Theory of Pure Type-II Superconductors in High Magnetic Fields. II. Ultrasonic Attenuation*

KAZUMI MAKI†
Department of Physics, University of California, San Diego, La Jolla, California
(Received 26 September 1966)

We propose here a theory of pure type-II superconductors in high magnetic fields. Making use of the guess that the effect of the magnetic field on a pure type-II superconductor in a high field is similar to that of a transport current, we circumvent the difficulty associated with the expansion in powers of the order parameter Δ. As an application of this conjecture, we calculate here the ultrasonic attenuation coefficients in the gapless region of a pure type-II superconductor. The attenuation coefficients decrease sharply in the superconducting region as $(H_{c2}-H)^{1/2}$, where H is the external field, and are strongly anisotropic—a simple manifestation of the anisotropy in the excitation spectrum of quasiparticles.

I. INTRODUCTION

IN the preceding paper[1] (which will be referred to as I), by making use of the fact[2] that the order parameter $\Delta(\mathbf{r})$ is given by the Abrikosov solution

$$\Delta(\mathbf{r})=\sum_n C_n e^{inky}\psi_n(x)\,,$$

$$\psi_n(x)=\exp[-eH(x-kn/2eH)^2]\,, \tag{1}$$

we have shown that the perturbation expansion in powers of $\Delta(\mathbf{r})$ leads to unphysical results when it is applied to the study of the dynamical properties of a pure type-II superconductor in high magnetic fields. In this connection, it is noted that the second-order diagrams in powers of $\Delta(\mathbf{r})$ have expressions equivalent to those in a current-carrying case, except for the difference in the spectral function introduced there.[1] Therefore, it is quite natural to ascribe the origin of the above difficulty to the confluence of singularities in the density of states, as is the case for current-carrying states. In fact we can show (see Appendix) that the density of states at small excitation energy ω has the asymptotic form

$$\frac{N(\omega)}{N(0)}=1-\frac{\langle|\Delta|^2\rangle_{\text{av}}}{\epsilon^2}\ln\left(\frac{\epsilon}{4\gamma\omega}\right)+0.77\times\frac{1}{8}\frac{\langle|\Delta|^4\rangle_{\text{av}}}{\epsilon^2\omega^2}+\cdots, \tag{2}$$

and $\epsilon=v(\frac{1}{2}eH_{c2})^{\frac{1}{2}}$, where v is the Fermi velocity. The coefficient of $\langle|\Delta|^2\rangle_{\text{av}}$ has been discussed previously. The above expression indicates that the expansion of $N(\omega)$ in powers of $|\Delta|^2$ breaks down for $\omega\lesssim(\langle|\Delta|^2\rangle_{\text{av}})^{1/2}$.

In order to circumvent this difficulty, we shall make the following assumption: The analogy between a pure type-II superconductor in a high field and a current-carrying state holds not only for the second-order diagrams (see I) for $\Delta(\mathbf{r})$, but also for the higher-order diagrams. For example, we have the following expression for the density of states[1]:

$$\frac{N(\omega)}{N(0)}=1+\frac{\langle|\Delta|^2\rangle_{\text{av}}}{2}\int_{-\infty}^{\infty}\rho_0(\alpha)\,d\alpha\frac{1}{(\omega-\alpha)^2}, \tag{3}$$

which is equivalent to one obtained by Juranek *et al.*[3] Here $\rho_0(\alpha)$ is given by

$$\rho_0(\alpha)=\int\frac{d\Omega}{4\pi}\rho(\alpha,\Omega)=\frac{1}{\epsilon}\int_{|\alpha|/\epsilon}^{\infty}e^{-t^2}dt\,,$$
$$\rho(\alpha,\Omega)=[(\pi)^{1/2}\epsilon\sin\theta]^{-1}\exp[-(\alpha/\epsilon\sin\theta)^2]. \tag{4}$$

It is easy to see that Eq. (3) diverges logarithmically for small ω. The above assumption allows us to write down the expression for the density of states, which we may expect to be valid for small ω,

$$\frac{N(\omega)}{N(0)}=\int_{-\infty}^{\infty}\rho_0(\alpha)\,d\alpha\,\text{Re}\left(\frac{\omega-\alpha}{[(\omega-\alpha)^2-\Delta^2]^{1/2}}\right). \tag{5}$$

We shall hereafter set $\Delta^2\equiv\langle|\Delta|^2\rangle_{\text{av}}$. If we expand Eq. (5) in powers of Δ^2, the first two terms agree with Eq. (3) and the third term reproduces correctly the divergence of the coefficients of $\langle|\Delta|^4\rangle_{\text{av}}$ in Eq. (1), though the numerical coefficient is slightly different.

II. EXPRESSIONS FOR THE ULTRASONIC ATTENUATION COEFFICIENTS

As a simple application of the analogy between a pure type-II superconductor in high fields and a pure superconductor carrying a uniform current, we shall discuss here the attenuation coefficients of ultrasound. In the following we assume $l/\xi_0\gg1$, though we impose no restriction on ql, where l is the electronic mean free path, ξ_0 is the coherence distance, and q is the wave vector of the sound wave.

* Research sponsored by the U. S. Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under Grant No. AF-AFOSR-610-64.

† On leave of absence from the Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan.

[1] K. Maki and M. Cyrot, preceding paper, Phys. Rev. 156, 433 (1967).

[2] K. Maki and T. Tsuzuki, Phys. Rev. 139, A868 (1965).

[3] H. J. Juranek, L. Neumann, and L. Tewordt, Z. Physik 173, 459 (1966).

