Theory of Pure Type-II Superconductors in High Magnetic Fields. I. Difficulties Associated with Expansions in Powers of the Order Parameter

MICHEL CYROT AND KAZUMI MAKI*

Laboratoire de Physique des Solides, associé au Centre National de la Recherche Scientifique, Faculté des Sciences d'Orsay, Orsay, France

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We consider here the transport properties of a pure type-II superconductor in a field close to the upper critical field H_{c2}, where the order parameter $\Delta(\mathbf{r})$ is small and given by Abrikosov's solution. By analyzing the second-order correction in $\Delta(\mathbf{r})$ to the ultrasonic attenuation coefficient, it is established that in the calculation the expansion in powers of $\Delta(\mathbf{r})$ will not be valid. The above situation is in strong contrast to the case of dirty type-II superconductors.

I. INTRODUCTION

FROM a theoretical point of view, it is often convenient to classify type-II superconductors in two groups: clean (or pure) ones, e.g., pure intrinsic superconductors such as Nb and V, and dirty ones, including most extrinsic superconductors such as alloys and metallic compounds. In the field region close to the upper critical field where the order parameter is small, it is natural to consider $\Delta(\mathbf{r})$ as a small parameter in discussions of equilibrium as well as nonequilibrium properties of type-II superconductors. In fact, in the dirty limit such expansions in powers of $\Delta(\mathbf{r})$ are possible and the transport properties are explained in terms of gapless superconductors.[1–4] However, in the clean limit we expect there are differences in electronic properties from those in the dirty limit. For example, we know that the κ_2 parameter, which appears in the expression of the magnetization,

$$-4\pi M=(H_{c2}-H_0)/(2\kappa_2{}^2(t)-1)\beta_A,\quad \beta_A=1.16 \quad (1)$$

diverges like $\ln t^{-1}$ as $t=T/T_{c0}$ approaches zero in the clean limit,[5] whereas in the dirty limit, κ_2 converges a finite value at $t=0$.

Recently de Gennes *et al.*,[6] in their calculation of the density of states up to the second-order terms in the order parameter, found that the density of states in a pure type-II superconductor in the high-field region has a logarithmic singularity at zero excitation energy, and suggested a possible way of detecting this anomaly through the measurement of the nuclear spin relaxation rate. Later, Juranek *et al.*[7] derived a similar density of states by a more direct calculation.

The purpose of the present paper is to point out a peculiar difficulty in the perturbation expansion in powers of $\Delta(\mathbf{r})$, which was masked to a certain extent in the calculation of the equilibrium properties[5–8] and now is revealed in a striking way in the calculation of the nonequilibrium properties (e.g., ultrasonic attenuation coefficients). We assume here for simplicity that the electronic mean free path l is infinite, though we believe the general conclusion retains validity if $l\gg\xi_0$, where ξ_0 is the coherence length.

As is well known, in the high field region the order parameter $\Delta(\mathbf{r})$ is given by Abrikosov's solution[5] (for $\mathbf{H}\|z$ axis):

$$\Delta(\mathbf{r})=\sum_{n=-\infty}^{\infty} C_n e^{ikny}\psi_n(x)\,,$$
$$\psi_n(x)=\exp[-eH(x-kn/2eH)^2]\,, \quad (2)$$

where C_n and k are constants.

A simple spectral representation of the second-order diagrams in $\Delta(\mathbf{r})$, which represent correction terms to thermal products, is obtained in the next section. Making use of this representation, it is not difficult to calculate various transport coefficients. As a simple example, we consider the ultrasonic attenuation coefficient of the longitudinal wave. It turns out that the coefficient of $\langle|\Delta(\mathbf{r})|^2\rangle_{\rm av}$ in the attenuation coefficient is exactly zero. A similar result can be obtained for the case of thermal conductivity. We interpret the above results as an indication of the fact that the power-series expansion does not apply even when T is close to T_c.

