Theory of Electromagnetic Properties of Superconducting and Normal Systems. I*

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General expressions for the current density for superconductors in a transverse electromagnetic field are derived from a Green's-function formulation. They are sufficiently general to apply to cases of strong coupling, to superconductors with magnetic impurities, and to anisotropic materials. It is explicitly shown that the response function satisfies

$$\lim_{q\to0}\lim_{\omega\to0} K(q,\omega)=K(0,0)=\lim_{\omega\to0}\lim_{q\to0} K(q,\omega).$$

The condition for superconductivity is that $K(0,0)$ be nonvanishing. An expression for the current density in real space in a form similar to that of Mattis and Bardeen is obtained. Various limiting forms of the expressions are derived to facilitate applications to various problems. An expression for the Josephson tunnelling current which can be applied to strong-coupling and impure superconductors is derived under the assumption that the tunnelling matrix element is constant.

1. INTRODUCTION

THE main purpose of this paper is to discuss the electromagnetic properties of strong-coupling and impure superconductors based on generalized pairing schemes.

On the basis of the theory of Bardeen, Cooper, and Schrieffer (BCS),[1] the electromagnetic properties of isotropic weak-coupling superconductors have been discussed by Mattis and Bardeen.[2] Miller[3] applied the theory to calculate the surface impedance of aluminum and tin, and also to determine the effect of a finite-scattering mean free path on the penetration depth λ, generally in good agreement with experiment. He has given limiting forms of the response function that apply in the Pippard ($\lambda\ll\xi_0$) and London ($\lambda\gg\xi_0$) limits. Here ξ_0 is the coherence length. Waldram[4] has measured the surface impedance of superconducting tin and tin alloys, and has obtained good agreement with the theory, except for anisotropic effects not described by the theory.

The electromagnetic properties of pure and impure isotropic weak-coupling superconductors have been discussed by Abrikosov, Gor'kov, and Khalatnikov[5] by deriving the response function $K(q,\omega)$ from a Green's-function formulation. Rickayzen and others[6] have given rather general derivations and shown that they reduce to those of Mattis and Bardeen in appropriate limits. Effects of collective excitations on the electromagnetic response have been discussed by Anderson, Rickayzen, Tsuneto, Larkin, and others.[7,8] Abrikosov and Gor'kov have discussed the electromagnetic properties of an isotropic weak-coupling superconductor containing magnetic impurities.[9] More detailed calculations, particularly for the limit $q=0$, have been given by Skalski, Betbeder-Matibet, and Weiss.[10]

A general formulation for the response functions for many-body systems obtained by Martin and Schwinger[11] has been applied to superconductors in the weak-coupling limit by Kandanoff and Martin.[12] By using a field-theoretical viewpoint Nambu[13,14] was able to show the gauge invariance of the pairing scheme, and helped justify calculations of the electromagnetic properties of superconductors based on the BCS wave functions.

For a general treatment of the electromagnetic properties of superconductors, a generalization of the original BCS pairing scheme is necessary. In the original BCS theory, an energy-gap parameter Δ was introduced to describe pairing the ground state. In the later Green's-

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[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957); **108**, 1175 (1957); J. Bardeen and J. R. Schrieffer, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1961), Vol. 3, p. 1.
[2] D. C. Mattis and J. Bardeen, Phys. Rev. **111**, 412 (1958).
[3] P. B. Miller, Phys. Rev. **113**, 1208 (1958); **118**, 928 (1960).
[4] J. R. Waldram, Advan. Phys. **13**, 1 (1964).
[5] A. A. Abrikosov, L. P. Gor'kov, and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. **35**, 265 (1958) [English transl.: Soviet Phys.—JETP **8**, 182 (1958)]—the corrected one; I. M. Khalatnikov and A. A. Abrikosov, Advan. Phys. **8**, 45 (1958); A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **35**, 1558 (1958) [English transl.: Soviet Phys.—JETP **8**, 1090 (1959)].
[6] G. Rickayzen, in *Lecture Notes on The Many Body Problem*, edited by C. Fronsdal (W. A. Benjamin, Inc., New York, 1962), p. 85; G. Rickayzen, *Theory of Superconductivity* (John Wiley & Sons, Inc., New York, 1965).
[7] P. W. Anderson, Phys. Rev. **112**, 1900 (1958); G. Rickayzen, *ibid.* **115**, 795 (1958); T. Tsuneto, *ibid.* **118**, 1029 (1960).
[8] A. I. Larkin, Zh. Eksperim. i Teor. Fiz. **46**, 2188 (1964) [English transl.: Soviet Phys.—JETP **19**, 1478 (1964)]; V. G. Vaks, V. M. Galitskii, and A. I. Larkin, Zh. Eksperim. i Teor. Fiz. **41**, 1655 (1961) [English transl.: Soviet Phys.—JETP **14**, 1177 (1961)].
[9] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **39**, 1781 (1960) [English transl.: Soviet Phys.—JETP **12**, 1243 (1960)].
[10] S. Skalski, O. Betbeder-Matibet, and P. R. Weiss, Phys. Rev. **136**, A1500 (1964).
[11] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).
[12] L. P. Kadanoff and P. C. Martin, Phys. Rev. **124**, 670 (1961).
[13] Y. Nambu, Phys. Rev. **117**, 648 (1960).
[14] See, for example, P. W. Anderson, Phys. Rev. **110**, 827 (1958); G. Rickayzen, *ibid.* **111**, 817 (1958).

function calculations of Gor'kov and others,[15] the gap parameter is in general a function of both momentum **k** and energy ω, $\Delta(\mathbf{k},\omega)$. In an isotropic system the momentum dependence is unimportant, and may be neglected. However for strong-coupling superconductors, such as Pb, Hg, etc., it is necessary to take the frequency dependence into account in order to treat properly the retarded nature of the electron-phonon interaction.[16] This is also true for a superconductor containing magnetic impurities. In the anisotropic systems, the angular dependence on **k** must be included, but the magnitude of **k** may be replaced by the value on the Fermi surface in the same direction.

It should be noted that a gap in the energy spectrum is not necessarily a characteristic of the superconducting state. In fact, the superconducting state is in general characterized by infinite dc conductivity (zero dc resistivity), and perfect diamagnetism (the Meissner effect). To describe the above phenomena we require a general relation between the current density and field. Following the usual field-theoretical treatment the expectation value of the current operator in the presence of the vector potential $A(x)$ can be written [we choose the gauge in such a way that the scalar field is zero, that is, the transverse field is described by the vector potential $A(x)$]

$$J_\mu(x)=-\frac{1}{4\pi}\int d^4x'\, K_{\mu\nu}(x,x')A^\nu(x'). \tag{1.1}$$

Here the response function $K_{\mu\nu}$ can be described in terms of a four-point function, a two-particle Green's function, or a current-current correlation function. Throughout this paper we use the units $\hbar=c=k_B=1$. In the momentum space the current density may be written in a form[17]

$$J_\mu(\mathbf{q},\omega)=-\frac{1}{4\pi}K_{\mu\nu}(\mathbf{q},\omega)A^\nu(\mathbf{q},\omega). \tag{1.2}$$

For the system to be perfectly conducting and perfectly diamagnetic, Maxwell's equations require that the response function $K_{\mu\nu}(\mathbf{q},\omega)$ have a finite value $K_{\mu\nu}(0,0)$ in two distinct limits: $\omega\to 0$, $q\to 0$ and $q\to 0$, $\omega\to 0$. This finite value is directly related to the superfluid density of a two-fluid model, and may be taken as one of the essential parameters to describe the superconducting state. For the case of so-called gapless superconductivity, the effective energy gap vanishes in that there are excitations of abritrarily low energy, but $K_{\mu\nu}(0,0)$ does not. It will be made clear later that this limiting value is finite if the gap parameter Δ is different from zero. Thus it is fair to say that a nonzero gap parameter is essential for superconductivity and superfluidity, but an energy gap in the excitation spectrum is not required.

It is our main object, based on the generalized pairing scheme, to obtain an explicit form for the response function $K_{\mu\nu}(\mathbf{q},\omega)$ sufficiently general for strong-coupling and impure superconductors. We show explicitly that the response function satisfies the important condition

$$\lim_{q\to 0}\lim_{\omega\to 0}K_{\mu\nu}(\mathbf{q},\omega)=K_{\mu\nu}(0,0)=\lim_{\omega\to 0}\lim_{q\to 0}K_{\mu\nu}(\mathbf{q},\omega) \tag{1.3}$$

for the theory of superconductivity and superfluidity. We also obtain an expression for the current density in real space that can be applied to strong-coupling and impure systems.

There are two contributions to $K_{\mu\nu}(\mathbf{q},\omega)$: the paramagnetic $K_{\mu\nu}{}^p$ and the diamagnetic $K_{\mu\nu}{}^d$ parts. The diamagnetic term is proportional to the density of electrons, and thus is not different from that in the normal state. In the normal state the total current induced by a static field is negligibly small, so that

$$K_{\mu\nu}{}^n(\mathbf{q},0)=K_{\mu\nu}{}^{np}(\mathbf{q},0)+K_{\mu\nu}{}^d=0.$$

Thus we can write the current density in the superconducting state as[18]

$$J_\mu(\mathbf{q},\omega)=-\frac{1}{4\pi}\{K_{\mu\nu}{}^{sp}(\mathbf{q},\omega)-K_{\mu\nu}{}^{np}(\mathbf{q},0)\}A^\nu(q,\omega). \tag{1.4}$$

We see that only the paramagnetic part need be calculated for our purpose. An alternative way is to write the current density as

$$\begin{aligned}J_\mu{}^s(\mathbf{q},\omega)-J_\mu{}^n(\mathbf{q},\omega)&=-\frac{1}{4\pi}\{K_{\mu\nu}{}^s(\mathbf{q},\omega)-K_{\mu\nu}{}^n(\mathbf{q},\omega)\}A^\nu(q,\omega)\\&=-\frac{1}{4\pi}\{K_{\mu\nu}{}^{sp}(\mathbf{q},\omega)-K_{\mu\nu}{}^{np}(\mathbf{q},\omega)\}A^\nu(q,\omega).\end{aligned} \tag{1.5}$$

After the calculations we can identify the parts corresponding to the superconducting and normal states.

In Sec. 2 the response function in terms of the Green's functions is discussed. The Green's functions for strong-coupling and impure superconductors are discussed in Sec. 3. In Sec. 4, an explicit calculation of the response function is carried out. We have obtained an expression for the current density in real space, which can be applied to strong-coupling and impure systems,

[15] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **34**, 735 (1958) [English transl.: Soviet Phys.—JETP **7**, 505 (1958)].

[16] D. J. Scalapino, J. R. Schrieffer, and J. W. Walkins, Phys. Rev. Letters **10**, 336 (1963); Phys. Rev. **148**, 263 (1966).

[17] For the anisotropic system, we assume that the interaction potential, spectrum, etc. are anisotropic. In most cases the penetration depth and the coherence distance are large compared with the lattice constant, and then the field A and the gap parameter Δ are not much changed within the periodic distance. We then use the Fourier transform of $A(q)$, disregarding the fact that a quasimomentum but not a real momentum exists in the lattice.

[18] J. Bardeen has pointed out to the author that Eq. (1.4) is valid within an approximation in which the Landau diamagnetism is neglected.

in a form similar to that of Mattis and Bardeen:

$$\mathbf{J}(\mathbf{r},t)=C\sum_{\omega} e^{i\omega t}\int d\mathbf{r}'\frac{\mathbf{R}[\mathbf{R}\cdot\mathbf{A}_\omega(\mathbf{r}')]}{R^4}I(\omega,R,T)\,, \quad (1.6)$$

where $\mathbf{R}=\mathbf{r}-\mathbf{r}'$, and $C=N(0)e^2v_0/2\pi^2$. Here $N(0)$ and v_0 are the normal-state density of states and the velocity averaged appropriately on the Fermi surface. In the weak-coupling isotropic systems the effect of scattering by nonmagnetic impurities is to multiply the kernel by $\exp[-R/l]$, where l is the mean free path. We show that more generally the mean-free-path effect (lifetime effect) is to introduce a factor $\exp[-R/L]$ with

$$L^{-1}=\{\Gamma(k_1)+\Gamma(k_2)\}/|v|\,, \quad (1.7)$$

where the variables k_1 and k_2 stand for (Ω_1,ω') and $(\Omega_2,\omega+\omega')$, respectively. Here Ω_1 and Ω_2 are angular variables in appropriate directions. The function $\Gamma(k)$ is directly related to the cut (imaginary part) of the self-energy resulting from the various scatterings;

$$\Gamma(k)=\mathrm{Im}\{Z(k)[k_0^2-\Delta^2(k)]^{1/2}\}\,. \quad (1.8)$$

The velocity v on the Fermi surface is given by[19]

$$v=\nabla_{\mathbf{k}}\epsilon_{\mathbf{k}}\,. \quad (1.9)$$

The wave-function renormalization $Z(k)$ and the gap parameter $\Delta(k)$ are solutions of integral equations discussed in Sec. 3. For the isotropic weak-coupling system but with magnetic impurities the effective mean-free-path L is

$$\frac{1}{L}=\frac{1}{l}+\frac{1}{l_s}\,,$$

where l and l_s are the effective mean free paths resulting from non–spin-flip (ordinary impurity) and spin-flip (magnetic impurity) scatterings, respectively. In Sec. 5 we discuss the various limiting values of the response function, especially

$$K_{\mu\nu}(0,0)=\frac{e^2}{\pi^2}\int\frac{dA}{|v|}v_\mu v_\nu\,\mathrm{Re}\sum_n\frac{2\pi}{\beta}\frac{\Delta_n^2}{(\omega_n^2+\Delta_n^2)^{3/2}Z_n}\,, \quad (1.10)$$

where $\omega_n=(2n+1)\pi/\beta$, n being an integer, $Z_n=Z(\Omega,i\omega_n)$, $\Delta_n=\Delta(\Omega,i\omega_n)$, and $\beta=1/T$. Here dA donates the area element on the Fermi surface. For the isotropic system the Ω dependence drops out; we then obtain

$$K(0,0)=\frac{4\pi}{\Lambda}\frac{\rho_s}{\rho}=\frac{4\pi}{\Lambda}\,\mathrm{Re}\sum_n\frac{2\pi}{\beta}\frac{\Delta_n^2}{[\omega_n^2+\Delta_n^2]^{3/2}Z_n}\,, \quad (1.11)$$

where the London parameter Λ is

$$\Lambda^{-1}=\frac{ne^2}{m}=\tfrac{2}{3}N(0)e^2v_0^2\,, \quad (1.12)$$

and ρ_s is the superfluid density in a two-fluid model. In the appropriate limits, that is, the weak coupling, the result of Eq. (1.11) reduces to those of Mattis and Bardeen, Abrikosov *et al.*,[20] and Rickayzen[6] for the case of a superconductor with nonmagnetic impurities, and to those of Abrikosov and Gor'kov[21] and Weiss *et al.*[10] for the case of a superconductor with magnetic impurities.