Following Tsuneto,[4] we shall express the attenuation coefficients in terms of various correlation functions. We shall adopt here the method developed by Kadanoff and Falko,[5] which allows us to treat the effect of the Coulomb interaction in a simple way. For the low-frequency limit $\omega<\pi T_{c0}$ (we adopt hereafter the unit system $\hbar=c=k_B=1$), the attenuation coefficient for the longitudinal sound wave is given by[5]

$$\alpha_L=\mathrm{Re}\frac{q^2}{i\omega\rho_{\mathrm{ion}}v_s}\left\{\langle[\tau_{z'z'},\tau_{z'z'}]\rangle_{(\mathbf{q},\omega)}-\frac{2p_0^2}{3m}\langle[\tau_{z'z'},n]\rangle_{(\mathbf{q},\omega)}+\left(\frac{p_0^2}{3m}\right)^2\langle[n,n]\rangle_{(\mathbf{q},\omega)}\right\}, \quad (6)$$

where p_0 and $\mathbf{q}$ are the Fermi momentum and the wave vector of the sound wave, respectively. Here $\tau_{z'z'}$ is the stress tensor and n is the density operator of electrons:

$$\tau_{ij}=\sum_{\mathrm{spin}}\left\{\frac{(\nabla-\nabla')_i}{2i}\frac{(\nabla-\nabla')_j}{2im}\psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r},t)\right\}_{\mathbf{r}'=\mathbf{r}}, \quad (7)$$

and

$$n=\sum_{\mathrm{spin}}\psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t). \quad (8)$$

The primed coordinate system is taken relative to the sound-wave propagation (we take z' as the direction of the propagation). Similarly, the expression for the transverse wave is given by[5,6]

$$\alpha_T=\mathrm{Re}\frac{q^2}{i\omega\rho_{\mathrm{ion}}v_s}\left\{\langle[\tau_{x'z'},\tau_{z'x'}]\rangle_{(\mathbf{q},\omega)}-\frac{(\langle[\tau_{x'z'},j_{x'}]\rangle_{(\mathbf{q},\omega)})^2}{\langle[j_{x'},j_{x'}]\rangle_{(\mathbf{q},\omega)}}\right\}, \quad (9)$$

where $j_{x'}$ is a component of the current operator. Here we take x' as the direction of the polarization vector of the sound wave. The second term in Eq. (9) may be called "the electromagnetic term," in contrast to the first, or "collision-drag" term.[7] Except in the extremely pure case with a rather high-frequency sound wave, the second term is always negligible; we shall consider here only the first term.

III. EVALUATION OF THE CORRELATION FUNCTIONS

The correlation functions are obtained by using the techniques of thermal Green's functions. In particular we shall see later that the analogy to a current-carrying case simplifies our analysis enormously. Since we are interested here in those correlation functions for arbitrary ql value, we shall formulate the problem in the presence of random impurity atoms. Scattering effects due to impurities are taken account of by carrying out the renormalization of the self-energy as well as that of vertex functions.[8] First we shall consider the renormalization of the self-energy. Here we must replace ω and $\Delta(\mathbf{r})$ by

$$\tilde{\omega}=\omega(1+1/2\tau|\omega|) \quad \text{and} \quad \tilde{\Delta}(r)=\eta_{\omega H}\Delta(r),$$

respectively,[9] where

$$\eta_{\omega H}=\{1-(1/2\tau|\tilde{\omega}|)[1-C(v(eH)^{1/2}/|\tilde{\omega}|)]\}^{-1} \quad (10)$$

and

$$C(a)=\frac{2}{a}\int_0^\infty due^{-u^2}\arctan(au).$$

Here we have made use of the fact that $\Delta(\mathbf{r})$ is given by Eq. (1). The above renormalization factor for Δ has been found by Helfand and Werthamer.[10] Fortunately, we can show that the effect of the renormalization factor of $\Delta(\mathbf{r})$ is negligible as long as we are concerned in the limit $l/\xi_0\gg1$. We shall neglect this factor (i.e., put $\eta_{\omega H}=1$) hereafter. Second, the renormalizations of the vertex functions are far more important in the present case. Since the related problem has been already discussed in great detail by Kadanoff and Falko,[5] we shall make use of their results here without going into further discussions. After these preliminaries it is easy to write down various correlation functions:

$$\mathrm{Im}\langle[\tau_{z'z'},\tau_{z'z'}]\rangle_{(\mathbf{q},\omega)}=\frac{1}{2\pi^2}\frac{p_0^2}{m}\int\frac{d\Omega}{4\pi}\int_{-\infty}^{\infty}\rho(\alpha,\Omega)d\alpha\frac{z'^2(z'^2+y^{-2})}{1-iyz'}J(\omega,\alpha), \quad (11)$$

$$\mathrm{Im}\langle[\tau_{z'z'},n]\rangle_{(\mathbf{q},\omega)}=\frac{1}{2\pi^2}p_0^3\int\frac{d\Omega}{4\pi}\int_{-\infty}^{\infty}\rho(\alpha,\Omega)\,d\alpha\left\{\frac{z'^2}{1-iyz'}\cdot\frac{1}{1-y^{-1}\arctan y}+\frac{1}{y^2}\right\}\frac{J(\omega,\alpha)}{2}, \quad (12)$$

$$\mathrm{Im}\langle[n,n]\rangle_{(\mathbf{q},\omega)}=\frac{m}{2\pi^2}p_0\int\frac{d\Omega}{4\pi}\int_{-\infty}^{\infty}\rho(\alpha,\Omega)d\alpha\frac{1}{1-iyz'}\frac{J(\omega,\alpha)}{1-y^{-1}\arctan y}, \quad (13)$$

[4] T. Tsuneto, Phys. Rev. **121**, 402 (1961); see also T. Tsuneto (unpublished).
[5] L. P. Kadanoff and I. I. Falko, Phys. Rev. **136**, A1170 (1964).
[6] K. Maki, Phys. Rev. **148**, 370 (1966).
[7] L. T. Claiborne, Jr., and R. W. Morse, Phys. Rev. **136**, A893 (1964).
[8] K. Maki and P. Fulde, Phys. Rev. **140**, A1586 (1965).
[9] K. Maki, Physics **1**, 21 (1964).
[10] E. Helfand and N. R. Werthamer, Phys. Rev. Letters **13**, 686 (1964).