II. SECOND-ORDER EFFECT IN $\Delta(r)$

The transport properties of the system are usually expressed in terms of retarded products (e.g., electric conductivity is given in terms of retarded products of the current operator). Since retarded products are obtained

* On leave of absence from Department of Physics, University of California, San Diego, La Jolla, California, and from Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan. Present address: Department of Physics, University of California, San Diego, La Jolla, California.

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[3] C. Caroli and M. Cryot, Physik Kondensierten Materie 4, 285 (1965).

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[6] Groupe de Superconductivité d'Orsay, Physik Kondensierten Materie **5**, 141 (1966).

[7] H. J. Juranek, L. Neumann, and L. Tewordt, Z. Physik **173**, 459 (1966).

[8] In Ref. 5, it is noted that the expansion of the free energy in powers of $\Delta(\mathbf{r})$ breaks down at $T=0$.

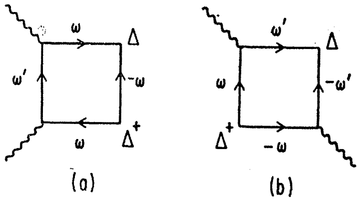

FIG. 1. Typical diagrams which appear in the calculation of the various correlation functions.

from corresponding thermal products by analytical continuation,[9] we shall consider here the correction terms due to the existence of small $\Delta(\mathbf{r})$ to the thermal products. In the following, we shall restrict our consideration to the thermal products of the density operator for definiteness, though similar reasoning applies to any of the thermal products. The thermal products of the density operator are expressed in terms of Green's functions as

$$\langle[n,n]\rangle(\mathbf{q},\nu)=\frac{T}{V}\sum_n\int d^3r\int d^3r'\,e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}[G_{\omega_n}(\mathbf{r},\mathbf{r}')\times G_{\omega_n'}(\mathbf{r}'\cdot\mathbf{r})+F_{\omega_n}(\mathbf{r}\cdot\mathbf{r}')F_{\omega_n'}{}^*(\mathbf{r}'\cdot\mathbf{r})],\quad(3)$$

where $G_\omega(\mathbf{r},\mathbf{r}')$ and $F_\omega(\mathbf{r},\mathbf{r}')$ can be expanded in powers of Δ as[10]

$$G_\omega(\mathbf{r},\mathbf{r}')=G_\omega{}^0(\mathbf{r},\mathbf{r}')-\int G_\omega{}^0(\mathbf{r},\mathbf{l})\Delta(\mathbf{l})G_{-\omega}{}^0(\mathbf{s},\mathbf{l})\times\Delta^\dagger(\mathbf{s})G_\omega{}^0(\mathbf{s},\mathbf{r}')d^3l d^3s,\quad(4)$$

$$F_\omega(\mathbf{r},\mathbf{r}')=\int G_\omega{}^0(\mathbf{r},\mathbf{l})\Delta(\mathbf{l})G_{-\omega}{}^0(\mathbf{r}',\mathbf{l})d^3l,$$

and

$$G_\omega{}^0(\mathbf{r},\mathbf{s})=\exp\left[ie\int_{\mathbf{r}}^{\mathbf{s}}\mathbf{A}(\mathbf{l})\cdot d\mathbf{l}\right]\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}}{i\omega-\xi},\quad(5)$$

which is the Green's function in the normal metal.

The vector potential $\mathbf{A}(\mathbf{r})$ is given now by $\mathbf{A}(\mathbf{r})=(0,Hx,0)$ and

$$\omega_n=2\pi T(n+\tfrac{1}{2}),\quad \omega_n'=\omega_n-\omega_\nu\quad(6)$$

and $\xi=p^2/2m-\mu$. We have assumed here that the external field $\mathbf{H}$ is directed along the z axis.

Substituting Eqs. (4) and (5) in Eq. (3), we have two different integrals, which give rise to the second-order terms in $\Delta(\mathbf{r})$ as given in the diagrams in Fig. 1.