Some useful normal-state response-function and conductivity formulas can be obtained from those in the superconducting state by setting the gap parameter $\Delta(k)$ to be zero. We have rederived known formulas for the normal-state response as well as some new ones.

Finally in the last section we discuss various features of the calculations and possible extensions of them. In the Appendix an expression for the Josephson tunneling current which can be applied to strong-coupling and impure superconductors is derived under the assumption that the tunneling matrix element is constant.

2. RESPONSE FUNCTION

As pointed out in the previous section, the paramagnetic response function is needed for our purpose. In this section we discuss a brief review of a derivation of the response function as discussed formally by many authors.[22]

The paramagnetic response function in general can be expressed in terms of a four-point function, that is, a two-particle Green's function.

It is convenient to express the paramagnetic response function $K_{\mu\nu}{}^p(x,x')$ in terms of a time-ordered current-current correlation function defined by

$$P_{\mu\nu}(x,x')=-4\pi i\langle Tj_\mu{}^p(x)j_\nu{}^p(x')\rangle\,, \quad (2.1)$$

where j^p is the usual paramagnetic-current operator. The Fourier transforms of $K_{\mu\nu}{}^p$ and $P_{\mu\nu}$ with respect to time are closely related to each other according to

$$\mathrm{Re}K_{\mu\nu}{}^p(\omega)=\mathrm{Re}P_{\mu\nu}(\omega)\,,$$
$$\mathrm{Im}K_{\mu\nu}{}^p(\omega)=\mathrm{sgn}(\omega)\,\mathrm{Im}P_{\mu\nu}(\omega)\,.$$

Once $P_{\mu\nu}(\omega)$ is known, then $K_{\mu\nu}{}^p(\omega)$ is obtained, and vice versa since they are equal for positive frequencies and are complex conjugate to each other for the negative frequencies. Hereafter we calculate $P_{\mu\nu}$, but refer it to the response function $K_{\mu\nu}{}^p$ assuming that we are working with positive frequencies.

[19] We neglect the interband transitions. The double band effect on the transition temperature is discussed by V. A. Moskalenko and M. E. Palistrant, Zh. Eksperim. i Teor. Fiz. **49**, 770 (1965) [English transl.: Soviet Phys.—JETP **22**, 536 (1966)].

[20] See the last paper in Ref. 5.

[21] In Ref. 9 they have obtained the result only for T near T_c.

[22] J. R. Schrieffer, *Theory of Superconductivity* (W. A. Benjamin, Inc., New York, 1964), Chap. 8, p. 203; A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics*, translated by R. A. Silverman (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963). L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962); R. Kubo, J. Phys. Soc. Japan **12**, 570 (1957); T. Matsubara, Progr. Theoret. Phys. (Kyoto) **14**, 351 (1955); D. N. Zubarev, Usp. Fiz. Nauk **71**, 71 (1960) [English transl.: Soviet Phys.—Usp. **3**, 320 (1960)]. The general formulation for the many-body problem is discussed in Ref. 11.

In order to take into account the retarded nature of the electron-phonon interaction, it is convenient to work in the spinor representation of Nambu[13] and Eliasberg.[23] In this scheme spinor wave-field operators $\Psi_{\mathbf{k}}(t)$ and $\Psi_{\mathbf{k}}^\dagger(t)$ are defined by

$$\Psi_{\mathbf{k}}^\dagger(t)=(\psi_{\mathbf{k}\uparrow}^\dagger(t),\psi_{-\mathbf{k}\downarrow}(t)),$$

$$\Psi_{\mathbf{k}}(t)=\begin{pmatrix}\psi_{\mathbf{k}\uparrow}(t)\\ \psi_{-\mathbf{k}\downarrow}^\dagger(t)\end{pmatrix},$$

where $\psi_{\mathbf{k}\sigma}$ and $\psi_{\mathbf{k}\sigma}^\dagger$ are the usual field operators. The current operator $j_\mu{}^p$ can be written

$$j_\mu{}^p(\mathbf{q},t)=e\sum_{\mathbf{k}}\Psi_{\mathbf{k}}^\dagger(t)\gamma_\mu(\mathbf{k},\mathbf{k}+\mathbf{q})\Psi_{\mathbf{k}+\mathbf{q}}(t)\,, \quad (2.2)$$

where the bare vertex function $\gamma_\mu(\mathbf{k},\mathbf{k}+\mathbf{q})$, with use of the Pauli matrices τ_1, τ_2, and τ_3, may be written

$$\gamma_\mu(\mathbf{k},\mathbf{k}+\mathbf{q})=v_\mu[1-\delta_{\mu,0}]+\tau_3\delta_{\mu,0}. \quad (2.3)$$

Here the velocity component v_μ is given by

$$v_\mu=(1/m)(\mathbf{k}+\tfrac{1}{2}\mathbf{q})_\mu \quad (2.4a)$$

for the isotropic system, and may be written as[24]

$$v_\mu=(\nabla_{\mathbf{k}}\epsilon_{\mathbf{k}})_\mu+\tfrac{1}{2}[\nabla_{\mathbf{k}+\mathbf{q}}\epsilon_{\mathbf{k}+\mathbf{q}}-\nabla_{\mathbf{k}}\epsilon_{\mathbf{k}}]_\mu \quad (2.4b)$$

for the anisotropic system within the approximation in which interband transitions are neglected.

Inserting the current operator Eq. (2.2) into the response function Eq. (2.1), and taking the Fourier transform, we obtain the following gauge-invariant form[25]:

$$K_{\mu\nu}{}^p(q)=-4\pi ie^2\times\int_k \mathrm{Tr}\{\gamma_\mu(k,k+q)G(k+q)\Gamma_\nu(k+q,k)G(k)\}. \quad (2.5)$$

We have here used the notation $q=(\mathbf{q},q_0)=(\mathbf{q},\omega)$, and

$$\int\frac{d^4k}{(2\pi)^4}(T=0)\leftrightarrow\int_k\leftrightarrow\int\frac{d^3k}{(2\pi)^3}\times\sum_n\frac{i}{\beta}(k_0\to z_n)(T\neq 0)\,, \quad (2.6)$$

where $z_n=n\pi i/\beta$, n being an odd (even) integer for the fermion (boson) system. The electronic matrix Green's function is defined by

$$G(k)=-i\int d(t-t')\langle T\Psi_{\mathbf{k}}(t)\Psi_{\mathbf{k}}^\dagger(t')\rangle\exp[ik_0(t-t')]=\begin{pmatrix}G & F\\ F^+ & -G\end{pmatrix},$$

where G and F are the Green's functions introduced by Gor'kov, and $\langle\ \rangle$ denotes the usual grand canonical average. The Fourier transform with respect to time becomes the usual Fourier-series sum at finite temperatures. The dressed vertex function Γ_ν satisfies the integral equation defined by Fig. 1. As shown by Migdal,[26] the vertex correction resulting from the electron-phonon interaction is of the order of the square root of the mass ratio of electron and ion, $(m/M)^{1/2}$, and thus can be neglected. The response function can then be approximated by replacing the dressed vertex function Γ_ν by the bare vertex function γ_ν:

$$K_{\mu\nu}{}^p(q)=-4\pi ie^2\times\int_k \mathrm{Tr}\{\gamma_\mu(k,k+q)G(k+q)\gamma_\mu(k+q,k)G(k)\}. \quad (2.7)$$

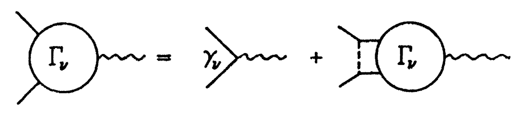

FIG. 1. The integral equation for the dressed vertex function Γ_ν. The solid and wavy lines stand for electrons and photons, respectively. The dotted line denotes the interaction.

For the impure system, the randomness of impurities leads to a similar treatment after averaging all quantities over random impurity configurations. We omit vertex corrections resulting from the impurity scattering. Presumably the only change would be to replace the relaxation time with an effective relaxation time for transport. We can then use the above response function for the impure system with the appropriate Green's function. Equation (2.7) is our starting point to calculate explicitly the response function as will be discussed in Sec. 4. To do this we need to know the formal structure of the electronic Green's function which will be discussed in the next section.

3. GREEN'S FUNCTION

To proceed with the calculation of the repsonse function, we need the formal structure of the Green's functions for strong-coupling and impure superconductors.

In this section, we give a brief review of the relevant expressions as derived by Eliasberg, Abrikosov, Gor'kov, and others.[9,13,16,21,22]

[23] G. E. Eliasberg, Zh. Eksperim. i Teor. Fiz. **38**, 966 (1960) [English transl.: Soviet Phys.—JETP **11**, 696 (1960)].

[24] If one rewrites the current operator of Eq. (2.2) as $j_\mu{}^p(q)=\sum_{\mathbf{k}}\psi_{\mathbf{k}-\mathbf{q}/2}{}^\dagger\gamma_\mu(\mathbf{k}-\frac{1}{2}\mathbf{q},\mathbf{k}+\frac{1}{2}\mathbf{q})\psi_{\mathbf{k}+\mathbf{q}/2}$, then the velocity component v_μ is $v_\mu=(\nabla_{\mathbf{k}}\epsilon_{\mathbf{k}})_\mu$ of Eq. (1.9) within the approximation as stated in Ref. 19.

[25] One should not confuse the dressed vertex function $\Gamma_\nu(k_1,k_2)$ here and the cut of self-energy $\Gamma=\mathrm{Im}\{Z(k)(k_0{}^2-\Delta^2(k))^{1/2}\}$ later from Eq. (4.9).

[26] A. B. Migdal, Zh. Eksperim. i Teor. Fiz. **34**, 1438 (1958) [English transl.: Soviet Phys.—JETP **7**, 996 (1958)].

Following the usual equation-of-motion method, the electronic matrix Green's function for a pure strong-coupling superconductor can be written as

$$G^{-1}(k)=Z(k)[k_0-\Delta(k)\tau_1]-\bar{\epsilon}_k\tau_3\,. \tag{3.1}$$

The wave-function renormalization factor $Z(k)$, the gap parameter $\Delta(k)$, and the renormalized quasiparticle energy $\bar{\epsilon}_k$ are solutions of the integral equations

$$[1-Z(k)]k_0=\int_q Z(q)T(k,q)q_0\,, \tag{3.2a}$$

$$\bar{\epsilon}_k=\epsilon_k+\int_q \bar{\epsilon}_q T(k,q)=Z_3(k)\epsilon_k\,, \tag{3.2b}$$

$$Z(k)\Delta(k)=\int_q Z(q)\Delta(q)T(k,q)\,, \tag{3.2c}$$

and

$$T(k,q)=[T_{\mathrm{e-e}}+T_{\mathrm{e-p}}]/\{Z^2(q)[q_0{}^2-\Delta^2(q)]-\bar{\epsilon}_q{}^2\}\,.$$

Here $T_{\mathrm{e-e}}$ and $T_{\mathrm{e-p}}$ are the kernels representing the electron-electron and electron-phonon interactions, respectively. In the usual approximation in which the symmetry between electrons and holes is valid, Z_3 is equal to unity. For simplicity, we shall assume this to be the case in the following. The above integral equations have been solved numerically, for lead, at zero temperature by Schrieffer *et al.*,[16] and at finite temperatures by Swihart *et al.*,[27] for the isotropic system. The anisotropic effects have been discussed by Bennett and others.[28]

For the case of the system containing nonmagnetic and magnetic impurities, the scattering may be described by the interaction

$$V(\mathbf{x})=\sum_a \{V_1(\mathbf{x}-\mathbf{R}_a)+\boldsymbol{\sigma}\cdot\mathbf{S}_aV_2(\mathbf{x}-\mathbf{R}_a)\}\,, \tag{3.3}$$

where $\mathbf{S}_a$ and $\mathbf{R}_a$ are the spin and position, respectively, of the ath impurity atom and σ is the electron-spin operator. After averaging over all impurity configurations, the same procedure as employed for the pure system leads to the Green's function

$$G_I^{-1}(k)=Z_I(k)[k_0-\Delta_I(k)\tau_1]-\bar{\epsilon}_{Ik}\tau_3\,, \tag{3.4}$$

where Z_I, Δ_I, and $\bar{\epsilon}_{Ik}$ are solutions of the integral equations

$$[1-Z_I(k)]k_0=\int_q Z_I(q)q_0T_I{}^+(k,q)\,, \tag{3.5a}$$

$$\bar{\epsilon}_{Ik}=\epsilon_k+\int_q \bar{\epsilon}_{Iq}T_I{}^+(k,q)=Z_{3I}(k)\epsilon_k\,, \tag{3.5b}$$

$$Z_I(k)\Delta_I(k)=\int_q Z_I(q)\Delta_I(q)T_I{}^-(k,q)\,, \tag{3.5c}$$

and

$$T_I{}^\pm(k,q)=[T_{\mathrm{e-e}}+T_{\mathrm{e-p}}+T_I{}^0\pm T_I{}^S]/\{Z_I{}^2(q)[q_0{}^2-\Delta_I{}^2(q)]-\bar{\epsilon}_{Ik}{}^2\}\,.$$

Here $T_I{}^0$ and $T_I{}^S$ are the kernels representing the ordinary (non-spin-flip) and magnetic (spin-flip) impurity scatterings, respectively. When $T_I{}^{0,S}=0$, that is, no impurity scattering, the integral equations (3.5) become those of Eq. (3.2) for the pure system. The renormalization factor $Z_{3I}(k)$ for the quasiparticle energy is introduced in the same manner as that for $Z_3(k)$, and it is nearly unit in the approximation which is used for $Z_3(k)$.