and

$$\mathrm{Im}\langle[\tau_{x'z'},\tau_{x'z'}]\rangle_{(\mathbf{q},\omega)}=\frac{1}{2\pi^2}p_0^3\int\frac{d\Omega}{4\pi}\int_{-\infty}^{\infty}\rho(\alpha,\Omega)d\alpha\frac{z'^2x'^2}{1-iyz'}J(\omega,\alpha)\,, \tag{14}$$

where $\rho(\alpha,\Omega)$ has been defined in Eq. (4),

$$J(\omega,\alpha)=\int_{\Delta-\alpha}^{\infty}d\omega'\left[\tanh\frac{\omega'+\omega}{2T}-\tanh\frac{\omega'}{2T}\right]\frac{(\omega'+\omega+\alpha)(\omega'+\alpha)-\Delta^2}{[(\omega'+\omega+\alpha)^2-\Delta^2]^{1/2}[(\omega'+\alpha)^2-\Delta^2]^{1/2}}, \tag{15}$$

$$y=ql,\ z'=\mathbf{q}\cdot\mathbf{v}/qv,\ x'=\mathbf{e}\cdot\mathbf{v}/v\,, \tag{16}$$

and $\mathbf{e}$ is the polarization vector of the sound wave.

Here we have made use of the analogy to a current-carrying state, which can be seen explicitly in the coherence factor in Eq. (15). Especially in the low-frequency limit $(\omega<\Delta)$, $J(\omega,\alpha)$ reduces to

$$J(\omega,\alpha)=\frac{\omega}{2T}\int_{\Delta-\alpha}^{\infty}d\omega'\cosh^{-2}\left[\frac{\omega'}{2T}\right]=2\omega f(\Delta-\alpha)\,, \tag{17}$$

where $f(x)=(1+e^{x/T})^{-1}$, the Fermi factor.

Substituting the above expressions in Eqs. (6) and (9), we obtain the expressions of the attenuation coefficients for arbitrary ql.

IV. SOME LIMITING CASES

In the following we shall discuss some limiting situations where further calculations are feasible.

A. $ql\gg1$

1. *Longitudinal Wave*

In this limit only $\langle[n,n]\rangle_{(\mathbf{q},\omega)}$ contributes to the attenuation coefficient and we have

$$\frac{\alpha_L^s}{\alpha_L^n}=\int_{-\infty}^{\infty}d\alpha\,\phi_1(\alpha,k)\times2f(\Delta-\alpha)\,, \tag{18}$$

where

$$\phi_1(\alpha,k)=2qv\int\frac{d\Omega}{4\pi}\delta(\mathbf{q}\cdot\mathbf{v})\rho(\alpha,\Omega)$$

$$=\frac{2}{(\sqrt{\pi})\epsilon}\int\frac{d\Omega}{4\pi}\delta[(1-k^2)^{1/2}\cos\theta+k\sin\theta\cos\phi]\frac{e^{-(\alpha/\epsilon\sin\theta)^2}}{\sin\theta},$$

$$=\frac{2}{(\pi^3)^{1/2}\epsilon}\int_0^1\frac{dz}{(1-z^2)^{1/2}(1-k^2z^2)^{1/2}}\exp\left[-\left(\frac{\alpha}{\epsilon}\right)^2(1-k^2z^2)^{-1}\right]. \tag{19}$$

$k=\cos\Theta$ and Θ is the angle between $\mathbf{q}$ and the external field $\mathbf{H}$, which we take in the direction of z axis. Since we are interested in the region where Δ is small, we simplify Eq. (18) as

$$\frac{\alpha_L^s}{\alpha_L^n}=1-\frac{\Delta}{2T}\int_{-\infty}^{\infty}\phi_1(\alpha,k)\cosh^{-2}\left(\frac{\alpha}{2T}\right)+O(\Delta^3). \tag{20}$$

It is interesting to note that the shift in the attenuation coefficients is proportional to Δ, a behavior which cannot be reproduced by a simple expansion of the kind discussed in I. The following asymptotic expansions are worth noticing:

$$\frac{\alpha_L^s}{\alpha_L^n}=1-\frac{\Delta}{2T}\left\{1-\frac{1}{2}\left(\frac{\epsilon}{2T}\right)^2\left(1-\frac{k^2}{2}\right)+\frac{1}{2}\left(\frac{\epsilon}{2T}\right)^4(1-k^2+\tfrac{3}{8}k^4)\cdots\right\},\quad\text{for}\quad T\lesssim T_{c0}, \tag{21}$$

$$=1-\frac{4}{\sqrt{\pi^3}}\frac{\Delta}{\epsilon}\left\{K(k)-\frac{1}{3}\left(\frac{\pi T}{\epsilon}\right)^2(K(k)+kK'(k))+\frac{7}{30}\left(\frac{\pi T}{\epsilon}\right)^4(K(k)+(5/3)kK'(k)+\tfrac{1}{3}k^2K''(k))\right\},\quad\text{for}\quad T\ll T_{c0}, \tag{22}$$

where $K(z)$ is the complete elliptic integral. The temperature dependence of $\epsilon(T)$ is given by[2,11]

$$-\ln\frac{T}{T_{c0}}=\int_{-\infty}^{\infty}\rho_0(\alpha)\left\{\psi\left(\tfrac{1}{2}+\frac{i\alpha}{2\pi T}\right)-\psi(\tfrac{1}{2})\right\}, \quad (23)$$

where $\rho_0(\alpha)$ has been defined in Eq. (4), and $\psi(z)$ is the digamma function. In view of the fact that the theoretical $H_{c2}(t)$ does not agree with the experimental one[2,12] for pure Nb, it is more convenient to express $\epsilon(T)$ in terms of the experimentally observed $H_{c2}(t)$;

$$\epsilon(t)=v(\tfrac{1}{2}eH_{c2}(t))^{1/2}. \quad (24)$$

Δ is given as a function of external field[2]:

$$\Delta^2=\frac{m(2\pi T)^2}{6\pi eH_{c2}(t)}\frac{(H_{c2}(t)-H)}{(2\kappa_2^2(t)-1)\beta}g^{-1}(\rho),\ \beta=1.16. \quad (25)$$