First, we shall discuss the integral corresponding to Fig. 1 (a), which comes from the $G_\omega G_{-\omega'}$ term in Eq. (3):

$$I_a(\mathbf{q},\nu)=\frac{T}{V}\sum_n\int d^3r\int d^3s\int d^3l\times\int d^3r'\,e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}G_{\omega_n}{}^0(\mathbf{r},\mathbf{s})\Delta(\mathbf{s})G_{-\omega_n}{}^0(\mathbf{l},\mathbf{s})\times\Delta^\dagger(\mathbf{l})(G_{\omega_n}(\mathbf{l},\mathbf{r}')G_{\omega_n}{}'(\mathbf{r}',\mathbf{r}).\quad(7)$$

In the above expression $I_a(\mathbf{q},\nu)$ is the Fourier transform of $I_a(\mathbf{r},\mathbf{r}')$, for $\mathbf{r}-\mathbf{r}'$, which is averaged over the bulk specimen of volume V.

We transform the integral as

$$I_a(\mathbf{q},\nu)=\frac{T}{V}\sum_n\int d^3r\int d^3s\int d^3l\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\times(i\omega_n-\xi)^{-1}(i\omega_n+\xi+\mathbf{v}\cdot\mathbf{k})^{-1}[i\omega_n-\xi-\mathbf{v}\cdot(\mathbf{k}+\mathbf{k}')]^{-1}[i\omega_n'-\xi-\mathbf{v}\cdot(\mathbf{k}+\mathbf{k}'+\mathbf{q})]^{-1}\times\exp\left[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{s})+i\mathbf{k}'\cdot(\mathbf{r}-\mathbf{l})+2ie\int_{\mathbf{l}}^{\mathbf{s}}\mathbf{A}(\mathbf{l}')d\mathbf{l}'\right]\Delta(\mathbf{s})\Delta^\dagger(\mathbf{l})$$

$$=T\sum_n\int\frac{d^3p}{(2\pi)^3}\int_{-\infty}^{\infty}d\alpha_1\int_{-\infty}^{\infty}d\alpha_2(i\omega_n-\xi)^{-1}(i\omega_n+\xi+2\alpha_1)^{-1}[i\omega_n-\xi-2(\alpha_1+\alpha_2)]^{-1}\times[i\omega_n'-\xi-2(\alpha_1+\alpha_2)-\mathbf{v}\cdot\mathbf{q}]^{-1}I(\alpha_1,\alpha_2),\quad(8)$$

where

$$I(\alpha_1,\alpha_2)=\frac{4}{V}\int d^3r\int d^3s\int d^3l\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\times\left[\exp\left(i\mathbf{k}(\mathbf{r}-\mathbf{s})+ik\cdot'(\mathbf{r}-\mathbf{l})+2ie\int_{\mathbf{l}}^{\mathbf{s}}\mathbf{A}(\mathbf{l}')\cdot d\mathbf{l}'\right)\Delta(\mathbf{s})\Delta^\dagger(\mathbf{l})\delta(2\alpha_1-\mathbf{v}\cdot\mathbf{k})\delta(2\alpha_2-\mathbf{v}\cdot\mathbf{k}')\right]$$

$$=\frac{4|C|^2}{V}\int d^3r\int ds_x\int dl_x\int\frac{dk_x}{2\pi}\int\frac{dk'_x}{2\pi}e^{ik_x(r_x-s_x)+ik'_x(r_x-l_x)}\times\delta(2\alpha_1-v\sin\theta\,[k_x\cos\phi+eH\sin\phi(s_x+l_x)])\delta(2\alpha_2-v\sin\theta\,[k_x'\cos\phi-eH\sin\phi(s_x+l_x)])e^{-eH(s_x{}^2+l_x{}^2)}.\quad(9)$$

[9] See, for example, A. A. Abrikosov, L. G. Prokov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1963).