For the isotropic weak-coupling limit, that is, within the BCS tkeory ($T_{\mathrm{e-e}}+T_{\mathrm{e-p}}=-V$), the above integral equations reduce to algebraic equations of Abrikosov and Gorkov,

$$\begin{aligned}Z_I(\omega)\omega&=\omega+i(\Gamma+\Gamma_s)\omega/[\omega^2-\Delta^2{}_I(\omega)]^{1/2}\\&=\bar{\omega}+i(\Gamma-\Gamma_s)\bar{\omega}/[\bar{\omega}^2-\Delta^2]^{1/2}\,,\end{aligned} \tag{3.6a}$$

$$\bar{\epsilon}_{Ik}\approx\epsilon_k\,, \tag{3.6b}$$

$$\begin{aligned}\Delta_I(\omega)&=\Delta-2i\Gamma_s\Delta_I(\omega)/[\omega^2-\Delta_I{}^2(\omega)]^{1/2}\\&=\Delta-2i\Gamma_s\Delta/[\bar{\omega}^2-\Delta^2]^{1/2}\,,\end{aligned} \tag{3.6c}$$

where

$$\begin{aligned}\Delta&\approx N(0)V\int d\omega\,(\tanh\tfrac{1}{2}\beta\omega)\,\mathrm{Re}\{\Delta_I(\omega)/[\omega^2-\Delta_I{}^2(\omega)]^{1/2}\}\\&=N(0)V\int d\omega\,(\tanh\tfrac{1}{2}\beta\omega)\,\mathrm{Re}\{\Delta/[\bar{\omega}^2-\Delta^2]^{1/2}\}\,,\end{aligned} \tag{3.7}$$

$$\Gamma\approx n_I\int d\Omega\,|v_1(k,q)|^2\,, \tag{3.8}$$

$$\Gamma_s\approx n_I\tfrac{1}{4}S_a(S_a+1)\int d\Omega\,|v_2(k,q)|^2\,. \tag{3.9}$$

We have here assumed that the matrix elements $v_1(k,q)$ and $v_2(k,q)$ are dependent only on the angle on the Fermi surface. For convenience we have introduced the renormalized frequency $\bar{\omega}$ as

$$\bar{\omega}(\omega)=\omega+2\Gamma_si\bar{\omega}(\omega)/[\bar{\omega}^2(\omega)-\Delta^2]^{1/2}\,. \tag{3.10}$$

In this limit we observe that all calculations can be carried out by solving Eq. (3.6c), or Eq. (3.10) which

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[27] J. S. Swihart (private communication); D. J. Scalapino, Y. Wada, and J. S. Swihart, Phys. Rev. Letters **14**, 102 (1965); **14**, 106 (1965).

[28] A. J. Bennett, Phys. Rev. **140**, A1902 (1965); V. L. Pokrovskii, Zh. Eksperim. i Teor. Fiz. **40**, 641 (1961); **40**, 898 (1961) [English transls.: Soviet Phys.—JETP **13**, 447 (1961); **13**, 628 (1961)]; V. L. Pokrovskii, and M. S. Ryvkin, Zh. Eksperim. i Teor. Fiz. **40**, 1859 (1961); **43**, 92 (1962) [English transls.: Soviet Phys.—JETP **13**, 1306 (1961); **16**, 67 (1963)]; P. Hohenberg, Zh. Eksperim. i. Teor. Fiz. **45**, 1208 (1963) [English transl.: Soviet Phys.—JETP **18**, 834 (1964)]; D. Markowitz and L. P. Kadanoff, Phys. Rev. **131**, 563 (1963); J. R. Clem, *ibid.* **148**, 392 (1966); M. A. Biondi, M. P. Garfunkel, and W. A. Thompson, *ibid.* **136**, A1471 (1964); L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **45**, 1943 (1963) [English transl.: Soviet Phys.—JETP **18**, 1031 (1964)].

turns out to be a fourth power algebraic equation. The explicit solutions of $\Delta_I(\omega)$ of Eq. (3.6c) and of $\bar{\omega}$ of Eq. (3.10) are given elsewhere.[29] When $\Gamma_s=0$, that is, no magnetic impurity (spin-flip) scattering, then $\bar{\omega}(\omega)=\omega$ and $\Delta_I(\omega)=\Delta$ as one expects.

At zero temperature the solution of Eq. (3.7) can be written as[9]

$$\ln\frac{\Delta(\Gamma_s,0)}{\Delta(0,0)}\approx-\frac{\pi}{4}x+\frac{x_0}{2x}+\frac{x}{2}\tan^{-1}x_0-\sinh^{-1}x_0\,, \quad (3.11)$$

where $x=2\Gamma_s/\Delta(\Gamma_s,0)$ and $x_0=\theta(x-1)[x^2-1]^{1/2}$.

When $T\to T_c$, that is, $\Delta_I\to 0$ or $\Delta\to 0$, then from Eq. (3.6c) we obtain

$$\Delta_I(\omega)=[\omega/(\omega+2i\Gamma_s)]\Delta\,. \quad (3.12)$$

Equation (3.7) then leads to the equation of the transition temperature as

$$[N(0)V]^{-1}\approx\int d\omega\frac{\omega}{\omega^2+(2\Gamma_s)^2}\tanh(\tfrac{1}{2}\beta_c\omega)\,. \quad (3.13)$$

Combining this with the BCS equation ($\Gamma_s=0$), we obtain the well-known result of Abrikosov and Gor'kov

$$\ln\frac{T_c}{T_c{}^p}=\psi(\tfrac{1}{2})-\psi(\tfrac{1}{2}+\eta)=\sum_n\left(\frac{1}{n+\frac{1}{2}+\eta}-\frac{1}{n+\frac{1}{2}}\right), \quad (3.14)$$

where $\psi(x)$ is the digamma function,[30] and $\eta=\Gamma_s/\pi T_c$. T_c and $T_c{}^p$ denote the transition temperatures with and without magnetic impurity scattering. In other words, $T_c=T_c(\Gamma_s)$ and $T_c{}^p=T_c(\Gamma_s=0)$.

It is noted that the formal structure of the Green's functions for all the cases is the same, so that similar calculations may be used for the electromagnetic response. For this purpose we write the Green's function of the system in the following form for all cases:

$$\begin{aligned}G(k)&=\{Z(k)[k_0-\Delta(k)\tau_1]-\bar{\epsilon}_k\tau_3\}^{-1}\\&=\frac{1}{2}\left\{\frac{N(k)+P(k)\tau_1+\tau_3}{\epsilon(k)-\bar{\epsilon}_k}+\frac{N(k)+P(k)\tau_1-\tau_3}{\epsilon(k)+\bar{\epsilon}_k}\right\},\end{aligned} \quad (3.15)$$

where

$$\epsilon(k)=Z(k)[k_0-\Delta^2(k)]^{1/2}, \quad (3.16a)$$

$$N(k)=Z(k)k_0/\epsilon(k)=k_0/[k_0{}^2-\Delta^2(k)]^{1/2}, \quad (3.16b)$$

$$P(k)=Z(k)\Delta(k)/\epsilon(k)=\Delta(k)/[k_0{}^2-\Delta^2(k)]^{1/2}. \quad (3.16c)$$

Since the dominant contributions come from electrons near the Fermi surface we expect that the $|\mathbf{k}|$ dependence is unimportant, but the dependences of frequencies and of angles on the Fermi surface are important for strong-coupling and anisotropic systems. We may then write the variable k in Eq. (3.16) as $k=(\mathbf{k},\omega)=(\Omega,\omega)$, where Ω denotes the angular variables on the Fermi surface. Hereafter, we use the variable k as (Ω,ω) unless otherwise specified.

We introduce the densities of states $n(k)$ and pairs $p(k)$ as[31]

$$\mathrm{Im}\frac{1}{\pi}\sum_{\mathbf{k}}\tfrac{1}{2}\,\mathrm{Tr}G(k)\approx\frac{1}{(2\pi)^3}\int\frac{dA}{|v|}n(k)\,, \quad (3.17a)$$

$$\mathrm{Im}\frac{1}{\pi}\sum_{\mathbf{k}}\tfrac{1}{2}\,\mathrm{Tr}\tau_1G(k)\approx\frac{1}{(2\pi)^3}\int\frac{dA}{|v|}p(k)\,, \quad (3.17b)$$

and

$$n(k)=\mathrm{Re}\{N(k)/Z_3(k)\}\approx\mathrm{Re}\{N(k)\}\,, \quad (3.18a)$$

$$p(k)=\mathrm{Re}\{P(k)/Z_3(k)\}\approx\mathrm{Re}\{P(k)\}\,. \quad (3.18b)$$

We have here used the approximation for the integration on the Fermi surface

$$\sum_{\mathbf{k}}=\int\frac{d^3k}{(2\pi)^3}=\frac{1}{(2\pi)^3}\times\int dA\int\frac{d\epsilon_k}{|v|}\approx\frac{1}{(2\pi)^3}\int\frac{dA}{|v|}\int d\epsilon_k\,, \quad (3.19)$$

where dA denotes an area element on the Fermi surface. For an isotropic system, the angular dependence drops out in Eqs. (3.17) and (3.18), and one obtains the usual results for the densities of states and pairs.

The effective energy gap may be defined as the frequency ω_g at which the density of states first begins to have finite value, that is,

$$n(\omega<\omega_g)=0\,. \quad (3.20)$$

It should be noted that ω_g is not the same as the gap parameter Δ except for the weak-coupling isotropic system with no magnetic impurities. When magnetic impurities are present, the density of states is proportional to the imaginary part of $\bar{\omega}$ from Eq. (3.10):

$$\mathrm{Im}\bar{\omega}=2\Gamma_s\,\mathrm{Re}\{\bar{\omega}/[\bar{\omega}^2-\Delta^2]^{1/2}\}=2\Gamma_s n(\omega)\,.$$

In other words the density of states first begins to have finite value when $\bar{\omega}$ becomes complex. Thus the effective energy gap is obtained by setting $\partial\omega/\partial\bar{\omega}=0$ in Eq. (3.10), to give

$$\omega_g=\{\Delta^{2/3}-(2\Gamma_s)^{2/3}\}^{3/2}\theta(\Delta-2\Gamma_s)\,, \quad (3.21)$$

where $\theta(x)$ is the usual step function, 1 for positive x, and zero otherwise. The region of gapless superconductivity[32,33] is that for $0<\Delta(\Gamma_s,T)<2\Gamma_s$, such that $\omega_g=0$.

[29] S. B. Nam, Phys. Rev. (to be published).

[30] E. T. Whittakar and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, London, 1952), Chap. 12, p. 247.

[31] Especially in the normal state $n(k)=\mathrm{Re}\{1/Z_3{}^N(k)\}$.

[32] Similar phenomena have been discussed by many authors: K. T. Rogers, Ph.D. thesis, University of Illinois, 1960 (unpublished); J. Bardeen, Rev. Mod. Phys. **34**, 667 (1962); K. Maki, Progr. Theoret. Phys. (Kyoto) **29**, 603 (1963); **31**, 731 (1964).

[33] A. I. Larkin, Zh. Eksperim. i Teor. Fiz. **48**, 232 (1965) [English transl.: Soviet Phys.—JETP **21**, 153 (1965)].

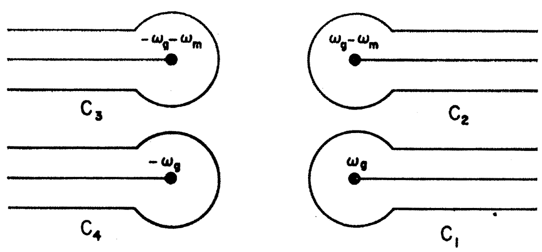

FIG. 2. The contours for the ω integration.

Superconductivity requires that Δ be different from zero, but not ω_g.

In closing this section it is noted again that hereafter we use a Green's function of the form (3.15) for all cases (pure, impure, strong-coupling, and weak-coupling systems), with the appropriate $Z(k)$, $\Delta(k)$, and $Z_3(k)$.

4. CURRENT DENSITY

In this section starting from Eq. (2.7) we calculate explicitly the response function in the transverse gauge utilizing the Green's function of the form Eq. (3.15). The current density, which can be applied to strong-coupling and impure superconductors, in a form similar to that of Mattis and Bardeen, is obtained in the real space.

