Quantum Theory of a Gas Laser*

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We derive the equation of motion for the quantum-mechanical radiation density matrix of a gas laser to lowest order in the dimensionless coupling constant. Our derivation is fully quantum mechanical and we can calculate the coherence properties from the radiation density matrix. Our model consists of N two-level systems interacting with radiation in a cavity in the presence of dissipation, pumping, and collisions. The method we use is a generalization of the Bogoliubov derivation of the kinetic equation for a small parameter. Our derivation holds for any physically realizable pump power, and near threshold reduces to Lamb's near-threshold theory. With the equation of motion for the radiation density matrix, we obtain solutions both for when the average field is nonzero and for when it is zero. The steady-state electromagnetic density is the same in both cases except for a small spontaneous-emission term. We show that the reason gas lasers do not satisfy rate equations is the existence of zeroth-order correlations between the internal atomic variables and atomic center-of-mass variables. It is these same zeroth-order correlations which are responsible for the Lamb dip. Our derivation includes collisions and reduces the calculations of their effect to quadrature.

I. INTRODUCTION

In this paper we derive the equation of motion for the quantum-mechanical radiation density matrix of a gas laser to lowest order in the dimensionless coupling constant. Our deviation holds for all physically realizable pump power and includes Lamb's theory[1] as a special case. We find exact solutions in special cases for arbitrary power levels.

In a previous paper[2] we derived and partially solved the kinetic equations for the single-particle density matrix and the electromagnetic-field density matrix for a system of N two-level systems interacting with radiation in a cavity. We included dissipation, pumping, and center-of-mass motion. The method we used was a generalization of the Bogoliubov[3] derivation of the kinetic equations for a small parameter. However, all methods of derivation yield the same result at least to lowest order in the dimensionless radiation-matter coupling constant γ. In I we assumed that the center-of-mass variables were initially uncorrelated with both the internal atomic variables and the radiation variables. We further assumed that the velocity distribution of the center-of-mass motion was given, which meant that we neglected recoil on absorption and emission. One of the consequences of the present paper is to show explicitly that the recoil terms are small. The assumption of no zeroth-order initial correlation between center-of-mass variables and internal atomic variables led directly to the result that the average electromagnetic energy and average particle occupation numbers satisfied rate equations.

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[1] W. E. Lamb, Phys. Rev. **134**, A1429 (1964).

[2] C. R. Willis, Phys. Rev. **147**, 406 (1966); hereafter referred to as I.

[3] N. N. Bogoliubov, in *Studies in Statistical Mechanics*, edited by J. de Boer and G. E. Uhlenbeck (North-Holland Publishing Company, Amsterdam, 1962), pp. 5–118.

We now assume only that the matter and radiation variables are initially uncorrelated; i.e., there are no initial zeroth-order correlations between center-of-mass variables and internal atomic variables. Two of the consequences of this zeroth-order correlation are that the average electromagnetic energy and particle occupation numbers do not satisfy rate equations and that there is a Lamb dip in the power-versus-detuning curve.

We use the generalized Bogoliubov derivation of the kinetic equations developed in I. However, now the single-particle density matrix ρ is an operator in the two-dimensional internal variable space and also a density matrix in the center-of-mass variables. A typical matrix element is $\rho_{-+}(x,x',t)$, where $+$ represents the excited state, $-$ the ground state of the two-level system, and (x,x') indicates the dependence of matrix elements on the center-of-mass variables x. Although the center-of-mass motion is classical, it is convenient to treat it quantum mechanically and in the last steps of the derivation take the classical limit of the center-of-mass motion. As a consequence we have the operator equation of motion for $\rho(x,v,t)$, where v is the classical velocity of the center of mass.

