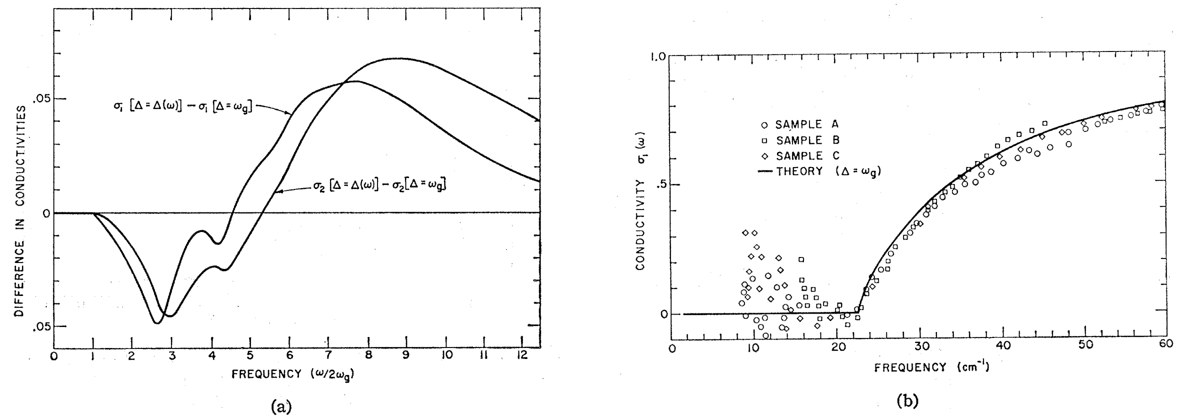


FIG. 2. (a) The difference between conductivities $\sigma_{1,2}[\Delta=\Delta(\omega)]$ calculated by using $\Delta=\Delta(\omega)$ and $\Delta=\omega_g$. The strong-coupling effect reduces the conductivity at low frequencies, especially, about 5% near frequency $2.5\sim3.0(2\omega_g)$. On the other hand, the conductivity is enhanced at high frequencies, and approaches the normal-state value at very high frequencies. (b) The real part of the conductivity ratio $\sigma_1(\omega)$ from Palmer's data (Ref. 10) for lead. The theoretical curve here is calculated by using $\Delta=\omega_g$. Here the gap determined experimentally is $\omega_g=\frac{1}{2}(22.5\pm0.6)$ cm^{-1}$=(1.39\pm0.06)\times10^{-3}$ eV.

$<\frac{1}{2}\Delta(0,0)]$. Here the parameter $2\Gamma_s$ is the inverse relaxation time resulting from spin-flip scattering by magnetic impurities. We have carried out calculations of the densities of states and pairs, the conductivity and surface impedance for various values of Γ_s and T. We present here only a typical result for $2\Gamma_s=0.35\Delta(0,0)$. The more complete calculations for $2\Gamma_s=0.10$, 0.20, 0.35, 0.45, 0.49, etc. are given in the author's thesis.[13]

We first discuss a general description for calculations of superconducting properties of the system with magnetic impurities.

To proceed with the calculation we need to obtain the effective gap parameter $\bar{\Delta}(\omega)$, which is a solution of the following equation[14]:

$$\bar{\Delta}(\omega)=\Delta-2\Gamma_s i\bar{\Delta}(\omega)/[\omega^2-\bar{\Delta}^2(\omega)]^{1/2} \qquad (3.1)$$
$$=\Delta-2\Gamma_s i\Delta/[\bar{\omega}^2(\omega)-\Delta^2]^{1/2},$$

where the usual gap parameter Δ is a solution of a BCS-like integral equation;

$$\Delta\approx[N(0)V_{\rm BCS}]\int d\omega\,\tanh(\tfrac{1}{2}\beta\omega)\,{\rm Re}\{P(\omega)\}, \qquad (3.2)$$
$$P(\omega)=\bar{\Delta}(\omega)/[\omega^2-\bar{\Delta}^2(\omega)]^{1/2}=\Delta/[\bar{\omega}^2(\omega)-\Delta^2]^{1/2}.$$

We have here introduced, for convenience, a renormalized frequency $\bar{\omega}$ defined by

$$\bar{\omega}(\omega)=\omega+2\Gamma_s i\bar{\omega}(\omega)/[\bar{\omega}^2(\omega)-\Delta^2]^{1/2}. \qquad (3.3)$$

When $\Gamma_s=0$ then $\bar{\Delta}(\omega)=\Delta$ and $\bar{\omega}(\omega)=\omega$.