[10] L. P. Gor'kov, Zh. Eksperim. i. Teor. Fiz. 36, 1918 (1959) [English transl.: Soviet Phys.—JETP 9, 1364 (1959)].

Here we have substituted for $\Delta(\mathbf{r})$ the expression

$$\Delta(\mathbf{r})=Ce^{-eHx^2}, \quad (10)$$

without loss of generality, thanks to the orthogonality of the function. v is the Fermi velocity and θ,φ are polar coordinates describing $\mathbf{v}$ where the polar axis is taken along the z axis (i.e., in the field direction). Furthermore, in the above transformation we have made use of the expression

$$2ie\int_{1}^{s}\mathbf{A}(\mathbf{l}')d\mathbf{l}'=ieH(s_x+l_x)(s_y-l_y). \quad (11)$$

The final integrations are easily carried out and we have

$$I(\alpha_1,\alpha_2)=\frac{|C|^2}{V\pi}\int d^3r\frac{1}{eHv^2\sin^2\theta\cos\phi}\times\exp\left(-\frac{2i(\alpha_1+\alpha_2)r_x}{v\sin\theta\cos\phi}-\frac{\alpha_1^2e^{-i\phi}+\alpha_2^2e^{i\phi}}{v^2eH\sin^2\theta\cos\phi}\right)$$
$$=\frac{|C|^2}{Lx}\delta(\alpha_1+\alpha_2)\frac{1}{veH}\frac{1}{\sin\theta}\times\exp[-2(\alpha_1/v(eH)^{1/2}\sin\theta)^2], \quad (12)$$

where we set $V=L_xL_yL_z$.

For the more general expression of $\Delta(\mathbf{r})$ given in Eq. (2), we have

$$I_a(\mathbf{q},\nu)=\langle|\Delta|^2\rangle_{\mathrm{av}}T\sum_n\int\frac{d^3p}{(2\pi)^3}\int_{-\infty}^{\infty}\rho(\alpha,\Omega)d\alpha\times(i\omega_n+\alpha-\xi)^{-1}(i\omega_n+\alpha+\xi)^{-1}\times(i\omega_n+\alpha-\xi^{-1})(i\omega_n'+\alpha-\xi-\mathbf{v}\cdot\mathbf{q})^{-1}, \quad (13)$$

where the angular-dependent spectral function is given by

$$\rho(\alpha,\Omega)=\frac{1}{(\sqrt{\pi})\epsilon\sin\theta}\exp\left[-\left(\frac{\alpha}{\epsilon\sin\theta}\right)^2\right],\quad \epsilon=(v/\sqrt{2})(eH_{c2})^{1/2}. \quad (14)$$

Here we have replaced ξ in Eq. (8) by $(\xi-\alpha)$, which corresponds to the change of integral variable $\mathbf{p}$ to $\mathbf{p}+\mathbf{k}$.

From Eq. (8) we immediately see that the above integral has the same form as that corresponding to the case of a superconductor carrying a uniform current, except that the spectral function $\rho(\alpha,\Omega)$ is now given by Eq. (14). In the case of a uniform current we have[11]

$$\rho_u(\alpha,\Omega)=\delta(\alpha-\mathbf{v}\cdot\mathbf{q}_s), \quad (15)$$

where $\mathbf{q}_s$ is the momentum of the condensed pair. We note that the spectral function given in Eq. (14) has a sharp peak for $\theta=0$ (or, in the direction of the field), which signifies that the quasiparticle running parallel to the field has a singular density of states similar to the one in the BCS state.