Using Eqs. (1.4), (1.5), (2.7), and (3.19), we write the transverse response function

$$K_{\mu\nu}(q,\omega_m)=\frac{e^2}{\pi^2}\int dA\int\frac{d\epsilon_k}{|v|}v_\mu v_\nu\int_c\frac{d\omega}{2\pi i}f(\omega)\operatorname{Tr}\{\ \}, \tag{4.1}$$

$$\{\ \}=\{G(\mathbf{k}+\mathbf{q},\,\omega+\omega_m)G(\mathbf{k},\omega)-G_N(\mathbf{k}+\mathbf{q},\,\omega)G_N(\mathbf{k},\omega)\},$$

where G_N is the normal-state Green's function. The subtraction term corresponds to the diamagnetic term. We have here used the analytic continuation identity

$$\sum_n\frac{1}{\beta}F(z_n)\leftrightarrow\int_c\frac{dz}{2\pi i}F(z)f(z),$$

where $f(z)$ is the fermi (boson) function for an odd (even) integer n from Eq. (2.6). $F(z)$ is an arbitrary function except for possible poles, assuming that the poles of $F(z)$ do not coincide with those of $f(z)$. On the other hand, if $zf(z)F(z)\to 0$ as $z\to\infty$, we can distort the contour in the convenient form depending on the poles of $F(z)$.

We now need to calculate the following type of integral:

$$I=\int\frac{d\epsilon_k}{2\pi i}\int_c d\omega\, f(\omega)\operatorname{Tr}\{G(\mathbf{k}+\mathbf{q},\,\omega+\omega_m)G(\mathbf{k},\omega)\}. \tag{4.2}$$

We suppose, even though we do not write it down explicitly, that there is always a subtraction in such a way that the formally divergent term is eliminated as discussed in Sec. 1. We then proceed with all calculations by interchanging freely the order of integrations with respect to d^3k and $d\omega$.

For the ω integration, we consider the contours shown in Fig. 2. We can obtain the contribution from each contour, and write

$$I=I_1+I_2+I_3+I_4, \tag{4.3}$$

where I_i is the contribution from the contour C_i, and

$$I_1=\int\frac{d\epsilon_k}{2\pi i}\int_{\omega_g}^{\infty}d\omega\, f(\omega)$$
$$\times\operatorname{Tr}\{[G_+(\mathbf{k},\omega)-G_-(\mathbf{k},\omega)]G_-(\mathbf{k}+\mathbf{q},\,\omega+\omega_m)\} \tag{4.4}$$

and similar terms for I_2, I_3, and I_4. We have here used the notation

$$G_\pm(\mathbf{k},\omega)\equiv G(\mathbf{k},\,\omega\pm i0). \tag{4.5}$$

Using the Green's function in the form of Eq. (3.15), we obtain

$$\operatorname{Tr}\{G_\alpha(k_1)G_\beta(k_2)\}=\tfrac{1}{2}[g_{\alpha\beta}(1,2)+1]$$
$$\times\{I\}+\tfrac{1}{2}[g_{\alpha\beta}(1,2)-1]\{II\}, \tag{4.6}$$

$$\{I\}=\frac{1}{Q-\epsilon_1{}^\alpha+\epsilon_2{}^\beta}\left(\frac{1}{\epsilon_1{}^\alpha+\epsilon_{k1}}-\frac{1}{\epsilon_2{}^\beta+\epsilon_{k2}}\right)$$
$$-\frac{1}{Q+\epsilon_1{}^\alpha-\epsilon_2{}^\beta}\left(\frac{1}{\epsilon_1{}^\alpha-\epsilon_{k1}}-\frac{1}{\epsilon_2{}^\beta-\epsilon_{k2}}\right),$$

$$\{II\}=\frac{-1}{Q-\epsilon_1{}^\alpha-\epsilon_2{}^\beta}\left(\frac{1}{\epsilon_1{}^\alpha+\epsilon_{k1}}+\frac{1}{\epsilon_2{}^\beta-\epsilon_{k2}}\right)$$
$$+\frac{1}{Q+\epsilon_1{}^\alpha+\epsilon_2{}^\beta}\left(\frac{1}{\epsilon_1{}^\alpha-\epsilon_{k1}}+\frac{1}{\epsilon_2{}^\beta+\epsilon_{k2}}\right),$$

where the coherence factor $g_{\alpha\beta}(1,2)$ is given by

$$g_{\alpha\beta}(1,2)=N^\alpha(1)N^\beta(2)+P^\alpha(1)P^\beta(2), \tag{4.7}$$

and

$$Q=\epsilon_{k2}-\epsilon_{k1}\approx(\mathbf{k}_2-\mathbf{k}_2)\cdot\nabla_k\epsilon_k=\mathbf{q}\cdot v.$$

The subscripts 1 and 2 stand for the variables $k_1=(\mathbf{k}_1,\omega_1)$ and $k_2=(\mathbf{k}_2,\omega_2)$, respectively, and the superscripts α and β denote $(+)$ or $(-)$ which indicates the upper, or lower, across the cut on the real axis of the frequency. We have used the renormalized quasiparticle energy $\bar{\epsilon}_k$ as ϵ_k. We can put $\bar{\epsilon}_k$ in place of ϵ_k in Eq. (4.6), so that we obtain

$$\operatorname{Tr}\{G_\alpha(k_1)G_\beta(k_2)\}=\frac{1}{Z_3{}^\alpha(k_1)Z_3{}^\beta(k_2)}\quad\{\text{same form}\} \tag{4.8}$$

replacing $\epsilon(k)$ by $\epsilon(k)/Z_3(k)$. Equation (4.6) can be also rewritten in terms of $\epsilon_0(k)$ and $\Gamma(k)$ introduced as

$$\epsilon(k)=Z(k)[k_0{}^2-\Delta^2(k)]^{1/2}\equiv\epsilon_0(k)+i\Gamma(k). \tag{4.9}$$

Inserting this into Eq. (4.6), we obtain a typical term in Eq. (4.6) as

$$\frac{1}{Q+\epsilon_1{}^\alpha-\epsilon_2{}^\beta}\frac{1}{\epsilon_1{}^\alpha-\epsilon_{k1}}=\frac{1}{Q+\epsilon_{01}-\epsilon_{02}+i[\alpha\Gamma_1-\beta\Gamma_2]}\frac{1}{\epsilon_{01}-\epsilon_{k1}+i\alpha\Gamma_1}, \tag{4.10}$$

where $\Gamma_{1,2}=\Gamma(k_{1,2})$, $\epsilon_{01,2}=\epsilon_0(k_{1,2})$, and so forth.

Using Eq. (4.6), we can carry out the integration with respect to ϵ_k in Eq. (4.2). For this it should be mentioned that the causal requirement eliminates some terms in Eq. (4.6). Equation (4.10) is helpful for this. We choose the same causal sign as that of Mattis and Bardeen. After a little calculation, we obtain the response function

$$K_{\mu\nu}(\mathbf{q},\omega)=\frac{e^2}{\pi^2}\int\frac{dA}{|v|}v_\mu v_\nu I(Q,\omega,T), \tag{4.11}$$

where the kernel $I(Q,\omega,T)$ is given by

$$I(Q,\omega,T)=\int_{\omega_g-\omega}^{\omega_g}d\omega'\ \{I\}\tanh[\tfrac{1}{2}\beta(\omega+\omega')]+\int_{\omega_g}^{\infty}d\omega'\ [\{I\}\tanh[\tfrac{1}{2}\beta(\omega+\omega')]-\{II\}\tanh(\tfrac{1}{2}\beta\omega')], \tag{4.12}$$

$$\{I\}=\frac{g_{+-}(1,2)+1}{Q+\epsilon_{02}-\epsilon_{01}-i(\Gamma_1+\Gamma_2)}+\frac{g_{++}(1,2)-1}{Q-\epsilon_{02}-\epsilon_{01}-i(\Gamma_1+\Gamma_2)},$$

$$\{II\}=\frac{g_{+-}(1,2)+1}{Q+\epsilon_{02}-\epsilon_{01}-i(\Gamma_1+\Gamma_2)}+\frac{g_{--}(1,2)-1}{Q+\epsilon_{01}+\epsilon_{02}-i(\Gamma_1+\Gamma_2)}.$$

Here the subscripts 1 and 2 stand for $k_1=(\Omega,\omega')$ and $k_2=(\Omega_2,\omega+\omega')$, respectively. We have here used the analytic continuation identity, $f(\omega'\pm\omega_m)=f(\omega')$, with $\omega_m=2\pi mi/\beta$.

The conductivity then becomes

$$\sigma_{\mu\nu}(\mathbf{q},\omega)=\frac{1}{4\pi i}\frac{1}{\omega}K_{\mu\nu}(\mathbf{q},\omega). \tag{4.13}$$

The normal-state response function can be obtained from Eq. (4.11) by setting the gap parameter Δ to be zero;

$$K_{\mu\nu}{}^N(\mathbf{q},\omega)=\frac{e^2}{\pi^2}\int\frac{dA}{|v|}v_\mu v_\nu I^N(Q,\omega,T),$$

$$I(Q,\omega,T)=\left\{\int_{-\omega}^{0}\tanh[\tfrac{1}{2}\beta(\omega+\omega')]+\int_0^\infty[\tanh[\tfrac{1}{2}\beta(\omega+\omega')]-\tanh(\tfrac{1}{2}\beta\omega')]\right\}\frac{2d\omega'}{Q+\epsilon_{02}{}^N-\epsilon_{01}{}^N-i(\Gamma_1{}^N+\Gamma_2{}^N)}, \tag{4.14}$$

where $\epsilon_0{}^N$ and Γ^N are defined for the normal state in a way similar to Eq. (4.9), that is,

$$\epsilon^N(k)=Z^N(k)k_0\equiv\epsilon_0{}^N(k)+i\Gamma^N(k). \tag{4.15}$$

The normal-state conductivity becomes

$$\sigma_{\mu\nu}{}^N(\mathbf{q},\omega)=\frac{1}{4\pi i\omega}K_{\mu\nu}{}^N(\mathbf{q},\omega). \tag{4.16}$$

For the isotropic systems after an angular integration, we obtain the current density

$$J_\mu(q,\omega)=-\frac{1}{4\pi}K_{\mu\nu}(q,\omega)A^\nu(q,\omega)=-\frac{1}{4\pi}K(q,\omega)A_\mu(q,\omega), \tag{4.17}$$

where the response function $K(q,\omega)$ is

$$K(q,\omega)=\frac{3\pi}{2\Lambda}\left\{\int_{\omega_g-\omega}^{\omega_g}d\omega'\ \{I\}\tanh[\tfrac{1}{2}\beta(\omega+\omega')]+\int_{\omega_g}^{\infty}\times[\{I\}\tanh[\tfrac{1}{2}\beta(\omega+\omega')]-\{II\}\tanh(\tfrac{1}{2}\beta\omega')]\right\},$$

$$\{I\}=F(q,\,\epsilon_{02}-\epsilon_{01},\,\Gamma_1+\Gamma_2)[g_{+-}(1,2)+1]+F(q,\,-\epsilon_{02}-\epsilon_{01},\,\Gamma_1+\Gamma_2)[g_{++}(1,2)-1],$$

$$\{II\}=F(q,\,\epsilon_{02}-\epsilon_{01},\,\Gamma_1+\Gamma_2)[g_{+-}(1,2)+1]+F(q,\,\epsilon_{02}+\epsilon_{01},\,\Gamma_1+\Gamma_2)[g_{--}(1,2)-1]. \tag{4.18}$$

The subscripts 1 and 2 here stand for (ω') and $(\omega+\omega')$. The function F is given by

$$F(q,E,\Gamma)=\frac{1}{qv_0}\left[2S+(1-S^2)\ln\frac{S+1}{S-1}\right], \tag{4.19}$$

where

$$S=(1/qv_0)[E-i\Gamma].$$

The conductivity then becomes

$$\sigma(q,\omega)=K(q,\omega)/4\pi i\omega. \tag{4.20}$$

The corresponding expressions for the normal state are

$$K^N(q,\omega)=\frac{3\pi}{\Lambda}\left\{\int_{-\omega}^{0}\tanh[\tfrac{1}{2}\beta(\omega+\omega')]+\int_0^\infty[\tanh[\tfrac{1}{2}\beta(\omega+\omega')]-\tanh(\tfrac{1}{2}\beta\omega')]\right\}d\omega'\times F(q,\,\epsilon_{02}{}^N-\epsilon_{01}{}^N,\,\Gamma_1{}^N+\Gamma_2{}^N), \tag{4.21}$$

and

$$\sigma^N(q,\omega)=K^N(q,\omega)/4\pi i\omega. \tag{4.22}$$

In the weak-coupling limit, $\epsilon_{02}{}^N-\epsilon_{01}{}^N=\omega$, and the normal-state response function and conductivity become

$$K^N(q,\omega)=(3\pi\omega/\Lambda)F(q,\omega,2\Gamma) \tag{4.23}$$

and

$$\sigma^N(q,\omega)=(3/4i\Lambda)F(q,\omega,2\Gamma), \tag{4.24}$$

where 2Γ is the inverse relaxation time. Utilizing Eq. (4.24), we can formally rewrite the response function Eq. (4.18) in terms of the normal-state conductivity, Eq. (4.24), of the weak-coupling case. This implies that for the isotropic system, the response function and conductivity in the superconducting state can formally be expressed in terms of the normal-state conductivity, taking properly into account the coherence factor $g_{\alpha\beta}(1,2)$.