We see that all calculations can be done by knowing $\bar{\Delta}(\omega)$ or $\bar{\omega}(\omega)$. The explicit solutions for $\bar{\Delta}(\omega)$ and $\bar{\omega}(\omega)$ can be written[15]

$$\bar{\Delta}(\omega)=\tfrac{1}{2}(\Delta+\Delta_0)+\tfrac{1}{2}\{(\Delta+\Delta_0)^2 - (2/\Delta_0)[y_0(\Delta+\Delta_0)+2\Delta\omega^2]\}^{1/2}, \qquad (3.4a)$$

$$\bar{\omega}(\omega)=\tfrac{1}{2}(\omega+\omega_0)+\tfrac{1}{2}\{(\omega+\omega_0)^2 - (2/\omega_0)[y_0(\omega+\omega_0)+2\Delta^2\omega]\}^{1/2}, \qquad (3.4b)$$

where

$$\Delta_0=\{y_0+\Delta^2+(2\Gamma_s)^2\}^{1/2},$$
$$\omega_0=\{y_0+\Delta^2-(2\Gamma_s)^2\}^{1/2},$$
$$y_0=y_1+\alpha,$$
$$y_1=A_++A_-,\quad\text{or}\quad A_+z+A_-z^2,\quad\text{or}\quad A_+z^2+A_-z,$$
$$z^3=1,$$
$$A_\pm=\{2D+\alpha^3\pm2(D\alpha^3+D^2)^{1/2}\}^{1/3},$$

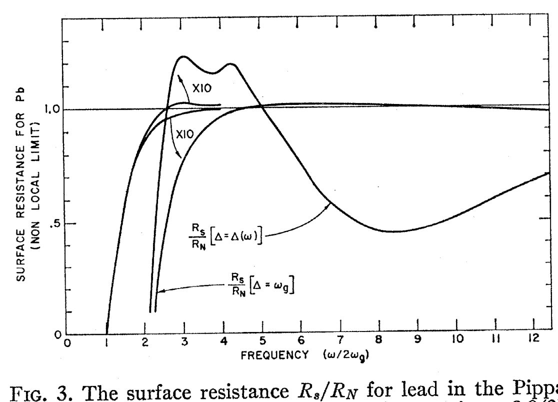


FIG. 3. The surface resistance R_s/R_N for lead in the Pippard limit. The characteristic peaks appear at frequencies $\sim3.0(2\omega_g)$ and $4.3(2\omega_g)$, and a small dip (~0.95) appears at frequency $\sim8.5(2\omega_g)$.

[13] S. B. Nam, thesis, University of Illinois, 1966 (unpublished).

[14] Equation (3.6c) in I.

[15] A quartic equation $x^4+bx^3+cx^2+dx+e=0$ can be solved as follows: we rearrange $(x^2+\frac{1}{2}bx)^2=(\frac{1}{4}b^2-c)x^2-ax-e$. We add $y^2+2y(x^2+\frac{1}{2}bx)$ to both sides, and rewrite it as $(x^2+\frac{1}{2}bx+y)^2=(\frac{1}{4}b^2+2y-c)x^2+(by-d)x+y^2-e$. Now we choose y in such a way that the right-hand side is a square form $(Ax+B)^2$. This reduces to a cubic equation for y;

$$y^3-\tfrac{1}{2}cy^2+(\tfrac{1}{4}cd-e)y+\tfrac{1}{8}(4ec-b^2c-d^2)=0.$$

This can be solved in the usual way. Once y is known, then x is obtained.

and

$$D=-\omega^2\Delta^2(2\Gamma_s)^2,\quad \alpha=\tfrac{1}{3}\{\Delta^2-(2\Gamma_s)^2-\omega^2\},\quad \text{for } \bar{\Delta}(\omega)$$

and

$$D=\omega^2\Delta^2(2\Gamma_s)^2,\quad \alpha=\tfrac{1}{3}\{\omega^2+(2\Gamma_s)^2-\Delta^2\},\quad \text{for } \bar{\omega}(\omega).$$

Here y_1 is to be chosen such that Δ_0, $\omega_0\neq 0$.