Second, the integral corresponding to Fig. 1 (b), which comes from $F_\omega F_{-\omega'}$ terms in Eq. (3), is computed in a similar way and we obtain

$$I_b(\mathbf{q},\nu)=\frac{T}{V}\sum_n\int d^3r\int d^3s\int d^3l\int d^3r'\, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}G_{\omega_n}(\mathbf{r}\cdot\mathbf{s})\Delta(\mathbf{s})G_{-\omega_n}(\mathbf{r}',\mathbf{s})G_{-\omega_n'}(\mathbf{l},\mathbf{r}')\Delta^\dagger(\mathbf{l})G_{\omega_n'}(\mathbf{l},\mathbf{r})$$
$$=\langle|\Delta|^2\rangle_{\mathrm{av}}T\sum_n\int\frac{d^3p}{(2\pi)^3}\int_{-\infty}^{\infty}\rho(\alpha,\Omega)d\alpha(i\omega_n+\alpha-\xi)^{-1}(i\omega_n+\alpha+\xi)^{-1}(i\omega_n'+\alpha+\xi+\mathbf{v}\cdot\mathbf{q})^{-1}(i\omega_n'+\alpha-\xi-\mathbf{v}\cdot\mathbf{q})^{-1}, \quad (16)$$

where $\rho(\alpha,\Omega)$ is again given in Eq. (14).

It might be worthwhile to note that the essential difference between I_a and I_b is due to the different ways ω and ω' appear in the denominators, which is obvious from the diagrams.

III. ULTRASONIC ATTENUATION COEFFICIENT

The above analysis can be carried out for any thermal product, such as that of the density correlation, current correlation, etc. Here we shall consider the ultrasonic attenuation coefficient of the longitudinal wave for $ql\gg1$, which is directly related to the thermal product of the density operators. Here q and l are the wave number of the sound wave and the electronic mean free path, respectively. In this case the attenuation coefficient is given by[12,13]

$$\alpha=\mathrm{Im}\left\{\frac{q^2}{\omega\rho_{\mathrm{ion}}v_s}\left(\frac{p_0^2}{3m}\right)^2\langle[n,n]\rangle(\mathbf{q},\omega)\right\}, \quad (17)$$

where p_0 is the Fermi momentum. $\langle[n,n]\rangle(\mathbf{q},\omega)$ can be obtained from $\langle[n,n]\rangle(\mathbf{q},\nu)$ by analytical continuation. Gathering here the results of the previous section and carrying out the integration over ξ (here we substitute

[11] The above spectral function is easily derived from the Green's functions given in K. Maki and T. Tsuneto, Progr. Theoret. Phys. (Kyoto) **27**, 228 (1962).

[12] T. Tsuneto, Phys. Rev. **121**, 402 (1961).

[13] L. P. Kadanoff and I. I. Falko, Phys. Rev. **136**, A1170 (1964).

$d^3p/(2\pi)^3 = [mp_0/(2\pi)^3]d\xi d\Omega$, we have

$$\langle[n,n]\rangle(q,\nu)=\frac{mp_0}{2\pi^2}\left\{1+\int\frac{\alpha\Omega}{4\pi}\frac{1}{i\mathbf{v}\cdot\mathbf{q}}\pi T\sum_n\left[1-\frac{\omega_n\cdot\omega_n'}{|\omega_n||\omega_n'|}-\frac{\langle|\Delta|^2\rangle_{\text{av}}}{2}\right.\right.$$
$$\left.\left.\times\int_{-\infty}^{\infty}\rho(\alpha,\Omega)d\alpha\left(\frac{1}{(\omega_n-i\alpha)^2}+\frac{1}{(\omega_n'-i\alpha)^2}-\frac{2}{(\omega_n-i\alpha)(\omega_n'-i\alpha)}\right)\right]\right\}$$
$$=\frac{mp_0}{2\pi^2}\left\{1+\int\frac{d\Omega}{4\pi}\pi\delta(\mathbf{v}\cdot\mathbf{q})\left[\omega\nu-\frac{2\langle|\Delta|^2\rangle_{\text{av}}}{(2\pi T)^2}I(\omega_\nu,\Omega)\right]\right\},\qquad(18)$$