We now discuss the current density in real space. Equations (4.11) and (4.12) lead to the current density in a form similar to that of Mattis and Bardeen:

$$\mathbf{J}(\mathbf{r},t)=C\sum_{\omega} e^{i\omega t}\int d\mathbf{r}'\frac{\mathbf{R}[\mathbf{R}\cdot\mathbf{A}_\omega(\mathbf{r}')]}{R^4}I(\omega,R,T), \tag{1.6}$$

where the kernel function $I(\omega,R,T)$ is given by

$$I(\omega,R,T)=\frac{\pi}{2i}\bigg\{\int_{\omega_g-\omega}^{\omega_g} e^{-RL}d\omega'\{I\}\tanh\tfrac{1}{2}\beta(\omega+\omega')$$
$$+\int_{\omega_g}^{\infty} e^{-R/L}d\omega'\,[\{I\}\tanh[\tfrac{1}{2}\beta(\omega+\omega')]$$
$$-\{II\}\tanh(\tfrac{1}{2}\beta\omega')\,]\bigg\}, \tag{4.25}$$

$$\{I\}=[g_{+-}(1,2)+1]\exp[i\alpha(\epsilon_{01}-\epsilon_{02})]+[g_{++}(1,2)-1]\exp[i\alpha(\epsilon_{01}+\epsilon_{02})],$$

$$\{II\}=[g_{+-}(1,2)+1]\exp[i\alpha(\epsilon_{01}-\epsilon_{02})]+[g_{--}(1,2)-1]\exp[-i\alpha(\epsilon_{01}+\epsilon_{02})].$$

Here the effective mean free path L is defined as

$$L=|v|/[\Gamma_1+\Gamma_2], \tag{4.26}$$

and

$$\alpha=R/|v|.$$

It is clear that for the weak-coupling isotropic system with no magnetic impurities, the current density of Eq. (1.6) becomes identical to that of Mattis and Bardeen with $\Delta(\omega)=\Delta$, $\omega_g=\Delta$,

$$g_{\alpha\beta}(1,2)=\frac{\omega'(\omega+\omega')+\Delta^2}{[\omega'^2-\Delta^2]^{1/2}[(\omega+\omega')^2-\Delta^2]^{1/2}}, \tag{4.27}$$

and $\Gamma(\omega)=\Gamma=v_0/2l$. The factor $\exp[-R/l]$ comes in here in a natural way as expected. For the weak-coupling system with magnetic impurities, the effective mean free path becomes

$$\frac{1}{L}=\frac{1}{l}+\frac{1}{l_s}, \tag{4.28a}$$

where l and l_s are the mean free paths resulting from non-magnetic and magnetic impurity scatterings, respectively. It is noted that in this case the effective gap parameter of Eq. (3.6c) is still dependent on Γ_s. When $\Gamma_s=0$, that is, no magnetic impurity present, the above result becomes identical to that of Mattis and Bardeen.

An expression for the normal-state current density in real space can be obtained by setting the gap parameter Δ to be zero from Eq. (1.6);

$$\mathbf{J}^N(\mathbf{r},t)=C\sum_{\omega} e^{i\omega t}\int d\mathbf{r}'\frac{\mathbf{R}[\mathbf{R}\cdot\mathbf{A}_\omega(\mathbf{r}')]}{R^4}I^N(\omega,R,T), \tag{4.29}$$

where the normal-state kernel is

$$I^N(\omega,R,T)=\frac{\pi}{i}\bigg\{\int_{-\omega}^{0}\tanh[\tfrac{1}{2}\beta(\omega+\omega')]$$
$$+\int_0^\infty [\tanh[\tfrac{1}{2}\beta(\omega+\omega')]-\tanh(\tfrac{1}{2}\beta\omega')\,]\bigg\}$$
$$\times d\omega'\,e^{-R/L-i\alpha(\epsilon_{02}^N-\epsilon_{01}^N)}, \tag{4.30}$$

where the effective mean free path in the normal state L is defined as

$$L=|v|/[\Gamma_1^N+\Gamma_2^N]. \tag{4.31}$$

For the weak-coupling isotropic system with non-magnetic impurities, $\epsilon_{02}^N-\epsilon_{01}^N=\omega$, and the normal-state current density of Eq. (4.29) becomes a Chambers expression with

$$I^N(\omega,R,T)=-\pi i\omega e^{-i\alpha\omega-R/l}. \tag{4.32}$$

In the weak-coupling system with magnetic impurities, the kernel $I^N(\omega,R,T)$ becomes

$$I^N(\omega,R,T)=-\pi i\omega e^{-i\alpha\omega-R/L}, \tag{4.33}$$

and

$$\frac{1}{L}=\frac{1}{l}+\frac{1}{l_s}. \tag{4.28b}$$

Here l and l_s are the mean free paths resulting from non-magnetic and magnetic impurity scatterings, respectively, in the normal state. They are in general different from those of Eq. (4.28a) in the superconducting state, and will be the same when the difference in the self-energies in two states is neglected.

We consider now a system in which the penetration depth is small compared with the coherence length (Pippard limit, i.e., $\lambda\ll\xi_0$) so that to a sufficient approximation, we can replace the kernel $I(\omega,R,T)$ by $I(\omega,0,T)$ as done by Mattis and Bardeen. This replacement can be done also when the mean free path is small compared with the coherence length $(L\ll\xi_0)$.[34] We then expect that the ratio of conductivities in superconduc-

[34] J. Bardeen (private communication) has pointed out to us that the limit $(L\ll\xi_0)$ is equivalent to the local limit.

ing and normal states is equal to the corresponding ratio of the kernel $I(\omega,0,T)$. From Eq. (4.25), we obtain

$$I(\omega,0,T)=-\pi i\omega[\sigma_1-i\sigma_2], \tag{4.34}$$

where

$$\sigma_1=\frac{1}{\omega}\int_{\omega_g-\omega}^{-\omega_g} d\omega'\, g_1(1,2)\tanh[\tfrac{1}{2}\beta(\omega+\omega')]+\frac{1}{\omega}\int_{\omega_g}^{\infty} \times d\omega'\, g_1(1,2)[\tanh[\tfrac{1}{2}\beta(\omega+\omega')]-\tanh(\tfrac{1}{2}\beta\omega')], \tag{4.35a}$$

$$\sigma_2=\frac{1}{\omega}\int_{[\omega_g-\omega,-\omega_g]}^{\omega_g} d\omega'\, g_2(1,2)\tanh[\tfrac{1}{2}\beta(\omega+\omega')] +\frac{1}{\omega}\int_{\omega_g}^{\infty} d\omega'\,[g_2(1,2)\tanh[\tfrac{1}{2}\beta(\omega+\omega')] +g_2(2,1)\tanh(\tfrac{1}{2}\beta\omega')]. \tag{4.35b}$$

Here $[\omega_g-\omega, -\omega_g]$ denotes that the algebraically largest of two numbers is to be used, the functions g_1 and g_2 correspond to the coherence factors, and are given by

$$g_1(1,2)=\mathrm{Re}\{N(1)\}\,\mathrm{Re}\{N(2)\} +\mathrm{Re}\{P(1)\}\,\mathrm{Re}\{P(2)\}, \tag{4.36a}$$

$$g_2(1,2)=\mathrm{Im}\{N(1)\}\,\mathrm{Re}\{N(2)\} +\mathrm{Im}\{P(1)\}\,\mathrm{Re}\{P(2)\}, \tag{4.36b}$$

where the functions $N(k)$ and $P(k)$ are defined by Eq. (3.16), and the real parts of them correspond to the densities of states and pairs defined by Eq. (3.18). From Eq. (4.30), we obtain the normal-state kernel

$$I^N(\omega,0,T)=-\pi i\omega. \tag{4.37}$$

For an isotropic system the angular dependence drops out in Eq. (4.35), and we obtain

$$\frac{\sigma^s}{\sigma^N}=\frac{I(\omega,0,T)}{-\pi i\omega}=\sigma_1-i\sigma_2. \tag{4.38}$$

We see that in the weak-coupling limit with no magnetic impurity, the result of Eq. (4.38) becomes that of Mattis and Bardeen. We shall see later that the result of Eq. (4.38) can be obtained from the general expressions for the conductivity of Eq. (4.20) in the limits of $q\rightarrow\infty$ and of q, $L\rightarrow 0$.

We can also obtain the same formal results by using Eq. (4.8), that is, including $Z_3(k)$. Then all results are formally the same with the replacements

$$\epsilon(k)\rightarrow\epsilon(k)/Z_3(k), \tag{4.39a}$$

$$\{g_{\alpha\beta}(1,2)\pm1\}\rightarrow\{g_{\alpha\beta}(1,2)\pm1\}/Z_3{}^\alpha(1)Z_3{}^\beta(2). \tag{4.39b}$$

5. LIMITING VALUES OF THE RESPONSE FUNCTION

In this section we discuss various limiting values of the response functions (4.18) and (4.21) for an isotropic system, and show explicitly the conditions for the Meissner effect and perfect conductivity. The corresponding results for an anisotropic system can be obtained from Eqs. (4.11) and (4.14), and will be given at the end of this section.

For this purpose various limiting values of the function F of Eq. (4.19) are needed:

$$F(0,E,\Gamma)=\frac{4}{3}\frac{1}{E-i\Gamma}, \tag{5.1a}$$

$$F(q,0,\Gamma)=\frac{2i}{\Gamma}F_0\!\left(\frac{qv_0}{\Gamma}\right), \tag{5.1b}$$

$$F(q,E,0)=\frac{1}{qv_0}\left[F_1\!\left(\frac{E}{qv_0}\right)+iF_2\!\left(\frac{E}{qv_0}\right)\right], \tag{5.1c}$$

$$F(0,0,\Gamma)=\frac{4i}{3}\frac{1}{\Gamma}, \tag{5.1d}$$

$$F(q\rightarrow\infty, E, \Gamma)\approx\frac{\pi i}{qv_0}, \tag{5.1e}$$

where

$$F_0(x)=(1/x^3)[(1+x^2)\tan^{-1}x-x], \tag{5.2}$$

$$F_1(x)=2x+(1-x^2)\ln\left|\frac{1+x}{1-x}\right|, \tag{5.3}$$

$$F_2(x)=\pi(1-x^2)\theta(1-x). \tag{5.4}$$

A. Normal State

Inserting Eq. (5.1) into Eqs. (4.21) and (4.22), we obtain the following limiting values of conductivity in the normal state:

$$\sigma^N(0,\omega)=\frac{1}{\Lambda}\frac{1}{\omega}\left\{\int_{-\omega}^{0}\tanh[\tfrac{1}{2}\beta(\omega+\omega')] +\int_0^{\infty}[\tanh[\tfrac{1}{2}\beta(\omega+\omega')]-\tanh(\tfrac{1}{2}\beta\omega')]\right\} \times\frac{d\omega'}{\Gamma^N(\omega')+\Gamma^N(\omega+\omega')+i\tilde{\omega}}, \tag{5.5a}$$

$$\sigma^N(q,0)=\frac{3}{2}\frac{1}{\Lambda}\int_0^{\infty}\frac{\mathrm{sech}^2(\tfrac{1}{2}\beta\omega)}{2\Gamma^N(\omega)} \times\frac{\beta}{2}d\omega F_0\!\left(\frac{qv_0}{2\Gamma^N(\omega)}\right), \tag{5.5b}$$

$$\sigma^N(q,\omega)_{\Gamma\rightarrow0}=\frac{3}{4qv_0}\frac{1}{\Lambda\omega}\left\{\int_{-\omega}^{0}\tanh[\tfrac{1}{2}\beta(\omega+\omega')] +\int_0^{\infty}[\tanh[\tfrac{1}{2}\beta(\omega+\omega')]-\tanh(\tfrac{1}{2}\beta\omega')]\right\} \times d\omega'\left[F_2\!\left(\frac{\tilde{\omega}}{qv_0}\right)-iF_1\!\left(\frac{\tilde{\omega}}{qv_0}\right)\right], \tag{5.5c}$$

$$\sigma^N(0,0)=\frac{1}{\Lambda}\int_0^\infty \frac{\operatorname{sech}^2(\frac{1}{2}\beta\omega)}{2\Gamma^N(\omega)}\frac{\beta}{2}d\omega, \tag{5.5d}$$

$$\sigma^N(q\to\infty,\omega)\approx\frac{3\pi}{4qv_0}\frac{1}{\Lambda}, \tag{5.5e}$$

where

$$\tilde{\omega}=\epsilon_{02}{}^N-\epsilon_{01}{}^N=\epsilon_0{}^N(\omega+\omega')-\epsilon_0{}^N(\omega').$$

The formula (5.5d) is often used for the dc resistivity for $q=0$, when the self-energy resulting from various scatterings depends on frequency, such as a resonance scattering from the s-d interaction.[35]

In the weak-coupling system with nonmagnetic and magnetic impurities, Eqs. (5.5) reduce to

$$\sigma^N(0,\omega)=\frac{\sigma^N(0,0)}{1+i\omega\tau}, \tag{5.6a}$$

$$\sigma^N(q,0)=\tfrac{2}{3}\sigma^N(0,0)F_0(qL), \tag{5.6b}$$

$$\sigma^N(q,\omega)_{L\to\infty}=\frac{3}{4qv_0}\frac{1}{\Lambda}\left\{F_2\left(\frac{\omega}{qv_0}\right)-iF_1\left(\frac{\omega}{qv_0}\right)\right\}, \tag{5.6c}$$

$$\sigma^N(0,0)=\frac{1}{\Lambda}\frac{1}{2\Gamma}=\frac{ne^2}{m}\tau, \tag{5.6d}$$

$$\sigma^N(q\to\infty,\omega)\approx\frac{3\pi}{4qv_0}\frac{1}{\Lambda}, \tag{5.6e}$$

where

$$\frac{1}{\tau}=\frac{1}{\tau_0}+\frac{1}{\tau_s}=\frac{v_0}{L}.$$

Here τ_0 and τ_s are the relaxation times resulting from nonmagnetic (ordinary) and magnetic (spin-flip) impurity scatterings, respectively, in the normal state. The results of Eqs. (5.6) become the well-known results for the normal-state conductivity[36] when $\tau_s=\infty$, that is, no magnetic impurity present. It is noted that the above results are valid even for a system with only spin-flip scattering present. When there is no scattering, $L\to\infty$, we see from Eq. (5.6c) that the real part of the conductivity, which is related to the power absorption, vanishes for $\omega>qv_0$ as one expects (often called the Cerenkov condition). More generally one can see the above condition from Eq. (5.5c).