As can be seen from Eqs. (1.7) and (3.3), the density of states is proportional to the imaginary part of $\bar{\omega}$;

$$\mathrm{Im}\bar{\omega}(\omega)=2\Gamma_s\,\mathrm{Re}\{\bar{\omega}(\omega)/[\bar{\omega}^2(\omega)-\Delta^2]^{1/2}\}=2\Gamma_s n(\omega).\quad (3.5)$$

We thus can obtain the effective energy gap ω_g as the frequency at which $\bar{\omega}$ first becomes complex. By setting $\delta\omega/\delta\bar{\omega}=0$ in Eq. (3.3), we find[4]

$$\omega_g=\{\Delta^{2/3}-(2\Gamma_s)^{2/3}\}^{3/2}\theta(\Delta-2\Gamma_s).\qquad (3.6)$$

Here $\theta(x)$ is the usual step function and is 1 for positive x and zero otherwise. The region of gapless superconductivity is that for

$$0<\Delta(\Gamma_s,T)<2\Gamma_s\qquad (3.7a)$$

such that $\omega_g=0$. Superconductivity requires that Δ be different from zero, but it is not necessary that $\omega_g>0$. To find an upper bound for Γ_s for which the gap parameter Δ vanishes, we study Eq. (3.2). When $\Delta\to 0$, we find from Eq. (3.1) that $\bar{\Delta}(\omega)$ becomes

$$\bar{\Delta}(\omega)=[\omega/(\omega+2\Gamma_s i)]\Delta.\qquad (3.8)$$

Setting $\Delta=0$ in the integral equation (3.2) for Δ, we find that the transition temperature satisfies

$$[N(0)V_{\mathrm{BCS}}]^{-1}\approx\int d\omega\frac{\omega}{\omega^2+(2\Gamma_s)^2}\tanh\frac{\omega}{2T_c}.\qquad (3.9)$$

Combining this result with the corresponding BCS

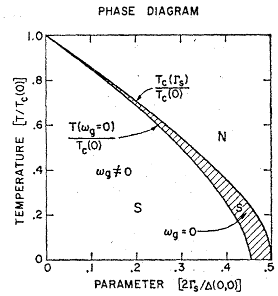


FIG. 5. The phase diagram constructed by the transition temperature $T_c(\Gamma_s)$ and the temperature corresponding to the onset of gapless superconductivlty, $\omega_g(\Gamma_s,T)=0$. There are three regions: nonzero gap, gapless, and normal states.

equation for $\Gamma_s=0$, we obtain the well-known result

$$\ln\frac{T_c(\Gamma_s)}{T_c(0)}=\sum_{n>0}\left\{\frac{1}{n+\frac{1}{2}+m}-\frac{1}{n+\frac{1}{2}}\right\},\qquad (3.10)$$

where $m=\Gamma_s/\pi T_c(\Gamma_s)$. We expect that since $T_c(\Gamma_s)$ is monotonically decreasing with increasing Γ_s, at a certain Γ_s^{cr} the transition temperature will vanish;

$$\ln\frac{2\omega_D}{\Delta(0,0)}\approx[N(0)V_{\mathrm{BCS}}]^{-1}\approx\int d\omega\frac{\omega}{\omega^2+(2\Gamma_s^{\mathrm{cr}})^2}\approx\ln\frac{\omega_D}{2\Gamma_s^{\mathrm{cr}}},$$

so that

$$2\Gamma_s^{\mathrm{cr}}\approx\tfrac{1}{2}\Delta(0,0).\qquad (3.11)$$

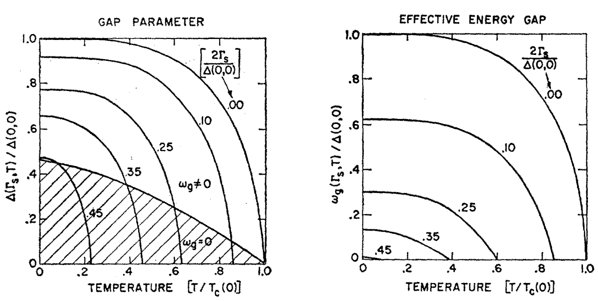
Thus we obtain the condition for the gapless superconductivity

$$0<\Delta(\Gamma_s,T)<2\Gamma_s<\tfrac{1}{2}\Delta(0,0).\qquad (3.7b)$$

Using Eq. (3.4), we carried out calculations of $\Delta(\Gamma_s,T)$ from Eq. (3.2). The results for $\Delta(\Gamma_s,T)$ are presented in Fig. 4(a), and the shady area corresponds to the gapless region, Eq. (3.7). The effective gap ω_g calculated from Eq. (3.6) is shown in Fig. 4(b).