The definitions of $\kappa_2(t)$ and $g(\rho)$ are given in Ref. 2. It might be much more convenient to rewrite the above expression as

$$\Delta^2=-(2/N(0))M\left(H_{c2}(t)-\tfrac{1}{2}T\frac{dH_{c2}(t)}{dT}\right), \quad (26)$$

where $N(0)=mp_0/2\pi^2$ is the density of states and M is the magnetization. Here we have made use of the identity[2]

$$\frac{1}{T}=-g(\rho)\frac{d}{dT}\left(\frac{v[eH_{c2}(t)]^{1/2}}{2\pi T}\right)^2. \quad (27)$$

Therefore, in the superconducting region the attenuation coefficients drop like $(H_{c2}-H)^{1/2}$ (i.e., with an infinite slope). The attenuation coefficients are strongly anisotropic in this limit. In particular, at $\Theta=\frac{1}{2}\pi$, the function $K(k)$ diverges like

$$\ln(4/(1-k^2)^{1/2})=\ln(4/|\tfrac{1}{2}\pi-\Theta|).$$

This feature is a simple reflection of the anisotropy in the excitation spectrum[1,13] of the quasiparticles. The quasiparticle feels the weaker order parameter when it travels perpendicular to the field (across the array of the vortex line) than when it travels along the field.

2. *Transverse Wave*

(The polarization vector is in the plane formed by **q** and **H**.) In this case we have

$$\frac{\alpha_T^s}{\alpha_T^n}=\frac{3\pi}{4ql}\int_{-\infty}^{\infty}\phi_2(\alpha,k)2f(\Delta-\alpha), \quad (28)$$

where

$$\phi_2(\alpha,k)=\frac{3}{2(\pi)^{1/2}}\int_0^1\frac{dz}{\epsilon}\{[(1-z^2)^{1/2}+k^2\times((1-z^2)^{-\frac{1}{2}}-\tfrac{3}{2}(1-z^2)^{1/2})]\}\times\exp\left(-\left(\frac{\alpha}{\epsilon}\right)^2(1-z^2)^{-1}\right). \quad (29)$$

The asymptotic expressions are

$$\frac{\alpha_T^s}{\alpha_T^n}=\frac{3\pi}{4ql}\left\{1-\frac{\Delta}{2T}\left(1-\frac{2}{5}\left(1-\frac{k^2}{4}\right)\left(\frac{\epsilon}{2T}\right)^2+\frac{21}{35}\left(1-\frac{k^2}{3}\right)\left(\frac{\epsilon}{2T}\right)^4\right)\right\}\ \text{for } T\lesssim T_{c0} \quad (30)$$

$$=\frac{3\pi}{4ql}\left\{1-\frac{3(\pi)^{1/2}}{4\epsilon}\Delta\left(1+\frac{k^2}{2}-\frac{8(\ln 2)}{\sqrt{\pi}}k^2\left(\frac{T}{\epsilon}\right)+\frac{24\zeta(3)}{\sqrt{\pi}}(1-k^2)\left(\frac{T}{\epsilon}\right)^3\right)\right\}\ \text{for } T\ll T_{c0}. \quad (31)$$

In the present case the coefficient of Δ is almost independent of temperature. This can be seen from the (theoretical) relation

$$\left.\frac{4\epsilon}{3\sqrt{\pi}}\right|_{T=0\cdot\mathrm{K}}\cong 1.20\times 2T_{c0}, \quad (32)$$

where we have made use of the theoretical expression $\epsilon(0)=\frac{1}{2}e\sqrt{\gamma}\Delta_{00}$ and Δ_{00} is the BCS order parameter for $H=0$ and $T=0$.

3. *Transverse Wave*

(The polarization vector is perpendicular to the plane formed by **q** and **H**.) In this case we have

$$\frac{\alpha_T^s}{\alpha_T^n}=\frac{3\pi}{4ql}\int_{-\infty}^{\infty}\phi_3(\alpha,k)2f(\Delta-\alpha), \quad (33)$$

where

$$\phi_3(\alpha,k)=\frac{3}{2\sqrt{\pi}}\int_0^1\frac{dz}{\epsilon}(1-z^2)^{1/2}\times\exp\left[-\left(\frac{\alpha}{\epsilon}\right)^2(1-z^2)^{-1}\right]. \quad (34)$$

It is easy to see that $\phi_3(\alpha,k)=\phi_2(\alpha,0)$. Thus we can use Eqs. (30) and (31) for the present case also. Contrary to the case of the longitudinal wave, the anisotropy in the attenuation coefficients is not very prominent.

B. $H\|q$

We shall consider here the case where the propagation vector of the sound wave is parallel to the external field (for arbitrary ql). In this case the angular integrals are much simplified.

---

[11] It is possible to transform the Eq. (14) in Ref. 2 to the form given in Eq. (23).

[12] C. K. Jones, J. K. Hulm, and B. S. Chandrasekhar, Rev. Mod. Phys. **36**, 74 (1964); T. McConville and B. Serin, Phys. Rev. **140**, A1169 (1965).

[13] The Orsay Group, Phys. Kondensierten Materie **5**, 141 (1966).