B. Superconducting State

We first give only a few limiting forms for the response function in the Pippard and London limits.

For the Pippard limit, $q\to\infty$, we insert Eq. (5.1e) into Eq. (4.18), and we obtain

$$K(q\to\infty,\omega)\approx-\frac{3\pi}{qv_0\Lambda}I(\omega,0,T). \tag{5.7}$$

We then obtain the conductivity,

$$\frac{\sigma^s}{\sigma^N}(q\to\infty,\omega)\approx\frac{I(\omega,0,T)}{-\pi i\omega}=\sigma_1-i\sigma_2 \tag{5.8}$$

as given by Eq. (4.38), with σ^N of Eq. (5.5e).

For the London limit, $q\to 0$, inserting Eq. (5.1a) into Eq. (4.18), we obtain

$$K(0,\omega)=\frac{2\pi}{\Lambda}\int_{\omega_t-\omega}^{\omega_g}d\omega'\,\{I\}\tanh[\tfrac{1}{2}\beta(\omega+\omega')]$$
$$+\int_{\omega_g}^{\infty}[\{I\}\tanh[\tfrac{1}{2}\beta(\omega+\omega')]-\{II\}\tanh(\tfrac{1}{2}\beta\omega')]d\omega', \tag{5.9}$$

$$\{I\}=\frac{g_{+-}(1,2)+1}{\epsilon_{02}-\epsilon_{01}-i(\Gamma_1+\Gamma_2)}+\frac{g_{++}(1,2)-1}{-\epsilon_{02}-\epsilon_{01}-i(\Gamma_1+\Gamma_2)},$$

$$\{II\}=\frac{g_{+-}(1,2)+1}{\epsilon_{02}-\epsilon_{02}-i(\Gamma_1+\Gamma_2)}+\frac{g_{--}(1,2)-1}{\epsilon_{02}+\epsilon_{01}-i(\Gamma_1+\Gamma_2)}.$$

If we make a further approximation, $\Gamma_{1,2}=\Gamma(\omega')$, $\Gamma(\omega'+\omega)\to\infty$, which corresponds to $L\ll\xi_0$, and assume that $\Gamma(\omega)$ is slowly varying with respect to frequency ω, Eq. (5.9) becomes

$$K(0,\omega)_{\Gamma\to\infty}\approx-4\sigma_0 I(\omega,0,T), \tag{5.10}$$

where $\sigma_0\approx(1/\Lambda)(1/2\Gamma_{\text{eff}})\approx\sigma^N(0,0)$. The conductivity can then be expressed by

$$\frac{\sigma^s}{\sigma^N}(0,\omega)_{\Gamma\to\infty}\approx\frac{I(\omega,0,T)}{-\pi i\omega}=\sigma_1-i\sigma_2 \tag{5.11}$$

as given by Eq. (4.38), with σ^N of Eq. (5.6d). Thus, from Eqs. (5.8) and (5.11), we obtain

$$\frac{\sigma^s}{\sigma^N}(q\to\infty,\omega)\approx\frac{I(\omega,0,T)}{-\pi i\omega}\approx\frac{\sigma^s}{\sigma^N}(0,\omega)_{\Gamma\to\infty}. \tag{5.12}$$

These results have also been obtained by the author in a quite different way.[37]

The results of Eq. (5.12) can be understood in the following way: In the Pippard limit, when the penetra-

[35] See for example, A. A. Abrikosov, Physics **2**, 5 (1965); 2, 61 (1965); H. Suhl, Phys. Rev. **138**, A515 (1965); **141**, 483 (1966); Y. Nagaoka, *ibid.* **138**, A1112 (1965); P. W. Anderson, *ibid.* **124**, 41 (1961); J. Kondo, Progr. Theoret. Phys. (Kyoto) **32**, 37 (1964); J. R. Schrieffer and D. C. Mattis, Phys. Rev. **140**, A1412 (1965).

[36] G. E. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1948).

[37] One can carry out the calculations using the spectral representation form for the Green's function from beginning [S. B. Nam, Ph.D. thesis, University of Illinois, 1966 (unpublished)].

tion depth is small compared with the coherence length, one may replace $I(\omega,R,T)$ by $I(\omega,0,T)$ as discussed earlier. In the local limit, $q\to 0$, $\Gamma\to\infty$, in other words, $L\to 0$ in the current density of Eq. (1.6), the factor $\exp[-R/L]$ causes the main contribution to come from $R\approx 0$, and again one may replace $I(\omega,R,T)$ by $I(\omega,0,T)$. These two simple arguments lead to Eq. (5.12) as one expects from Eq. (4.34).

It should be mentioned that in the weak-coupling system with magnetic impurities, σ^N in Eq. (5.11) is given by

$$\sigma_0\approx\sigma^N(0,0)=\frac{ne^2}{m}\frac{\tau_0\tau_s}{\tau_0+\tau_s}. \tag{5.13}$$

In this limit, the result of Eq. (5.9) leads to that of Weiss *et al.* To see this from Eqs. (3.6) and (4.9), we rewrite

$$\begin{aligned}\epsilon(\omega)&=Z(\omega)[\omega^2-\Delta^2(\omega)]^{1/2}\\&=[\omega^2-\Delta^2(\omega)]^{1/2}+i(\Gamma+\Gamma_s)\\&=[\bar{\omega}^2-\Delta^2]^{1/2}+i(\Gamma-\Gamma_s).\end{aligned} \tag{5.14}$$

If we use the notations v and Γ_2 of Weiss *et al.* in place of $\bar{\omega}$ and $\Gamma-\Gamma_s$, respectively, then we obtain from Eq. (5.9) their results in the weak-coupling limit. Our results of Eq. (5.11) differ from theirs in that they implicitly assume that $\tau_s\gg\tau_0$. Their result σ^N for the conductivity ratio at zero temperature, with $\omega/\Gamma_2\to 0$, becomes

$$\sigma_0\approx\sigma^N\approx\frac{ne^2}{m}\frac{1}{2\Gamma_2}=\frac{ne^2}{m}\frac{\tau_s\tau_0}{\tau_s-\tau_0}$$

instead of Eq. (5.13). This result is only valid for $\tau_s\gg\tau_0$. While this may be the usual situation, it is desirable to have the more general result Eq. (5.13), which is valid even for the system with only spin-flip scattering present.

We now calculate the limiting value $K(0,0)$ in two distinct limits: $q\to 0$, $\omega\to 0$ and $\omega\to 0$, $q\to 0$. To do this we observe that the function F of Eq. (4.19) is proportional to the normal-state conductivity in the weak-coupling limit from Eq. (4.24). We thus expect that the function F satisfies the usual sum rule

$$\begin{aligned}\mathrm{Im}F(q,0,\Gamma)&=\frac{2}{\pi}\int_0^\infty\frac{dE}{E}\,\mathrm{Re}F(q,E,\Gamma)\\&=\frac{2}{\pi}\,\mathrm{Re}\int_0^\infty\frac{dE}{E}F(q,E,\Gamma).\end{aligned} \tag{5.15}$$

We first consider the limit $q\to 0$, $\omega\to 0$. For this we take a limit $\omega\to 0$ in Eq. (5.9), using Eq. (5.15), and obtain

$$\begin{aligned}\lim_{\omega\to 0}K(0,\omega)&=\frac{6\pi}{\Lambda}\left\{\frac{\pi}{2}\Delta(\omega_g)\tanh[\tfrac{1}{2}\beta\Delta(\omega_g)]\mathrm{I\ m}F[0,0,2\Gamma(\omega_g)]-\mathrm{Re}\int_{\omega_g+0}^\infty d\omega\frac{\Delta^2(\omega)}{\omega^2-\Delta^2(\omega)}F[0,2\epsilon_0(\omega),2\Gamma(\omega)]\tanh(\tfrac{1}{2}\beta\omega)\right\}\\&=\frac{4\pi}{\Lambda}\,\mathrm{Re}\int_0^\infty\frac{d\epsilon_0}{\epsilon_0}\left\{\Delta(\omega_g)\tanh[\tfrac{1}{2}\beta\Delta(\omega_g)]-\frac{\Delta(\omega)}{D(\omega)}\tanh(\tfrac{1}{2}\beta\omega)\right\}\frac{1}{\epsilon(\omega)} &&(5.16\text{a})\\&=\frac{4\pi}{\Lambda}\,\mathrm{Re}\int_{\omega_g-0}^\infty\frac{d\omega\Delta^2(\omega)}{\Delta^2(\omega)-\omega^2}\frac{\tanh(\tfrac{1}{2}\beta)}{Z(\omega)[\omega^2-\Delta^2(\omega)]^{1/2}} &&(5.17\text{a})\\&=\frac{4\pi}{\Lambda}\,\mathrm{Re}\sum_n\frac{2\pi}{\beta}\frac{\Delta_n^2}{(\omega_n^2+\Delta_n^2)^{3/2}Z_n}. &&(1.11')\end{aligned}$$

We have here used

$$\frac{d\epsilon_0}{\epsilon_0}\frac{\Delta(\omega)}{D(\omega)}=\frac{\Delta^2(\omega)}{\omega^2-\Delta^2(\omega)}d\omega$$

for convenience, and have assumed that the gap parameter $\Delta(\omega)$ and the wave-function renormalization $Z(\omega)$ have no pole on the imaginary axis.

We consider now the reverse limits, that is, we let $\omega\to 0$ first, then we let $q\to 0$. Using Eq. (5.15), we obtain

$$\begin{aligned}K(q,0)&=\frac{6\pi}{\Lambda}\left\{\frac{\pi}{2}\Delta(\omega_g)\,\mathrm{Im}\{F[q,0,2\Gamma(\omega_g)]\}\tanh[\tfrac{1}{2}\beta\Delta(\omega_g)]-\mathrm{Re}\int_{\omega_g+0}^\infty d\omega\frac{\Delta^2(\omega)}{\omega^2-\Delta^2(\omega)}F[q,2\epsilon_0,2\Gamma(\omega)]\tanh(\tfrac{1}{2}\beta\omega)\right\}\\&=\frac{6\pi}{\Lambda}\,\mathrm{Re}\int_0^\infty\frac{d\epsilon_0}{\epsilon_0}\left\{\Delta(\omega_g)\tanh\tfrac{1}{2}\beta\Delta(\omega_g)-\frac{\Delta(\omega)}{D(\omega)}\tanh(\tfrac{1}{2}\beta\omega)\right\}F[q,2\epsilon_0,2\Gamma(\omega)] &&(5.18)\\&=\frac{6\pi}{\Lambda}\,\mathrm{Re}\int_{\omega_g-0}^\infty d\omega\frac{\Delta^2(\omega)}{\Delta^2(\omega)-\omega^2}F[q,2\epsilon_0,2\Gamma(\omega)]\tanh(\tfrac{1}{2}\beta\omega) &&(5.19)\\&=\frac{6\pi}{\Lambda}\,\mathrm{Re}\sum_n\frac{2\pi}{\beta}\frac{\Delta_n^2}{(\omega_n^2+\Delta_n^2)^{3/2}Z_n}F_0(S_n), &&(5.20)\end{aligned}$$

where the function $F_0(x)$ is given by Eq. (5.2), and

$$S_n=\frac{qv_0}{2\epsilon_n}=\frac{qv_0}{2(\omega_n{}^2+\Delta_n{}^2)^{1/2}Z_n}. \tag{5.21}$$

In the Pippard limit, $q\to\infty$, Eq. (5.20) becomes

$$K(q\to\infty,0)\approx\frac{6\pi^2}{\Lambda}\frac{1}{qv_0}\sum_n\frac{2\pi}{\beta}\frac{\Delta_n{}^2}{\omega_n{}^2+\Delta_n{}^2}. \tag{5.22}$$

This becomes the BCS result in the weak-coupling limit, with $\Delta_n=\Delta$;

$$K(q\to\infty,0)\approx\frac{3\pi^2}{q\xi_0\Lambda}\frac{\Delta(T)}{\Delta(0)}\tanh\frac{\Delta(T)}{2T}. \tag{5.23}$$

We now let $q\to 0$ in Eqs. (5.18), (5.19), and (5.20), and obtain

$$\lim_{q\to 0}K(q,0)=\frac{4\pi}{\Lambda}\operatorname{Re}\int_0^\infty\frac{d\epsilon_0}{\epsilon_0}\Delta(\omega_g)$$

$$\times\left[\tanh[\tfrac{1}{2}\beta\Delta(\omega_g)]-\frac{\Delta(\omega)}{D(\omega)}\tanh[\tfrac{1}{2}\beta\omega]\right]\frac{1}{\epsilon(\omega)} \tag{5.16b}$$

$$=\frac{4\pi}{\Lambda}\operatorname{Re}\int_{\omega_g-0}^\infty d\omega\frac{\Delta^2(\omega)}{\Delta^2(\omega)-\omega^2}\frac{\tanh(\tfrac{1}{2}\beta\omega)}{\epsilon(\omega)} \tag{5.17b}$$

$$=\frac{4\pi}{\Lambda}\operatorname{Re}\sum_n\frac{2\pi}{\beta}\frac{\Delta_n{}^2}{(\Delta_n{}^2+\omega_n{}^2)^{3/2}Z_n}. \tag{1.11''}$$