The transition temperature $T_c(\Gamma_s)$ evaluated from Eqs. (3.9) and (3.10) is shown in Fig. 5 with a curve indicating the temperature corresponding to the onset of gapless superconductivity, $\omega_g(\Gamma_s,T)=0$. Here the shady area also corresponds to the gapless region.

The densities of states $n(\omega)$ and pairs $p(\omega)$ have been computed from Eqs. (1.7) as functions of frequency ω for various values of Γ_s and T. The density of states has a maximum at a frequency above the effective energy gap, and has no singularity. When Γ_s goes to zero, the density of states approaches that of the BCS theory, and the effective energy gap also approaches the BCS gap. Finally, when $\Gamma_s=0$, they coincide with those of the BCS theory. On the other hand, when Γ_s increases and reaches $2\Gamma_s>\Delta(\Gamma_s,T)>0$, then the effective energy gap vanishes and the density of states has a finite value at zero frequency; for large value of Γ_s the density of states approaches the normal-state value. In Fig. 6, we give only the results for $2\Gamma_s=0.35\Delta(0,0)$. The curves $3n$ and $4n$ in Fig. 6 correspond to those in the gapless region.



(a) (b)

FIG. 4. (a) The gap parameter $\Delta(\Gamma_s,T)$. Here $2\Gamma_s$ is the inverse relaxation time resulting from spin-flip scattering by magnetic impurities. When $\Gamma_s=0$, the gap parameter $\Delta(0,T)$ corresponds to that of the BCS theory, and $\Delta(0,0)=\Delta_{\mathrm{BCS}}(0)$. The shady area corresponds to the gapless region, $0<\Delta(\Gamma_s,T)<2\Gamma_s<\frac{1}{2}\Delta(0,0)$. The temperature T is normalized by the transition temperature $T_c(\Gamma_s=0)$, that is, the BCS value. (b) The effective energy gap $\omega_g(\Gamma_s,T)$ is defined by a frequency at which the density of states first begins to have finite value; $\omega_g(\Gamma_s,T)=\{\Delta^{2/3}(\Gamma_s,T)-(2\Gamma_s)^{2/3}\}^{3/2}$ for $\Delta(\Gamma_s,T)>2\Gamma_s$, and $\omega_g=0$ otherwise. The value of $\omega_g(0,T)$ is equal to $\Delta(0,T)$, that is, the BCS gap.

Using these results we have carried out calculations for the conductivity from Eq. (1.2) for various values of Γ_s and T. The real part of the conductivity $\sigma_1(\omega)$ in the gapless region has finite value near zero frequencies, and approaches the normal-state value at high frequencies as one expects. We present in Fig. 7 only results for $2\Gamma_s=0.35\Delta(0,0)$. The results of $3\sigma_1$, $3\sigma_2$, $4\sigma_1$, and $4\sigma_2$ are those in the gapless region. In the nonzero gap region the conductivity has a behavior similar to that based on the Mattis and Bardeen theory, except the effective gap ω_g is given by Eq. (3.6) instead of by Δ. The results of $1\sigma_1$, $1\sigma_2$, $2\sigma_1$, and $2\sigma_2$ are in the nonzero gap region.

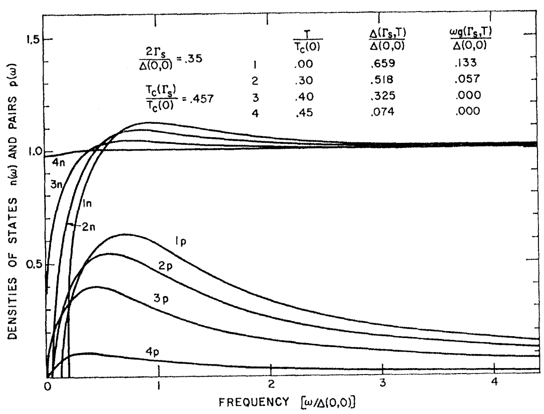


FIG. 6. The densities of states $n(\omega)$ and pairs $p(\omega)$ for the parameter $2\Gamma_s=0.35\Delta(0,0)$. There is a maximum at frequency above the effective energy gap, and no singularity. The curves of $3n$ and $4n$ correspond to the density of states in the gapless region, and have finite value at zero frequency.