First we obtain the coupled kinetic equations for ρ and the radiation density matrix R. The kinetic equation for ρ depends on the electromagnetic field variables through the average electromagnetic energy, $h\Omega\langle a^\dagger a\rangle$, where Ω is the cavity frequency and $a^\dagger$ and a are the usual creation and annihilation operators for the electromagnetic field. We formally solve the kinetic equation of motion for ρ, substitute the result in the kinetic equation for R and obtain a nonlinear equation for R alone. The nonlinearity arises because R now depends on $\langle a^\dagger a\rangle$ which is $\mathrm{Tr} a^\dagger a R$, where the trace is over a complete set of variables for the electromagnetic field. Next we find $\langle a^\dagger a\rangle$ by multiplying the equation of motion for R by $a^\dagger a$ and taking the trace. In this manner we obtain a closed nonlinear differential equation for $\langle a^\dagger a\rangle$. We are able to solve the equation of motion for $\langle a^\dagger a\rangle$ exactly in special cases.

Since we obtain the kinetic equation for the radiation density matrix R, we can evaluate all moments and solve for the coherence properties of our model. Furthermore, we show that the steady-state electromagnetic energy density is the same whether or not the electromagnetic fields $\langle a\rangle$ and $\langle a^\dagger\rangle$ are zero or nonzero. Our derivation is fully quantum mechanical and holds for all physically realizable pump powers. For pump power slightly above threshold our results reduce to Lamb's near-threshold theory. In particular, we show that if our results are expanded to first order in Lamb's "saturation parameter," then we obtain Lamb's near-threshold theory. We include collisions and reduce the treatment of collisions to quadrature.

In Sec. II we derive the kinetic equation for R. Section III contains the derivation of the kinetic equation for ρ. In Sec. IV we obtain the equation of motion for $\langle a^\dagger a\rangle$, and in Sec. V we find the stationary solutions for $\langle a^\dagger a\rangle$. We find the equations of motion for $\langle a^\dagger\rangle$ and $\langle a\rangle$ in Sec. VI and show our results reduce to Lamb's near threshold. Section VII is a comparison of our equation with rate equations. In Sec. VIII we discuss higher-order kinetic equations. Appendix A contains a derivation of the cavity frequency shift valid for all pump power. In Appendix B we show that the average electric field vanishes because of spontaneous emission.

II. DERIVATION OF THE KINETIC EQUATION FOR R

Our Hamiltonian for N two-level systems interacting with a single mode of the electromagnetic field is

$$H(N)=h(N)+H_f+H_{\text{c.m.}}+H_1\,, \tag{2.1}$$

where

$$h(N)=\frac{\hbar\omega_0}{2}\sum_{\alpha}^{N}\hat{\sigma}_\alpha\,;\quad H_f=\hbar\Omega(a^\dagger a+\tfrac{1}{2})\,,$$

$$H_1=\hbar\omega_0\gamma\sum_\alpha\Gamma(X_\alpha)[a^\dagger\sigma_\alpha+a\sigma_\alpha{}^\dagger]\,,$$

$$H_{\text{c.m.}}=\sum_\alpha\frac{P_\alpha{}^2}{2m}+\tfrac{1}{2}\sum_{\alpha,\beta}V(X_\alpha-X_\beta)+\sum_\alpha^N\sum_i^N U(X_\alpha-\eta_i)\,,$$

$$\gamma=(\hbar\omega_0)^{-1}(\hbar\Omega)^{1/2}e\langle a|\epsilon\cdot r|b\rangle(4\pi/V)^{1/2};$$

$$\Gamma(X_\alpha)=E(X_\alpha)V^{1/2}.$$

The normalized eigenfunction of the cavity corresponding to the frequency Ω evaluated at the position of the αth particle is $E(X_\alpha)$. A full discussion of the terms in the Hamiltonian is given in I.

In I we showed, by means of a straightforward generalization of the Bogoliubov[2] derivation of the kinetic equations, that the solution of the Liouville equation for our total system to order γ^2 is

$$\frac{\partial R}{\partial t}=-\frac{i}{\hbar}[H_f,R]+\mathcal{K}_rR-\gamma^2N\omega_0{}^2\,\mathrm{Tr}_\sigma\,\mathrm{Tr}_x\int_0^\infty d\tau \times[H_1,[H_1(\tau),R\rho\mathfrak{F}]]\,, \tag{2.2a}$$

$$\frac{\partial \rho}{\partial t}=-\frac{i}{\hbar}[h(1),\rho]+\mathcal{K}_{\text{int}}\rho-\gamma^2\omega_0{}^2\,\mathrm{Tr}_a\,\mathrm{Tr}_x\int_0^\infty d\tau \times[H_1,[H_1(\tau),R\rho\mathfrak{F}]]\,. \tag{2.2b}$$