1. Longitudinal Wave

$$\frac{\alpha_L{}^s}{\alpha_L{}^n}=1-\frac{\Delta}{2T}\int_{-\infty}^{\infty}d\alpha\Phi_1(\alpha,ql)\cosh^{-2}\left(\frac{\alpha}{2T}\right), \tag{35}$$

where

$$\Phi_1(\alpha,ql)=\frac{1}{3X(y)}\int\frac{d\Omega}{4\pi}\rho(\alpha,\Omega)(1-3z^2)\left\{\frac{1}{1+y^2z^2}\cdot\frac{1}{1-y^{-1}\arctan y}-\frac{3z^2}{y^2}\right\}$$

$$=\frac{1}{3X(y)}\frac{1}{\sqrt{\pi}}\int_0^1\frac{dz}{\epsilon}\frac{1-3z^2}{(1-z^2)^{1/2}}\exp\left(-\left(\frac{\alpha}{\epsilon}\right)^2(1-z^2)^{-1}\right)\left\{\frac{1}{1+y^2z^2}\cdot\frac{1}{1-y^{-1}\arctan y}-\frac{3z^2}{y^2}\right\}, \tag{36}$$

$$X(y)=\frac{\arctan y}{3(y-\arctan y)}-\frac{1}{y^2},\quad \text{and}\quad y=ql. \tag{37}$$

The asymptotic forms are given as

$$\frac{\alpha_L{}^s}{\alpha_L{}^n}=1-\frac{\Delta}{2T}\left\{1-\frac{1}{2}\left(\frac{\epsilon}{2T}\right)^2\left[1+\frac{1}{y^2}\left(1-\frac{4}{15X(y)}\right)\right]+\frac{1}{2}\left(\frac{\epsilon}{2T}\right)^4\left[1-\frac{1}{y^2}\left(\frac{26}{15}-\frac{124}{315X(y)}\right)+\frac{1}{y^4}\left(1-\frac{4}{15X(y)}\right)\right]\right\}$$

$$\text{for}\quad T\lesssim T_{c0}, \tag{38}$$

$$=1-\frac{(\sqrt{\pi})\Delta}{3X(y)\epsilon}\left\{\frac{1}{1-y^{-1}\arctan y}\left(\frac{1}{(1+y^2)^{1/2}}\left(1+\frac{3}{y^2}\right)-\frac{3}{y^2}\right)+\frac{3}{2y^2}\frac{4T}{\sqrt{\pi}}\frac{\ln 2}{\epsilon}\right.$$

$$\times\left(\frac{1}{1-y^{-1}\arctan y}\left[\frac{1}{1+y^2}\left(1+\frac{3}{y^2}\right)-\frac{3}{y^2}\right]+\frac{6}{y^2}\right)$$

$$\left.-\frac{\pi^2}{3}\left(\frac{T}{\epsilon}\right)^2\left[\frac{1}{1-y^{-1}\arctan y}\left(1+\frac{3}{y^2}\right)\frac{y^2}{(1+y^2)^{3/2}}-\frac{9}{y^2}\right]\right\}\quad \text{for}\quad T\ll T_{c0}. \tag{39}$$

2. Transverse Wave

$$\frac{\alpha_T{}^s}{\alpha_T{}^n}=g(y)\left[1-\frac{\Delta}{2T}\int_{-\infty}^{\infty}d\alpha\Phi_2(\alpha,ql)\cosh^{-2}\left(\frac{\alpha}{2T}\right)\right], \tag{40}$$

where

$$\Phi_2(\alpha.ql)=\frac{3}{1-g(y)}\int\frac{d\Omega}{4\pi}\rho(\alpha,\Omega)\frac{z^2(1-z^2)\cos^2\phi}{1+y^2z^2}=\frac{3y^2}{2[1-g(y)]\sqrt{\pi}}\int_0^1\frac{dz}{\epsilon}\frac{1}{(1-z^2)^{1/2}}$$

$$\times\exp\left(-\left(\frac{\alpha}{\epsilon}\right)^2(1-z^2)^{-1}\right)\frac{z^2(1-z^2)}{1+y^2z^2}, \tag{41}$$

and

$$g(y)=\tfrac{3}{2}y^{-3}(-y+(y^2+1)\arctan y). \tag{42}$$

The asymptotic expressions are

$$\frac{\alpha_T{}^s}{\alpha_T{}^n}=g(y)\left\{1-\frac{\Delta}{2T}\left[1-\frac{1}{2}\left(\frac{\epsilon}{2T}\right)^2\left((y^2+1)y^{-2}-\frac{1}{5}\frac{1}{[1-g(y)]}\right)+\frac{1}{2}\left(\frac{\epsilon}{2T}\right)^4\left((y^2+1)^2y^{-4}-\frac{1}{5}\frac{(11/7)+y^{-2}}{(1-g(y))}\right)\right]\right\}$$

$$\text{for}\quad T\lesssim T_{c0}, \tag{43}$$

$$=g(y)\left\{1-\frac{3\sqrt{\pi}\Delta}{2[1-g(y)]\epsilon}\left[\frac{y^2}{2[1+(1+y^2)^{1/2}]^2}+\frac{\pi^2}{3}\left(\frac{T}{\epsilon}\right)^2\frac{1}{(1+y^2)^{1/2}}\cdot\frac{y^2}{1+(1+y^2)^{1/2}}+\frac{12}{\sqrt{\pi}}\zeta(3)\left(\frac{T}{\epsilon}\right)^3\frac{y^2}{1+y^2}\right]\right\}$$

$$\text{for}\quad T\ll T_{c0}. \tag{44}$$

V. CONCLUDING REMARKS

We have thus far calculated the attenuation coefficients of ultrasound in the critical region by making use of the conjecture that the effect of fields in a pure type-II superconductor is similar to that of a uniform current. The result shows that the attenuation coefficient drops sharply in superconducting region (with an infinite slope), which is in sharp contrast to the behavior in a dirty type-II superconductor.