where again $\omega_n=2\pi T(\eta+\frac{1}{2})$, $\omega_n'=\omega_n-\omega_\nu$, and

$$I(\omega_\nu,\Omega)=\int_{-\infty}^{\infty}d\alpha\,\rho(\alpha,\Omega)\left\{\psi'\left(\frac{1}{2}+\frac{\omega_\nu}{2\pi T}+\frac{i\alpha}{2\pi T}\right)-2\frac{2\pi T}{\omega_\nu}\left[\psi\left(\frac{1}{2}+\frac{\omega_\nu}{2\pi T}+\frac{i\alpha}{2\pi T}\right)-\psi\left(\frac{1}{2}+\frac{i\alpha}{2\pi T}\right)\right]\right\}.\qquad(19)$$

For a low-frequency sound wave ($\omega\ll\pi T_c$), we have (after analytical continuation)

$$\langle[n,n]\rangle(\mathbf{q},\omega)=\frac{mp_0}{2\pi^2}\left\{1+\int\frac{\alpha\Omega}{4\pi}\pi\delta(\mathbf{v}\cdot\mathbf{q})\times\left[i\omega-\frac{2\langle|\Delta|^2\rangle_{\text{av}}}{(2\pi T)^2}I(0,\Omega)+O(\omega^2)\right]\right\},\qquad(20)$$

where

$$I(0,\Omega)=\int_{-\infty}^{\infty}d\alpha\,\rho(\alpha,\Omega)\psi'\left(\frac{1}{2}+\frac{i\alpha}{2\pi T}\right).\qquad(21)$$

We shall point out here two unsatisfactory features of the above result.

(a) $I(O,\Omega)$ is finite. Since the screening of the electric charge is described by the equation

$$-\mathbf{q}^2\Phi_\mathbf{q}=4\pi\langle[n,n]\rangle(\mathbf{q},0)\Phi_\mathbf{q},\qquad(22)$$

where $\Phi_\mathbf{q}$ is the Fourier transform of the scalar potential, the above result indicates that the screening behavior is modified drastically because of the set in of superconductivity contrary to our experience. For the BCS case with $\Delta=$constant, the coefficient of $|\Delta|^2$ vanishes identically for $\omega\to0$.

(b) The imaginary part of $I(\omega,\Omega)$, which is proportional to ω, vanishes identically. This implies [see Eq. (17)] that the attenuation coefficient is not affected up to Δ^4. However, we know that in the BCS state, because of the singularity at $\omega=2\Delta$ in $\langle[n,n]\rangle(q,\omega)$, the expansion as used above gives unphysical results, as is easily seen from the fact that in the BCS state the attenuation coefficient is given by

$$\alpha_L{}^s/\alpha_L{}^n=2f(\Delta/T),\qquad(23)$$
$$f(x)=(e^x+1)^{-1},$$

which cannot be expanded in powers of Δ^2.

IV. CONCLUDING REMARKS

We have seen above that the calculation based on the perturbation in powers of $\Delta(\mathbf{r})$, which is useful in the dirty type-II superconductors, gives rise to an unreasonable result for the ultrasonic attenuation coefficients of a pure type-II superconductor. In order to avoid this uncomfortable conclusion, we suggest that in the case of a pure type-II superconductor, the density of states has a strong irregularity for $\omega\sim\Delta$, which inhibits a simple perturbational approach as given above.

It is instructive to remember that in the case of the simple BCS state we have already encountered a similar situation; the perturbational expansion does not reproduce Eq. (23). Therefore, in the study of the transport properties of a pure type-II superconductor, a more careful analysis of the analytical behavior of higher-order terms in $\Delta(\mathbf{r})$ is required. We mention also that the difficulty we meet here is of a similar nature to those in deriving the time-dependent Ginzburg-Landau equations.[14]

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[14] E. Jakeman and E. R. Pike, Phys. Letters **20**, 533 (1966). Other references can be found in this letter.