We assumed the matter density matrix was a product $\rho\mathfrak{F}$ of the density matrix ρ for the internal variables and the center-of-mass density matrix $\mathfrak{F}$. The operator $\mathcal{K}_r$ refers to the interaction of the radiation with the radiation reservoir and $\mathcal{K}_{\text{int}}$ refers to the interaction of the internal variables with the pump and matter reservoir. The operators $\mathcal{K}_r$ and $\mathcal{K}_{\text{int}}$ are of the same general structure as the Wangsness-Bloch[4–6] reservoir operators. The symbols Tr_σ, Tr_x, Tr_a refer to traces over a complete set of variables for the internal atomic variables, the center of mass, and the electromagnetic field, respectively.

The product assumption $\rho(x,t)=\rho(t)\mathfrak{F}(x,t)$ in Eqs. (2.2a) and (2.2b) implies that there are no zeroth-order correlations between internal atomic variables and center-of-mass variables. It is necessary to treat the interaction with the reservoirs with more care when we retain correlations between internal atomic and center of mass variables. Consequently, when we repeat the derivation leading to Eqs. (2.2a) and (2.2b) without the product assumption on the matter density matrix we obtain

$$\frac{\partial R}{\partial t}=-\frac{i}{\hbar}[H_f,R]+\mathcal{K}_rR-(\gamma\omega_0)^2N\,\mathrm{Tr}_\sigma\,\mathrm{Tr}_x\,\mathrm{Tr}_{\text{res}} \times\int_0^\infty d\tau[H_1,[H_1(\tau),R\rho(x,t)\mathbf{P}]]\,, \tag{2.3a}$$

$$\frac{\partial \rho(x,t)}{\partial t}=-\frac{i}{\hbar}[h(1),\rho(x,t)]-\frac{i}{\hbar}\left[\frac{P^2}{2m},\rho(x,t)\right] +\mathcal{K}_{\text{int}}\rho(x,t)+\mathcal{L}_c(x,t)\rho(x,t)-(\gamma\omega_0)^2\,\mathrm{Tr}_a\,\mathrm{Tr}_{\text{res}} \times\int_0^\infty d\tau[H_1[H_1(\tau),R\rho(x,t)\mathbf{P}]]\,, \tag{2.3b}$$

where we have explicitly introduced the matter and radiation reservoir density matrix $\mathbf{P}$. The symbol Tr_{res} stands for a trace over a complete set of states of the matter and radiation reservoir. The operator $\mathcal{L}_c$ represents a collision operator for the center-of-mass variables. In a gas laser the density is sufficiently low so

[4] R. K. Wangsness and F. Bloch, Phys. Rev. **89**, 728 (1953).
[5] C. R. Willis and P. G. Bergmann, Phys. Rev. **128**, 391 (1962).
[6] W. Weidlich and F. Haake, Z. Physik **185**, 30 (1965); **186**, 203 (1965).

that $\mathcal{L}_c$ is a linear Boltzmann operator representing collisions with both system atoms and pump atoms. To be precise we have carried out the derivation of the kinetic equations γ^2 approaches zero at the same time as we let the density of atoms approach zero.

The symbol $H_1(\tau)$ represents the radiation-matter interaction at time τ where the development in time is generated by all terms in the Hamiltonian except the radiation-matter interaction potential. The reservoir Hamiltonian and the reservoir-system Hamiltonian contribute to $H_1(\tau)$. To clarify the meaning of $H_1(\tau)$ we consider the special case of no reservoirs and no collisions. Then $H_1(\tau)$ is

$$H_1(\tau)=\Gamma[X_{\mathrm{op}}+(P_{\mathrm{op}}/m)\tau][a^\dagger\sigma e^{i\Delta\tau}+a\sigma^\dagger e^{-i\Delta\tau}],$$

where

$$\exp(iH_f\tau)a^\dagger\exp(-iH_f\tau)=a^\dagger\exp(-i\Omega\tau),$$
$$\exp[ih(1)\tau]\sigma\exp[-ih(1)\tau]=\sigma\exp(i\omega_0\tau),$$
$$X(\tau)=X_{\mathrm{op}}+(P_{\mathrm{op}}/m)\tau;\quad \Delta=\omega_0-\Omega,$$

and P_{op} is the momentum operator for the center of mass. The mass m is the mass of the two-level system.