The attenuation coefficients in a pure type-II superconductor have already been discussed by Cooper *et al.*,[14] using a model in which $\Delta(\mathbf{r})$ varies periodically in space but without any phase change. They obtained results completely different from those described here. For example, they did not discover any difficulty associated with expansion in powers of $\Delta(\mathbf{r})$. The reason for this discrepancy appears to us to be that their model is too crude to describe Abrikosov's solution. Thus it seems to us that their result is irrelevant to the type-II superconductor, though there might be some physical situations where their formula applies. Recent measurements[15] of the ultrasonic attenuation coefficient on pure Nb specimens are in qualitative agreement with the present theory.

It is not difficult to apply the present technique to the calculation of other transport coefficients of a pure type-II superconductor in high magnetic fields.

Note added in proof. A recent measurement[16] of the ultrasonic attenuation, carried out by the UCLA group, on pure Nb samples in the mixed states in the parallel geometry (i.e., the propagation vector of the sound wave is parallel to the static field) is in fair agreement with the present theory. I would like to thank Professor I. Rudnick, Professor M. Levy, and Dr. H. Kagiwada and Dr. R. Kagiwada for interesting discussions on their experimental results and on their numerical results of the various integrals contained in the present text.

ACKNOWLEDGMENTS

I would like to thank M. Cyrot for several criticisms. It is a great pleasure to express gratitude to Professor de Gennes and Service de Physique des Solids in Orsay for hospitality extended me while this work was being carried out.

APPENDIX: THE DENSITY OF STATES

The density of states in a pure type-II superconductor in high magnetic field has been discussed previously[3,13]:

$$\frac{N(\omega)}{N(0)}=1+\frac{\langle|\Delta|^2\rangle_{\text{av}}}{2}\int_{-\infty}^{\infty}\rho_0(\alpha)\frac{1}{(\omega-\alpha)^2}d\alpha\,, \tag{A1}$$

where $\rho_0(\alpha)$ has been given in Eq. (4). We note here that the logarithmic singularity is a consequence of the sharpness of the spectral function at $\alpha=0$; we have the following asymptotic forms

$$\begin{aligned}\rho_0(\alpha)&=\frac{1}{\epsilon}\left(\frac{\sqrt{\pi}}{2}-\frac{|\alpha|}{\epsilon}+\frac{1}{3}\left(\frac{|\alpha|}{\epsilon}\right)^3-\cdots\right\}, \quad \frac{|\alpha|}{\epsilon}\ll 1\,,\\ &=\frac{1}{2|\alpha|}e^{-(\alpha/\epsilon)^2}, \quad \frac{|\alpha|}{\epsilon}\gg 1\,.\end{aligned} \tag{A2}$$

The approximation used by and de Gennes *et al.*[13] corresponds in the present formalism to replacing $\rho_0(\alpha)$ by an exponential function. In order to see the convergence of the above expansions in powers of Δ^2, we shall calculate here the coefficients of Δ^4 by assuming that $\Delta(\mathbf{r})$ is still given by Eq. (1). We have here

$$\begin{aligned}I_4=&\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\int\frac{d^3k_3}{(2\pi)^3}\int\frac{d^3k_4}{(2\pi)^3}\frac{1}{(\omega-\xi)^2}\frac{1}{(\omega+\xi+\mathbf{v}\cdot\mathbf{k}_1)}\frac{1}{(\omega-\xi-\mathbf{v}\cdot(\mathbf{k}_1+\mathbf{k}_2)}\frac{1}{\omega+\xi+\mathbf{v}\cdot(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)}\\ &\times\int d^3r\int d^3r_1\int d^3r_2\int d^3r_4\exp[i\mathbf{k}_1(\mathbf{r}-\mathbf{r}_1)+i\mathbf{k}_2(\mathbf{r}-\mathbf{r}_2)+i\mathbf{k}_3(\mathbf{r}-\mathbf{r}_3)+i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_4)+i\phi(\mathbf{r}_1,\mathbf{r}_2)\\ &\qquad+i\phi(\mathbf{r}_3,\mathbf{r}_4)]\cdot\Delta(\mathbf{r}_1)\Delta^\dagger(\mathbf{r}_2)\Delta(\mathbf{r}_3)\Delta^\dagger(\mathbf{r}_4)\\ =&\frac{1}{3}\frac{\partial}{\partial\omega}2\pi N(0)\int\frac{d\Omega}{4\pi}\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\int\frac{d^3k_3}{(2\pi)^3}\int\frac{d^3k_4}{(2\pi)^3}\left(\sum_{\text{cyclic}}\frac{1}{2\omega+\mathbf{v}\cdot\mathbf{k}_1}\frac{1}{2\omega-\mathbf{v}\cdot\mathbf{k}_2}\frac{1}{2\omega+\mathbf{v}\cdot\mathbf{k}_3}\right)\\ &\times\int d^3r\int d^3r_1\int d^3r_2\int d^3r_3\exp[i\mathbf{k}_1(\mathbf{r}-\mathbf{r}_1)+i\mathbf{k}_2(\mathbf{r}-\mathbf{r}_2)+i\mathbf{k}_3(\mathbf{r}-\mathbf{r}_3)+i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_4)+i\phi(\mathbf{r}_1\cdot\mathbf{r}_2)\\ &\qquad+i\phi(\mathbf{r}_3\cdot\mathbf{r}_4)]\Delta(\mathbf{r}_1)\Delta^\dagger(\mathbf{r}_2)\Delta(\mathbf{r}_3)\Delta^\dagger(\mathbf{r}_4)\,,\end{aligned} \tag{A3}$$

[14] L. N. Cooper, A. Houghton, and H. J. Lee, Phys. Rev. Letters **15**, 584 (1965).
[15] A. Ikushima, M. Fujii, and T. Suzuki, J. Phys. Chem. Solids **26**, 1 (1965); E. M. Forgan and C. E. Gough, Phys. Letters **21**, 133 (1966).
[16] R. Kagiwada, M. Levy, I. Rudnick, H. Kagiwada, and K. Maki, Phys. Rev. Letters **18**, 74 (1967).