Thus we have explicitly shown that the response function satisfies the important condition

$$\lim_{q\to 0}\lim_{\omega\to 0}K(q,\omega)=K(0,0)=\lim_{\omega\to 0}\lim_{q\to 0}K(q,\omega) \tag{1.3'}$$

for the theory of superconductivity and superfluidity. This limiting value different from zero implies the Meissner effect and perfect conductivity, and it is directly related to the superfluid density ρ_s in a two-fluid model:

$$\frac{4\pi}{\Lambda}\frac{\rho_s}{\rho}=K(0,0). \tag{5.24}$$

One can obtain $K(0,0)$ directly from the starting point by knowing that two limiting values are equal,

$$K(0,0)=(K^{sp}+K^d)_{0,0}=(K^{sp}-K^{np})_{0,0}+(K^{np}+K^d)_{0,0}$$
$$=(K^{sp}-K^{np})_{0,0}. \tag{5.25a}$$

Using the Green's function in a form of Eq. (3.15), we obtain

$$K(0,0)=\frac{2}{\Lambda}\int_c d\omega\ f(\omega)\int d\epsilon_k\ \mathrm{Tr}\{G(k,\omega)G(k,\omega)$$
$$-G_N(k,\omega)G_N(k,\omega)\}$$

$$=\frac{4\pi}{\Lambda}\operatorname{Re}\int_{\omega_g-0}^\infty\frac{d\omega}{\epsilon(\omega)}\frac{\Delta^2(\omega)}{\Delta^2(\omega)-\omega^2}\tanh(\tfrac{1}{2}\beta\omega) \tag{5.17c}$$

$$=\frac{4\pi}{\Lambda}\operatorname{Re}\sum_n\frac{2\pi}{\beta}\frac{\Delta_n{}^2}{(\omega_n{}^2+\Delta_n{}^2)^{3/2}Z_n}. \tag{1.11'''}$$

In the pure weak-coupling limit (BCS), Eq. (5.18) becomes exactly the BCS kernel [Eq. (C26) of the BCS paper (Ref. 1)];

$$K(q,0)=\frac{6\pi}{\Lambda}\frac{\Delta^2}{qv_0}\int_0^\infty\frac{d\epsilon}{\epsilon}$$

$$\times\left[\frac{\tanh(\tfrac{1}{2}\beta\Delta)}{\Delta}-\frac{\tanh(\tfrac{1}{2}\beta E)}{E}\right]F_1\left(\frac{2\epsilon}{qv_0}\right), \tag{5.26}$$

where the function $F_1(x)$ is given by Eq. (5.3), and $E=(\epsilon^2+\Delta^2)^{1/2}$, as one expects. Equations (5.16) and (1.11) lead to the BCS result

$$K(0,0)=\frac{4\pi}{\Lambda}\Delta^2\int_0^\infty\frac{d\epsilon}{\epsilon^2}\left[\frac{\tanh(\tfrac{1}{2}\beta\Delta)}{\Delta}-\frac{\tanh(\tfrac{1}{2}\beta E)}{E}\right] \tag{5.27}$$

$$=\frac{4\pi}{\Lambda}\sum_n\frac{2\pi}{\beta}\frac{\Delta^2}{[\omega_n{}^2+\Delta^2]^{3/2}}. \tag{5.28}$$

At zero temperature this becomes

$$K(0,0)=4\pi/\Lambda, \tag{5.29}$$

which is independent of the gap parameter.

In the weak-coupling system with nonmagnetic impurities, Eqs. (5.18) and (5.20) become

$$K(q,0)=\frac{6\pi}{\Lambda}\Delta^2\operatorname{Re}\int_0^\infty\frac{d\epsilon}{\epsilon}$$

$$\times\left\{\frac{\tanh(\tfrac{1}{2}\beta\Delta)}{\Delta}-\frac{\tanh(\tfrac{1}{2}\beta E)}{E}\right\}F(q,2\epsilon,2\Gamma) \tag{5.30}$$

$$=\frac{6\pi}{\Lambda}\sum_n\frac{2\pi}{\beta}\frac{\Delta^2F_0(S_n)}{(\omega_n{}^2+\Delta^2)[(\omega_n{}^2+\Delta^2)^{1/2}+\Gamma]}, \tag{5.31}$$

where

$$S_n=qv_0/2[(\omega_n{}^2+\Delta^2)^{1/2}+\Gamma].$$

The results of Eqs. (5.16) and (1.11) reduce to those of Mattis and Bardeen, Abrikosov *et al.*, and Rickayzen:

$$K(0,0)=\frac{4\pi}{\Lambda}\Delta^2\int_0^\infty\frac{d\epsilon}{\epsilon^2+\Gamma^2}\times\left\{\frac{\tanh(\frac{1}{2}\beta\Delta)}{\Delta}-\frac{\tanh(\frac{1}{2}\beta E)}{E}\right\} \quad (5.32)$$

$$=\frac{4\pi}{\Lambda}\sum_n\frac{2\pi}{\beta}\frac{\Delta^2}{(\omega_n{}^2+\Delta^2)[(\omega_n{}^2+\Delta^2)^{1/2}+\Gamma]}. \quad (5.33)$$

At zero temperature this becomes[38]

$$K(0,0)=\frac{4\pi}{\Lambda}\frac{1}{x}\left\{\frac{\pi}{2}-\frac{1}{(1-x^2)^{1/2}}\tan^{-1}[(1-x^2)^{1/2}/x]\right\}$$

$$=\frac{4\pi}{\Lambda}\frac{1}{x}\left\{\frac{\pi}{2}-\frac{1}{2}\frac{1}{(x^2-1)^{1/2}}\ln\left|\frac{x+(x^2-1)^{1/2}}{x-(x^2-1)^{1/2}}\right|\right\}, \quad (5.34)$$

where $x=\Gamma/2\Delta(0)=1/2\tau\Delta(0)=(\pi\xi_0/2l)$. For $x=0$, that is, $l=\infty$, this becomes the BCS result Eq. (5.29). When the mean free path is small compared with the coherence length ξ_0, Eq. (5.33) reduces to the well-known result

$$\frac{\rho_s}{\rho}\approx\left(\frac{l}{\xi_0}\right)\frac{\Delta(T)}{\Delta(0)}\tanh\frac{\Delta(T)}{2T}. \quad (5.35)$$

In the weak-coupling system with magnetic impurities, Eq. (1.11) leads to the result of Weiss *et al.*, and Abrikosov and Gor'kov:

$$K(0,0)=\frac{4\pi}{\Lambda}\sum_n\frac{2\pi}{\beta}\times\frac{\Delta_n{}^2}{(\omega_n{}^2+\Delta_n{}^2)[(\omega_n{}^2+\Delta_n{}^2)^{1/2}+\Gamma+\Gamma_s]}=\frac{4\pi}{\Lambda}\frac{\rho_s}{\rho}$$

$$=\frac{4\pi}{\Lambda}\sum_n\frac{2\pi}{\beta}\times\frac{\Delta^2}{(\bar{\omega}_n{}^2+\Delta^2)[(\bar{\omega}_n{}^2+\Delta^2)^{1/2}+\Gamma-\Gamma_s]}, \quad (5.36)$$

where the effective gap parameter Δ_n is

$$\Delta_n=\Delta-2\Gamma_s\Delta_n/[\omega_n{}^2+\Delta_n{}^2]^{1/2} \quad (5.37)$$

from Eq. (3.6c) and the renormalized frequency $\bar{\omega}_n$ is

$$\bar{\omega}_n=\omega_n+2\Gamma_s\bar{\omega}_n/[\bar{\omega}_n{}^2+\Delta^2]^{1/2} \quad (5.38)$$

from Eq. (3.10). It is clear that when $\Gamma_s=0$, that is, no magnetic impurity present, the above result reduces to Eq. (5.33) for the system with nonmagnetic impurities.

At zero temperature Eq. (5.36) can be evaluated by changing the variable ω_n into $\bar{\omega}_n$ of Eq. (5.38), or into Δ_n of Eq. (5.37);

$$\frac{\rho_s}{\rho}=\frac{1}{\delta}[S(0)-S(\delta)]+\frac{\gamma_s}{\delta^4}\left[\sum_{n=0}^{3}\frac{\delta^n}{n!}S^{(n)}(0)-S(\delta)\right]$$

$$=\frac{1}{\delta}[S(0)-S(\delta)]-\gamma_s\sum_{n=0}^{\infty}\frac{\delta^n}{(n+4)!}S^{(n+4)}(0), \quad (5.39)$$

where

$$\gamma_s=2\Gamma_s/\Delta(\Gamma_s,0)=\pi\xi_0/l_s,$$

$$\delta=\frac{\Gamma-\Gamma_s}{\Delta(\Gamma_s,0)}=\frac{\pi\xi_0}{2}\left[\frac{1}{l}-\frac{1}{l_s}\right],$$

$$\pi\xi_0=v_0/\Delta(\Gamma_s,0).$$

The function $S(\delta)$ is given by

$$S(\delta)=\frac{2}{(1+\delta)B}\{\tan^{-1}B-\tan^{-1}Bz_0\}$$

$$=\frac{1}{(1+\delta)\tilde{B}}\ln\left|\frac{1+\tilde{B}}{1+\tilde{B}}\bigg/\frac{1+\tilde{B}z_0}{1-\tilde{B}z_0}\right|, \quad (5.40)$$

where

$$z_0=x_0/(1+\gamma_s),\ x_0=\theta(\gamma_s-1)(\gamma_s{}^2-1)^{1/2},$$
$$B=[(1-\delta)/(1+\delta)]^{1/2},$$

and $\tilde{B}=[(\delta-1)/(\delta+1)]^{1/2}$. The various values of $S^{(n)}(0)$ are

$$S(0)=\tfrac{1}{2}\pi-\tan^{-1}x_0,$$

$$S'(0)=-(1-x_0/\gamma_s),$$

$$S''(0)=\tfrac{1}{2}\pi-[\tan^{-1}x_0+x_0/\gamma_s{}^2],$$

$$S'''(0)=-4\left[1-\frac{x_0}{\gamma_s}\left(1+\frac{1}{2\gamma_s{}^2}\right)\right],$$

$$S^{(4)}(0)=9\left[\tfrac{1}{2}\pi-\tan^{-1}x_0-\frac{x_0}{\gamma_s{}^2}\left(1+\frac{2}{3}\frac{1}{\gamma_s{}^2}\right)\right],$$

[38] One can write $\tan^{-1}(1/x^2-1)^{1/2}=\cos^{-1}(x)=\sin^{-1}(1-x^2)^{1/2}$.

and so forth. When $\gamma_s=0$, Eq. (5.39) leads to Eq. (5.34) as it should. It is not noted that ρ/ρ_s does not vanish unless δ, $\gamma_s\to\infty$, in other words, $\Delta(0,\Gamma_s)\to 0$ as $\Gamma_s\to\Gamma_s^{\mathrm{cr}}$, where Γ_s^{cr} is a critical value of Γ_s at which the system becomes normal.[39]

For a special case, $\delta=0$, Eq. (5.39) becomes

$$\frac{\rho_s}{\rho}=1-\tfrac{3}{8}\gamma_s[\tfrac{1}{2}\pi-\tan^{-1}x_0]-\frac{x_0}{\gamma_s}\left[\frac{5}{8}-\frac{1}{4}\frac{1}{\gamma_s^2}\right]. \quad (5.41)$$

When $\gamma_s=2\Gamma_s/\Delta(\Gamma_s,0)=2\Gamma/\Delta(\Gamma_s,0)=1$, then $\rho_s/\rho=1-3\pi/16$.

For a system with only spin-flip scattering, the superfluid density becomes

$$\frac{\rho_s}{\rho}=\frac{2}{\gamma_s}[S(-\tfrac{1}{2}\gamma_s)-S(0)]\left[1-\frac{8}{\gamma_s^2}\right]+\frac{16}{\gamma_s^3}\sum_{n=1}^{3}\frac{S^{(n)}(0)}{n!}\left(-\frac{\gamma_s}{2}\right)^n. \quad (5.42)$$

This may correspond to the superfluid density of a superconductor with nonmagnetic impurities but in a static magnetic field, with[33]

$$\gamma_s=2\tau_0\langle\mu^2\rangle H^2/\Delta(H),$$

where $\langle\mu^2\rangle$, H, and τ_0 are the average value of the square of magnetic moment, a static magnetic field, and the relaxation time.

In the limit, $L\ll\xi_0$, $\Delta\ll T$, and $\Gamma_s\ll T$, the superfluid density of Eq. (5.36) may be written as

$$\frac{\rho_s}{\rho}\approx\left(\frac{L}{\xi_0}\right)\frac{\Delta}{\Delta(\Gamma_s,0)}\tanh(\tfrac{1}{2}\beta\Delta)\left[1-\frac{\Delta\beta}{\sinh\beta\Delta}(2\Gamma_s\beta)(1+4\Gamma_s\beta)-\tfrac{1}{2}(2\Gamma_s\beta)^2(\Delta\beta)^2\operatorname{sech}^2(\tfrac{1}{2}\beta\Delta)\right]. \quad (5.43)$$

This reduces to the result of Eq. (5.35), when $\Gamma_s=0$. The more general result may be used for a system in which only spin-flip scattering is present, so that $L=l_s$. It is noted that the gap parameter Δ in Eq. (5.43) depends on T and Γ_s, $\Delta(\Gamma_s,T)$.