When we take the trace over the reservoir of the double commutators in Eqs. (2.3a) and (2.3b) we obtain

$$\mathrm{Tr}_{\mathrm{res}}[H_1,[H_1(\tau)R\rho\mathbf{P}]]=H_1\langle H_1(\tau)\rangle_P R\rho-\langle H_1(\tau)\rangle_P R\rho H_1$$
$$-H_1R\rho\langle H_1(\tau)\rangle_P+R\rho\langle H_1(\tau)\rangle_P H_1, \quad (2.4)$$

where

$$\langle H_1(\tau)\rangle_P\equiv\Gamma[X(\tau)](\langle a^\dagger(\tau)\sigma(\tau)\rangle_P+\langle a(\tau)\sigma^\dagger(\tau)\rangle_P)$$
$$=\Gamma[X(\tau)](\langle a^\dagger(\tau)\rangle_P\langle\sigma(\tau)\rangle_P+\langle a(\tau)\rangle_P\langle\sigma^\dagger(\tau)\rangle_P)$$

and the definition of the average $\langle O\rangle_P$ of any operator O is

$$\langle O\rangle_P\equiv\mathrm{Tr}_{\mathrm{res}}O\mathbf{P}.$$

The reservoir average $\langle a^\dagger(\tau)\sigma(\tau)\rangle_P$ can be written as a product $\langle a^\dagger(\tau)\rangle_P\langle\sigma(\tau)\rangle_P$ because the radiation and matter reservoirs are independent of each other.

The equations of motion satisfied by $\langle a^\dagger\rangle_P$, $\langle a\rangle_P$, $\langle\sigma^\dagger\rangle_P$, and $\langle\sigma\rangle_P$ are[7,8]

$$d\langle a^\dagger\rangle_P/dt=-i\Omega\langle a^\dagger\rangle_P-(\nu_r/2)\langle a^\dagger\rangle_P, \quad (2.5a)$$
$$d\langle a\rangle_P/dt=i\Omega\langle a\rangle_P-(\nu_r/2)\langle a\rangle_P, \quad (2.5b)$$
$$d\langle\sigma^\dagger\rangle_P/dt=i\omega_0\langle\sigma^\dagger\rangle_P-\nu_2\langle\sigma^\dagger\rangle_P, \quad (2.5c)$$
$$d\langle\sigma\rangle_P/dt=-i\omega_0\langle\sigma\rangle_P-\nu_2\langle\sigma\rangle_P. \quad (2.5d)$$

When we substitute the solutions of Eqs. (2.5) in the definition of $H_1(\tau)$ we obtain

$$\langle H_1(\tau)\rangle_P=\Gamma[X(\tau)]\exp-\tau[\nu_2+(\nu_r/2)]$$
$$\times\{a^\dagger\sigma e^{i\Delta\tau}+a\sigma^\dagger e^{-i\Delta\tau}\}, \quad (2.6)$$

where we have used

$$\langle a^\dagger(O)\rangle_P=a^\dagger,\ \langle a(O)\rangle_P=a,\ \langle\sigma^\dagger(O)\rangle_P=\sigma^\dagger,\ \langle\sigma(O)\rangle_P=\sigma.$$