where

$$\phi(\mathbf{l},\mathbf{s})=eH(l_x+s_x)(l_y-s_y).$$

Furthermore, by using a technique similar to that used in I, we find

$$I_4=2\pi N(0)\sum_{n,m,p}C_nC_mC_{n-p}^*C_{m+p}^*\exp\left\{-\frac{k^2}{4eH}[(n-m)^2+(n-m-2p)^2]\right\}$$
$$\times\frac{\partial}{\partial\omega}\left[\frac{1}{8}\iiint_{-\infty}^{\infty}d\alpha_1d\alpha_2d\alpha_3\prod_{i=1}^{3}\frac{1}{(\omega-\alpha_i)}I(\alpha_1,\alpha_2,\alpha_3)\right], \quad \text{(A4)}$$

where

$$I(\alpha_1,\alpha_2,\alpha_3)=\frac{1}{V}\int d^3r\int\frac{d\Omega}{4\pi}\frac{1}{v^4\sin^4\theta\cos^4\phi}\frac{\pi^2}{(eH)^2}\cos^2\phi\int_{-\infty}^{\infty}d\alpha_4$$
$$\times\exp\left[-\frac{2i(\alpha_1-\alpha_2+\alpha_3-\alpha_4)r_x}{v\sin\theta\cos\phi}-\frac{(\alpha_1^2+\alpha_3^2)e^{-i\phi}+(\alpha_2^2+\alpha_4^2)e^{i\phi}}{v^2eH\sin^2\theta\cos\phi}\right]$$
$$=\frac{1}{l_x}\int\frac{d\Omega}{4\pi}\frac{\pi^3v}{(v^2eH)^2\sin^3\theta\cos\phi}\int_{-\infty}^{\infty}d\alpha_4\,\delta(\alpha_1+\alpha_3-\alpha_2-\alpha_4)\exp\left[-\frac{(\alpha_1^2+\alpha_3^2)e^{-i\phi}+(\alpha_2^2+\alpha_4^2)e^{i\phi}}{v^2eH\sin^2\theta\cos\phi}\right]$$
$$=\frac{\pi^3}{l_x}v\int\frac{d\Omega}{4\pi}\frac{1}{(v^2eH)^2}\frac{1}{\sin^3\theta\cos\phi}\exp\left[-\frac{(\alpha_1^2+\alpha_3^2)e^{-i\phi}+[\alpha_2^2+(\alpha_1+\alpha_3-\alpha_2)^2]e^{i\phi}}{v^2eH\sin^2\theta\cos\phi}\right]. \quad \text{(A5)}$$

Using Feynman's identity we can transform the integral in (A4) as

$$\iint_{-\infty}^{\infty}\left(\int\prod_{i=1}^{3}\frac{d\alpha_i}{\omega-\alpha_i}\right)I(\alpha_1,\alpha_2,\alpha_3)=\int_{-\infty}^{\infty}d\alpha\frac{I(\alpha)}{(\omega-\alpha)^3}, \quad \text{(A6)}$$

where

$$I(\alpha)=2\iiint_0^1\prod_{i=1}^{3}du_1\,\delta(1-u_1-u_2-u_3)\iiint\prod_{-\infty}^{\infty}d\alpha_i\,\delta(\alpha-u_1\alpha_1-u_2\alpha_2-u_3\alpha_3)I(\alpha_1,\alpha_2,\alpha_3)$$
$$=\frac{\pi^3}{l_x}v\int\frac{d\Omega}{4\pi}\frac{1}{(v^2eH)^2}\frac{2}{\sin^3\theta\cos\phi}\iiint_0^1\prod_{i=1}^{3}du_i\,\delta(1-u_1-u_2-u_3)\int\frac{dk}{2\pi}e^{ik\alpha}\frac{(\sqrt{\pi})^3}{2}(v^2eH)^{3/2}$$
$$\times\cos\phi\left\{-\frac{v^2eH\sin^2\theta k^2}{4}[(u_1^2+u_3^2)(2+e^{2i\phi})+u_2^2(e^{-2i\phi}+2)+2(u_1+u_3)u_2-2u_1u_3e^{2i\phi}]\right\}$$
$$=\frac{\pi^3}{l_x}v\int\frac{d\Omega}{4\pi}\frac{2\sqrt{2}}{v^2eH}\frac{\pi^2}{\sin\theta}\int\prod_{i=1}^{3}du_i\,\delta(1-u_1-u_2-u_3)f(u_i)^{-1/2}\exp\left[-\frac{2\alpha^3}{v^2eH\sin^2\theta f(u_i)}\right], \quad \text{(A7)}$$

where

$$f(u_i)=f(u_1,u_3)=\tfrac{1}{2}[(1-u_1-u_3)^2(1+e^{-2i\phi})+(u_1-u_3)^2(1+e^{2i\phi})+1]. \quad \text{(A8)}$$

We finally arrive at

$$\frac{N(\omega)}{N(0)}=1+\frac{\langle|\Delta|^2\rangle_{\rm av}}{2}\int_{-\infty}^{\infty}\frac{\rho_0(\alpha)d\alpha}{(\omega-\alpha)^2}+\frac{3\langle|\Delta|^4\rangle_{\rm av}}{8}\int_{-\infty}^{\infty}\frac{\rho_1(\alpha)d\alpha}{(\omega-\alpha)^4}, \quad \text{(A9)}$$

where

$$\rho_1(\alpha)=\frac{2}{(\sqrt{\pi})\epsilon}\int\prod_{i=1}^{3}du_1\,\delta(1-u_1-u_2-u_3)\int\frac{d\Omega}{4\pi}\frac{1}{\sin\theta[f(u_i)]^{1/2}}\exp\left[-\left(\frac{\alpha}{\epsilon\sin\theta\sqrt{f}}\right)\right]^2. \quad \text{(A10)}$$