We now give a few limiting values of the response functions Eqs. (4.11) and (4.14) for an anisotropic system. The dc normal-state conductivity obtained from Eq. (4.14) is

$$\sigma_{\mu\nu}{}^N(0,0)=\frac{2e^2}{(2\pi)^3}\int\frac{dA}{|v|}v_\mu v_\nu\int_0^\infty\frac{\operatorname{sech}^2(\frac{1}{2}\beta\omega)}{2\Gamma^N(\Omega,\omega)}\frac{\beta}{2}d\omega. \quad (5.44)$$

In the weak-coupling limit this becomes

$$\sigma_{\mu\nu}{}^N(0,0)=\frac{2e^2}{(2\pi)^3}\int\frac{dA}{|v|}v_\mu v_\nu\tau(\Omega). \quad (5.45)$$

This is the usual expression for the normal dc conductivity of an anisotropic system in the relaxation-time approximation. Here we have used the relaxation time $\tau(\Omega)=1/2\Gamma^N(\Omega)$. Equation (5.44) reduces to Eq. (5.5d) for an isotropic system. Other limiting values of Eq. (4.14) can also be obtained.

For the superconducting state, in the local limit, $q\to 0$ and assuming that $\Gamma\to\infty$, which corresponds to $L\ll\xi_0$, we obtain from Eq. (4.11) the conductivity

$$\sigma_{\mu\nu}(0,\omega)_{L\to\infty}\approx\frac{2e^2}{(2\pi)^3}\int\frac{dA}{|v|}\frac{v_\mu v_\nu}{\Gamma_1+\Gamma_2}\{\sigma_1-i\sigma_2\}, \quad (5.46)$$

where σ_1 and σ_2 are defined by Eqs. (4.35). This expression reduces to Eq. (5.44) for the normal-state value, when the gap parameter Δ vanishes.

It can be shown that the anisotropic response function Eq. (4.11) satisfies the condition

$$\lim_{q\to0}\lim_{\omega\to0}K_{\mu\nu}(\mathbf{q},\omega)=K_{\mu\nu}(0,0)=\lim_{\omega\to0}\lim_{q\to0}K_{\mu\nu}(\mathbf{q},\omega), \quad (5.47)$$

where the limiting value $K_{\mu\nu}(0,0)$ is

$$K_{\mu\nu}(0,0)=\frac{e^2}{\pi^2}\int\frac{dA}{|v|}v_\mu v_\nu\operatorname{Re}\sum_n\frac{2\pi}{\beta}\frac{\Delta_n^2}{(\omega_n^2+\Delta_n^2)^{3/2}Z_n}. \quad (5.48)$$

This limiting value also can be obtained from the starting point for an anisotropic system in a way similar to Eq. (5.25):

$$K_{\mu\nu}(0,0)=K_{\mu\nu}{}^{sp}(0,0)-K_{\mu\nu}{}^{np}(0,0). \quad (5.25\text{b})$$

6. CONCLUSION

An expression has been given for the current density in real space, which can be applied to strong-coupling and impure systems. It is explicitly shown that the response function satisfies the conditions for infinite conductivity and the Meissner effect. The general expression Eq. (4.11) for the response function may be useful for calculating various properties, such as the surface impedance and the penetration depth.

Calculations have been made by the author for the Pippard limit, using Eq. (5.8) for the strong-coupling

[39] From Eq. (3.13) and the BCS gap equation, we can find the value Γ_s^{cr} at which T_c vanishes; $\Gamma_s^{\mathrm{cr}}\approx\frac{1}{2}\Delta_{\mathrm{BCS}}(0)$. See Ref. 29.

superconductor, Pb. In the London limit, using Eq. (5.11), he has calculated the conductivity, surface impedance, and penetration depth for an isotropic weak-coupling superconductor with magnetic impurities. These applications will be discussed in a subsequent paper.[29]

Utilizing the Green's function in a form of Eq. (3.15), one can easily obtain the thermal conductivity[40] and the ultrasonic attenuation coefficient[41] for strong-coupling and impure systems.

The theory discussed here can be extended to the case of a high-static magnetic field superimposed on an alternating field.

We have neglected vertex corrections associated with impurity scattering. In the weak-coupling case, the effect is to replace τ by τ_{tr}, and a similar effect is to be expected in strong coupling. However, the exact form of these corrections remains to be investigated.

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APPENDIX

In this Appendix we give a simple calculation of the Josephson tunneling current[42] under the assumption that the tunneling matrix element is constant.

Following the procedure employed by Ambegaokar and Baratoff,[43] we obtain an equation corresponding to their Eq. (16) as

$$\dot{N}\langle l\rangle=\sum_{\mathbf{kp}}\int\frac{d\omega}{2\pi}[f_{\mathbf{p}}^r(\omega+eV)-f_{\mathbf{k}}^l(\omega)]\{A_{\mathbf{k}}^l(\omega)A_{\mathbf{p}}^r(\omega+eV)|T_{\mathbf{kp}}|^2-B_{\mathbf{k}}^l(\omega)B_{\mathbf{p}}^r(\omega+eV)T_{-\mathbf{k},-\mathbf{p}}T_{\mathbf{k},\mathbf{p}}\cos\phi\}$$
$$+2\sum_{\mathbf{kp}}\int\frac{d\omega}{2\pi}\int\frac{d\omega'}{2\pi}\frac{P}{\omega-\omega'+eV}[f_{\mathbf{p}}^r(\omega')-f_{\mathbf{k}}^l(\omega)]T_{\mathbf{kp}}T_{-\mathbf{k},-\mathbf{p}}B_{\mathbf{k}}^l(\omega)B_{\mathbf{p}}^r(\omega')\sin\phi, \quad \text{(A1)}$$

where $\phi=\alpha_l+\alpha_r+(\mu_l-\mu_r+eV)t$. The functions A and B are the spectral functions of the Green's functions G and F of Gor'kov scheme. We have introduced an applied voltage eV into the calculation. The assumption that the tunneling matrix element be constant allows us to carry out the momentum integration. We then obtain

[40] In the calculation of the thermal conductivity, the formal structure of the response function is similar except that in this case the coherence factor $\bar{g}_{\alpha\beta}$ is

$$\bar{g}_{\alpha\beta}(1,2)=N^\alpha(1)N^\beta(2)-P^\alpha(1)P^\beta(2).$$

The calculations are otherwise identical. We obtain for the thermal conductivity,

$$K_S\approx A\beta^2\int_{\omega_g}^\infty d\omega\,[1+\bar{g}_{+-}(\omega,\omega)]\frac{\omega^2}{\Gamma(\omega)}\operatorname{sech}^2(\tfrac{1}{2}\beta\omega),$$

where

$$\bar{g}_{+-}(\omega,\omega)=\{\omega^2-|\Delta|^2\}/|\omega^2-\Delta^2(\omega)|,$$

and

$$\Gamma(\omega)=\operatorname{Im}\{Z(\omega)[\omega^2-\Delta^2(\omega)]^{1/2}\}$$

from Eq. (4.9). This result has been derived by V. Ambegaokar and L. Tewordt [Phys. Rev. **134**, A805 (1964)] for strong-coupling superconductors and by V. Ambegaokar and A. Griffin [Phys. Rev. **137**, A1151 (1965)] for superconductors containing magnetic impurities. For the latter, the gap parameter $\Delta(\omega)$ is given by Eq. (3.6c).

[41] Following the procedure employed by L. P. Kadanoff and I. I. Falko [Phys. Rev. **136**, A1170 (1964)], we find that the transverse ultrasonic attenuation coefficient for strong-coupling and impure superconductors at low frequencies is given by

$$\alpha^T(q,q_0\to 0)=A\left\{\int_{\omega_g}^\infty D_1(\omega)\frac{1-F(qL)}{(qL)}d\omega+\frac{q_0\alpha_2^2}{q_0\alpha_3+\alpha_d}\right\}, \quad (1)$$

where

$$\alpha_2=\int_{\omega_g}^\infty D_1(\omega)[1-F(qL)]d\omega,$$

$$\alpha_3=\int_{\omega_g}^\infty D_2(\omega)F(qL)(qL)d\omega,$$

and

$$\alpha_d=\int_{\omega_g}^\infty D_3(\omega)F(QL)(QL)d\omega,$$

and where

$$D_1(\omega)=\left[1+\frac{\omega^2-|\Delta(\omega)|^2}{|\omega^2-\Delta^2(\omega)|}\right]\frac{\beta}{2}\operatorname{sech}^2(\tfrac{1}{2}\beta\omega),$$

$$D_2(\omega)=\left[1+\frac{\omega^2+|\Delta(\omega)|^2}{|\omega^2-\Delta^2(\omega)|}\right]\frac{\beta}{2}\operatorname{sech}^2(\tfrac{1}{2}\beta\omega),$$

$$D_3(\omega)=2\left[\frac{2\Delta^2(\omega)}{\omega^2-\Delta^2(\omega)}\right]\tanh(\tfrac{1}{2}\beta\omega),$$

$$F(x)=\frac{3}{2}\frac{1}{x^3}[(1+x^2)\arctan(x)-x],$$

$$qL=qv_0/2\operatorname{Im}\{Z(\omega)[\omega^2-\Delta^2(\omega)]^{1/2}\},$$

$$QL=qv_0/2\{Z(\omega)[\omega^2-\Delta^2(\omega)]^{1/2}\},$$

and

$$A=nmqv_0/\rho_{\text{ion}}v_{\text{sound}}.$$

The second term in Eq. (1) vanishes at zero frequency, $q_0=0$, in the superconducting state since the Meissner current term (α_d) appears in the denominator. Equation (1) becomes that of Kadanoff and Falko in the weak coupling with magnetic impurities. In the normal state, however, the Meissner current term vanishes, and the second term is finite. In the weak-coupling limit, that is the effective mean free path L is independent of frequency, Eq. (1) reduces to that of Pippard [Phil. Mag. **46**, 1104 (1955)] in the normal state. The longitudinal correlation function can be obtained in the same way. (We thank Professor Leo P. Kadanoff for a discussion concerning the general response function.) In the limit $qL\gg 1$, the expression for the longitudinal ultrasonic attenuation coefficient simplifies, since the dominant contribution comes from the density-density correlation function:

$$\alpha^L(q,0)\approx A\int_{\omega_g}^\infty\frac{\beta}{2}d\omega\left[1+\frac{\omega^2-|\Delta(\omega)|^2}{|\omega^2-\Delta^2(\omega)|}\right]\arctan(qL)\operatorname{sech}^2(\tfrac{1}{2}\beta\omega).$$

This is the result given independently by V. Ambegaokar [Phys. Rev. Letters **16**, 1047 (1966)].

[42] B. D. Josephson, Phys. Letters **1**, 251 (1962); Advan. Phys. **14**, 419 (1965). We thank Dr. Josephson for the stimulating discussion about the origin of the current.

[43] V. Ambegaokar and A. Baratoff, Phys. Rev. Letters **10**, 486 (1963); **11**, 104 (1963).

the tunneling current as

$$\frac{J_T}{J_0}=\int d\omega\,[f^r(\omega+eV)-f^l(\omega)]\{n^l(\omega)n^r(eV+\omega)$$
$$-p^l(\omega)p^r(\omega+eV)\cos\phi\}+\frac{1}{\pi}\int d\omega\int d\omega'\frac{P}{\omega-\omega'+eV}$$
$$\times[f^r(\omega')-f^l(\omega)]p^l(\omega)p^r(\omega')\sin\phi, \quad \text{(A2)}$$

where $J_0=e(2\pi)[N(0)T_{\rm eff}]^2$. Here $T_{\rm eff}$ is the effective tunneling matrix element. We have here introduced the densities of the state $n(\omega)$ and pairs $p(\omega)$ defined by Eq. (3.18), that is,

$$N(0)n(\omega)=\frac{1}{2\pi}\sum_{\mathbf{k}} A(k)=N(0)\,\mathrm{Re}\{\omega/[\omega^2-\Delta^2(\omega)]^{1/2}\},$$

$$N(0)p(\omega)=\frac{1}{2\pi}\sum_{\mathbf{k}} B(k)=N(0)\,\mathrm{Re}\{\Delta(\omega)/[\omega^2-\Delta^2(\omega)]^{1/2}\}.$$

The first term in Eq. (A2) gives the usual tunneling current, and the second gives the Josephson tunneling current which may be expressed at zero voltage in the form

$$J_T(eV=0)=J_S\sin\phi, \quad \text{(A3)}$$

where the maximum value of the supercurrent J_S is given by

$$J_S=\frac{2}{\pi}J_0\int_0^\infty d\omega\int_0^\infty d\omega'\left\{\frac{f^l(\omega)-f^r(\omega')}{\omega-\omega'}+\frac{1-f^l(\omega)-f^r(\omega')}{\omega+\omega'}\right\}p^l(\omega)p^r(\omega'). \quad \text{(A4)}$$

This reduces to the result of Ambegaokar and Baratoff in the weak-coupling limit as it should, when variables are changed from ω and ω' to ϵ_1 and ϵ_2 according to $\omega^2=\epsilon_1^2+\Delta_1^2=E_1^2$, and $\omega'^2=\epsilon_2^2+\Delta_2^2=E_2^2$. Equation (A4) can be used more generally for strong-coupling and impure (nonmagnetic and magnetic) systems. In this equation the tunneling current is a function of the matrix element $T_{\rm eff}$ and the gap parameter $\Delta(\omega)$. One can obtain in principle $\Delta(\omega)$ and $T_{\rm eff}$ from the ordinary quasiparticle tunneling current. This would allow a check of Eq. (A4).