When we substitute Eq. (2.6) in the double commutator appearing in Eqs. (2.3a) and (2.3b), we obtain

$$[H_1,[\langle H_1(\tau)\rangle_P,R\rho]]=-\exp[-(\nu_2-i\Delta)\tau]a^\dagger Ra^\dagger(\Gamma\sigma\rho\sigma\Gamma(\tau)+\Gamma(\tau)\sigma\rho\sigma\Gamma)$$
$$-\exp[-(\nu_2+i\Delta)\tau]aRa(\Gamma\sigma^\dagger\rho\sigma^\dagger\Gamma(\tau)+\Gamma(\tau)\sigma^\dagger\rho\sigma^\dagger\Gamma)+\exp[-(\nu_2+i\Delta)\tau](\sigma\sigma^\dagger\Gamma\Gamma(\tau)\rho[a^\dagger,[a,R]]$$
$$+[\Gamma\sigma,\Gamma(\tau)\sigma^\dagger\rho][a,R]a^\dagger+\Gamma\sigma[\Gamma(\tau)\sigma^\dagger,\rho][a^\dagger,Ra]+[\Gamma\sigma,[\Gamma(\tau)\sigma^\dagger,\rho]]Raa^\dagger)+\exp[-(\nu_2-i\Delta)\tau]$$
$$\times(\sigma^\dagger\sigma\Gamma\Gamma(\tau)\rho[a,[a^\dagger,R]]+[\Gamma\sigma^\dagger,\Gamma(\tau)\sigma\rho][a^\dagger,R]a+\Gamma\sigma^\dagger[\Gamma(\tau)\sigma,\rho][a,Ra^\dagger]+[\Gamma\sigma^\dagger,[\Gamma(\tau)\sigma,\rho]]Ra^\dagger a), \quad (2.7)$$

where $\Gamma(\tau)=\Gamma[X(\tau)]$. The order of the factors Γ, $\Gamma(\tau)$, and ρ is important because they are noncommuting operator functions of the center-of-mass variables. In Eq. (2.7) we have neglected $\nu_r/2$ compared with ν_2.

To obtain the kinetic equation for R we take the trace of Eq. (2.7) over the internal atomic variables and the center-of-mass variables, and substitute the result in Eq. (2.4a):

$$\frac{\partial R}{\partial t}=-\frac{i}{\hbar}[H_f,R]+\mathcal{K}_rR-\gamma^2N\omega_0^2\int_0^\infty\{\exp[-(\nu_2+i\Delta)\tau]\big[(\langle\Gamma\Gamma(\tau)\rangle_--\langle\Gamma(\tau)\Gamma\rangle_+)[a^\dagger,Ra]+\langle\Gamma\Gamma(\tau)\rangle_-[a^\dagger,[a,R]]\big]$$
$$+\exp[-(\nu_2-i\Delta)\tau]\big[(\langle\Gamma\Gamma(\tau)\rangle_+-\langle\Gamma(\tau)\Gamma\rangle_-)[a,Ra^\dagger]+\langle\Gamma\Gamma(\tau)\rangle_+[a,[a^\dagger,R]]\big]\}d\tau, \quad (2.8)$$

where

$$\langle\Gamma\Gamma(\tau)\rangle_+=\mathrm{Tr}_{\sigma,x}\Gamma\Gamma(\tau)\sigma^\dagger\sigma\rho=\mathrm{Tr}_x(\Gamma\Gamma(\tau)\ \mathrm{Tr}_\sigma n_{\mathrm{op}}^+\rho),$$
$$\langle\Gamma\Gamma(\tau)\rangle_-=\mathrm{Tr}_{\sigma,x}\Gamma\Gamma(\tau)\sigma\sigma^\dagger\rho=\mathrm{Tr}_x(\Gamma\Gamma(\tau)\ \mathrm{Tr}_\sigma n_{\mathrm{op}}^-\rho).$$

The operators $n_{\mathrm{op}}^\pm$ are the number operators for the excited (+) and ground (−) states of a single two-level system. To get a self-contained equation of motion for R we need explicit expressions for $\langle\Gamma\Gamma(\tau)\rangle_\pm$ in terms of R. This we do in the next section.

At this point it is possible to see that the equations of motion for R depend on only the diagonal matrix elements of the internal variable part of ρ because

$$\mathrm{Tr}_\sigma n_{\mathrm{op}}^+\rho=\rho_{++}(x,x',t);\quad \mathrm{Tr}_\sigma n_{\mathrm{op}}^-=\rho_{--}(x,x't).$$

[7] M. Lax, Phys. Rev. **145**, 110 (1966).
[8] H. Haken, Z. Physik **190**, 327 (1966).