From (A10) it is easy to see that $\rho_1(\alpha)$ has almost the same form as $\rho_0(\alpha)$. In particular, for small α we have

$$\rho_1(\alpha)=\frac{1}{\epsilon}\left[\frac{\sqrt{\pi}}{2}C_0-\frac{|\alpha|}{\epsilon}C_1+\frac{1}{3}\left(\frac{|\alpha|}{\epsilon}\right)^3C_3-\cdots\right], \quad \text{(A11)}$$

where

$$C_n=\frac{1}{\pi}\int_0^{2\pi}d\phi\int_0^1\prod_{i=1}^3 du_i\,\delta(1-u_1-u_2-u_3)f(u_i)^{-(1+n)/2},\tag{A12}$$

Numerically we have

$$C_0\cong 0.90\,,$$
$$C_1=[\ln(1+\sqrt{2})]^2\cong 0.77\,,$$
$$C_3=\frac{1}{\sqrt{2}}\ln(1+\sqrt{2})\cong 0.622\,.\tag{A13}$$

For small ω we have

$$\frac{N(\omega)}{N(0)}=1+\frac{\langle|\Delta|^2\rangle_{\rm av}}{\epsilon^2}\ln\left(\frac{4\gamma\omega}{\epsilon}\right)+\frac{3}{8}\frac{\langle|\Delta|^4\rangle_{\rm av}}{\epsilon^2}\left[\frac{C_1}{3\omega^2}+\frac{2C_3}{3\epsilon^2}\ln\left(\frac{4\gamma\omega}{\epsilon}\right)\right]\quad\text{for}\quad \omega\ll\epsilon.\tag{A14}$$

The coefficients of $\langle|\Delta|^4\rangle_{\rm av}$ diverge more strongly than that of $\langle|\Delta|^2\rangle_{\rm av}$ does. Since C_1 is somewhat smaller than 1, we expect that the analogy we have discussed is not exact (valid only for $\Delta\to 0$).

It is interesting to note that the low-temperature behavior of $f_1(\rho)$, which appears in the definition of the parameter[2] κ_2, is also expressed in terms of C_1 and C_3 given above .We also point out here that the most divergent parts of the higher-order terms in Δ^2 behave like $(\langle|\Delta|^{2n}\rangle_{\rm av}/\epsilon^2)\omega^{-2(n-1)}$ for small ω.

PHYSICAL REVIEW VOLUME 156, NUMBER 2 10 APRIL 1967

Bilinear Reflection of a Double-Frequency Laser Beam from a Superconductor

K. K. GUPTA AND SUDHANSHU S. JHA
Tata Institute of Fundamental Research, Bombay, India
(Received 29 November 1966)

The bilinear current density induced in a superconducting metal by a laser beam with frequencies ω_1 and ω_2 has been calculated. The calculation is done within the framework of the BCS theory of superconductivity at temperature $T=0°$K. It is shown that in the superconducting state of the metal the component of the induced current density, varying with the difference frequency $\Omega=\omega_1-\omega_2$ and the wave vector $\mathbf{Q}=\mathbf{q}_1-\mathbf{q}_2$, where $\mathbf{q}_1$ and $\mathbf{q}_2$ are the wave vectors of the fundamental fields in the metal, differs considerably from the corresponding component in the normal state of the metal when $\hbar\Omega$ is of the order of the energy gap 2Δ. In this paper only that special case is considered where the wave vector $\mathbf{Q}$ is such that $\hbar Qv_f\ll 2\Delta$, v_f being the Fermi velocity of the electrons. If the collision frequency Ω_c^S of the electrons in the superconducting state is small compared to Ω, there is a sharp peak at $\hbar\Omega=2\Delta$ in the energy flux of the light wave of frequency Ω reflected from the surface of the superconductor. For $\hbar\Omega\gg 2\Delta$, the reflectivities are the same for both the normal and the superconducting states of the metal.

1. INTRODUCTION

RECENTLY there has been considerable interest in calculating[1,2] and measuring[3] the reflectivity of the second harmonic wave generated by an intense laser beam incident on the surface of a normal metal. If the incident wave contains a single frequency ω and if it is represented by a plane wave which is polarized perpendicular to the plane of incidence, the amplitude of the bilinear current density induced in the metal, varying as $e^{-2i\omega t}$, has the form

$$\mathbf{J}^{BLN}(2\omega)=-c\alpha(\omega+\omega)\nabla[\mathbf{E}(\omega)\cdot\mathbf{E}(\omega)],\tag{1.1}$$

where $\mathbf{E}(\omega)$ is the amplitude of the electric field in the metal, varying as $e^{-i\omega t}$. When the incident wave is polarized in any other direction, there is an additional term which is nonzero only at the surface[1] of the metal and which arises from the discontinuity in the normal component of the fundamental electric field at the surface. Except near a resonance for the interband transitions in a metal, it has been shown[2] further that one may write

$$\alpha(\omega+\omega)\approx\frac{-ie}{2\pi m^*c(\omega+\omega)}[\epsilon(\omega+\omega)-1],\tag{1.2}$$

[1] S. S. Jha, Phys. Rev. Letters, **15**, 412 (1965); Phys. Rev. **140**, A2020 (1965); **145**, 500 (1966); N. Bloembergen and Y. R. Shen, *ibid.* **141**, 298 (1966); **145**, 390 (1966).
[2] S. S. Jha and C. S. Warke, Phys. Rev. **153**, 751 (1967). In their Eq. (3.39) m should be replaced by m^*, where m^* is the effective mass of the conduction electrons.
[3] F. Brown, R. E. Parks, and A. M. Sleeper, Phys. Rev. Letters **14**, 1029 (1965); F. Brown and R. E. Parks, *ibid.* **16**, 507 (1966); N. Bloembergen, R. K. Chang, and C. H. Lee, *ibid.* **16**, 986 (1966).