We have checked the sum rule for the conductivity

$$\frac{2}{\pi}\int_{0^+}^{\infty}\{1-\sigma_1(\omega)\}d\omega=\lim_{\omega\to 0}\omega\sigma_2(\omega)$$

$$=\frac{K(q,0)}{4\pi\sigma^N}=\pi\Delta^p(0)S,\quad (3.12)$$

and have obtained agreement between both sides of Eq. (3.12); in other words, the value of S from the conductivity [Eq. (3.12)] agrees with that directly from Eq. (1.15). In this case Eq. (1.15) can be written as

$$S=\frac{2}{\pi}\frac{2\pi T}{\Delta(0,0)}\sum_n\frac{\Delta_n^2}{\omega_n^2+\Delta_n^2}$$

$$=\frac{2}{\pi}\frac{2\pi T}{\Delta(0,0)}\sum_n\frac{\Delta^2(\Gamma_s,T)}{\bar{\omega}_n^2+\Delta^2(\Gamma_s,T)},\quad (3.13)$$

where the effective gap parameter Δ_n is

$$\Delta_n=\Delta(\Gamma_s,T)-2\Gamma_s\Delta_n/[\omega_n^2+\Delta_n^2]^{1/2}$$

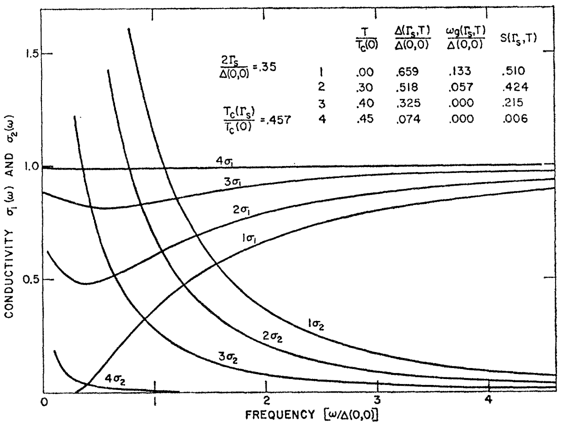


FIG. 7. The conductivity ratio $(\sigma^s/\sigma^N)(\omega)=\sigma_1(\omega)-i\sigma_2(\omega)$ for superconductors containing magnetic impurities [the parameter $2\Gamma_s=0.35\Delta(0,0)$]. The real part of the conductivity in the gapless region ($3\sigma_1$ and $4\sigma_1$) has finite value near zero frequencies. The parameter $S(\Gamma_s,T)$ is obtained from the sum rule:

$$S(\Gamma_s,T)=\frac{2}{\pi^2}\int_{0^+}^{\infty}[1-\sigma_1(\omega)]\frac{d\omega}{\Delta(0,0)}.$$

from Eq. (3.1), and the renormalized frequency $\bar{\omega}_n$ is

$$\bar{\omega}_n=\omega_n+2\Gamma_s\bar{\omega}_n/[\bar{\omega}_n^2+\Delta^2(\Gamma_s,T)]^{1/2}$$

from Eq. (3.3).

When $\Gamma_s=0$, then $\bar{\omega}_n=\omega_n$, $\Delta_n=\Delta$, and Eq. (3.13) becomes the corresponding BCS expression [Eq. (1.16)] as it should. The results of calculations of S presented in Fig. 8(a) are useful for calculations of the penetration depth of Eqs. (1.17) in the Pippard limit and of Eq. (1.18) in the London limit. In the London limit, S is directly proportional to the superfluid density of a two-fluid model.

The temperature at which S vanishes is the transition temperature $T_c(\Gamma_s)$. This is one way to obtain $T_c(\Gamma_s)$. These results of $T_c(\Gamma_s)$ agree with those from Eq. (3.9)

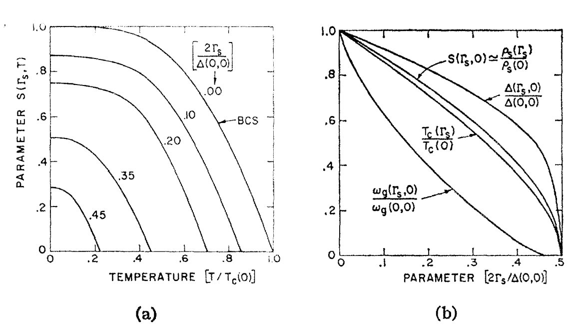


FIG. 8. (a) The parameter $S(\Gamma_s,T)$ defined by

$$S(\Gamma_s,T)=\frac{4T}{\Delta(0,0)}\sum_{n>0}\frac{\Delta^2(\Gamma_s,T)}{\bar{\omega}_n^2+\Delta^2(\Gamma_s,T)},$$

where $\bar{\omega}_n=\omega_n+2\Gamma_s\bar{\omega}_n/[\bar{\omega}_n^2+\Delta^2(\Gamma_s,T)]^{1/2}$ and $\omega_n=(2n+1)\pi T$, n being integer. In the London limit, $S(\Gamma_s,T)$ is directly related to the superfluid density of a two-fluid model; $\rho_s(\Gamma_s,T)/\rho_s(\Gamma_s,0)=S(\Gamma_s,T)$. (b) The gap parameter $\Delta(\Gamma_s,0)$, the effective energy gap $\omega_g(\Gamma_s,0)$, and the parameter $S(\Gamma_s,0)$ at zero temperature with the transition temperature $T_c(\Gamma_s)$.

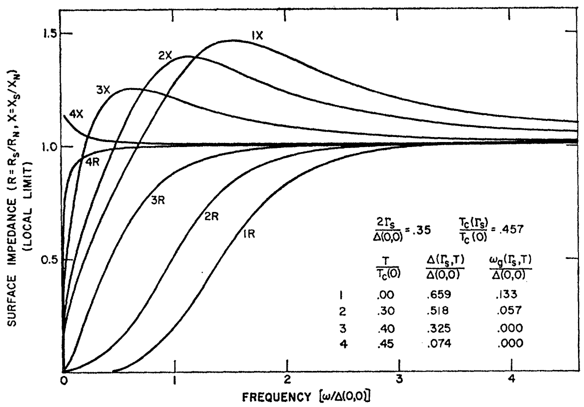


FIG. 9. The surface resistance $R=R_s/R_N$ and surface reactance $X=X_s/X_N$ for the parameter $2\Gamma_s=0.35\Delta(0,0)$.

shown in Fig. 5. For comparison, the results of S, ω_g, and Δ at zero temperature as functions of Γ_s are gathered with the transition temperature $T_c(\Gamma_s)$ in Fig. 8(b).

The surface impedance has been evaluated in the same manner as for Pb. In this case the local-limit values should be used, and they are presented in Fig. 9. More complete calculations are given in Ref. 13.

4. DISCUSSION

For the case of lead, we have carried out numerical calculations of the conductivity and of limiting values of the surface resistance at zero temperature only. It would be desirable to extend these calculations to finite temperatures, and to determine the temperature variations of the penetration depth λ. For the latter, it would be desirable to make a more complete calculation that includes the proper momentum dependence of $K(q,0)$ given in I.

For superconductors containing magnetic impurities, our calculations of $\Delta(\Gamma_s,T)$ and $T_c(\Gamma_s)$ agree with the previous results of Weiss *et al.* The additional calculations for the conductivity and surface impedance at finite temperatures may be useful for analysis of experimental results on such materials. The conductivity and surface impedances for various values of Γ_s and T in the nonzero gap region $[2\Gamma_s<\Delta(\Gamma_s,T)]$ have a behavior similar to that based on the Mattis and Bardeen theory. On the other hand, in the gapless region $[0<\Delta(\Gamma_s,T)<2\Gamma_s<\frac{1}{2}\Delta(0,0)]$ the real part of the conductivity has finite value near zero frequency, and approaches the normal-state value at high frequencies. The temperature range for gapless superconductivity depends on the concentration of magnetic impurities. It would be desirable to have measurements of the conductivity and the surface impedance that include the gapless region to compare with the theory.

The limiting case in which there is only spin-flip scattering is equivalent to a superconductor containing only nonmagnetic impurities but in a high static magnetic field H. The parameter Γ_s then corresponds to $\tau\langle\mu^2\rangle H^2$, where $\langle\mu^2\rangle$ and τ are the average value of the square of the magnetic moment and the relaxation time.[6]

Again, it would be desirable to make more general calculations that include the momentum dependence of the response function, making use of the general expression for the response function given in I.

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to Professor John Bardeen for suggesting this problem and for continual guidance in the course of this research. Without his constant advice this work would have been impossible. It is a great pleasure for the author to acknowledge discussions with Professor D. M. Ginsberg about the experimental information, and discussions with Dr. J. Kane and Dr. W. N. Mathews, Jr. about various topics. Finally, the author wishes to express his gratitude to Professor Gerald M. Almy who smoothed the way in many respects for the author.