However, ρ_{++} and ρ_{--} are nondiagonal operators in the center-of-mass space. Thus, even in the case that R has nondiagonal matrix elements (nonvanishing electric field), the radiation "sees" the matter internal degrees of freedom only through the diagonal matrix elements of ρ. We return to this point in Secs. VI and VII.

III. DERIVATION OF THE KINETIC EQUATION FOR $\rho(x,t)$

The derivation of the kinetic equation for ρ is more complicated than the derivation of the kinetic equation for R because ρ depends on two sets of variables, the internal variables and the center-of-mass variables. In order to obtain the kinetic equation for ρ we need the following trace of Eq. (2.7) over the electromagnetic field variables:

$$\begin{aligned}\mathrm{Tr}_a[\langle H_1(\tau)\rangle_P,R\rho]]=&\exp[-(\nu_2-i\Delta)\tau]([\Gamma\sigma^\dagger,\Gamma(\tau)\sigma\rho]\,\mathrm{Tr}_a[a^\dagger,R]a+[\Gamma\sigma^\dagger,[\Gamma(\tau)\sigma,\rho]]\langle a^\dagger a\rangle\\&-(\Gamma\sigma\rho\sigma\Gamma(\tau)+\Gamma(\tau)\sigma\rho\sigma\Gamma)\langle a^\dagger a^\dagger\rangle)+\exp[-(\nu_2+i\Delta)\tau]([\Gamma\sigma,\Gamma(\tau)\sigma^\dagger\rho]\,\mathrm{Tr}_a[a,R]a^\dagger\\&+[\Gamma\sigma,[\Gamma(\tau)\sigma^\dagger,\rho]]\langle aa^\dagger\rangle-(\Gamma\sigma^\dagger\rho\sigma^\dagger\Gamma(\tau)+\Gamma(\tau)\sigma^\dagger\rho\sigma^\dagger\Gamma)\langle aa\rangle)\,.\end{aligned}\tag{3.1}$$

The moments $\langle a^\dagger a^\dagger\rangle$ and $\langle aa\rangle$ satisfy linear homogeneous equations so if they are zero initially, they remain zero. In any case they vanish in a time ν_r^{-1} which is of the order of 10^{-6} sec; thus, we neglect them.

When we substitute Eq. (3.1) in Eq. (2.3b) and separate real and imaginary parts, we obtain

$$\frac{\partial\rho}{\partial t}=-\left(\frac{i}{\hbar}\right)[h(1),\rho]-\left(\frac{i}{\hbar}\right)\left[\frac{p^2}{2m},\rho\right]+\mathcal{L}_c\rho+\mathcal{K}_{\mathrm{int}}\rho+\mathcal{G}\,,\tag{3.2}$$

where

$$\begin{aligned}\mathcal{G}=&-\gamma^2\omega_0^2\int_0^\infty d\tau\exp[-\nu_2\tau]\cos\Delta\tau\,\{\langle a^\dagger a\rangle(2\Gamma\Gamma(\tau)[\sigma,[\sigma^\dagger,\rho]]+[\Gamma,[\Gamma(\tau),\rho]]+\Gamma[\sigma,[\Gamma(\tau),\rho]\sigma^\dagger]+\Gamma[\sigma^\dagger,[\Gamma(\tau),\rho]\sigma]\\&+[\Gamma,\Gamma(\tau)[\sigma^\dagger,\rho]]\sigma+[\Gamma,[\Gamma(\tau),[\sigma,\rho]]]\sigma^\dagger)+[\Gamma\sigma^\dagger,\Gamma(\sigma)\sigma\rho]-[\Gamma\sigma,\Gamma(\tau)\sigma^\dagger\rho]+[\Gamma\sigma,[\Gamma(\tau)\sigma^\dagger,\rho]]\}\\&-i\gamma^2\omega_0^2\int_0^\infty d\tau\exp[-\nu_2\tau]\sin\Delta\tau\{\langle a^\dagger a\rangle([\Gamma\sigma^\dagger,[\Gamma(\tau)\sigma,\rho]]-[\Gamma\sigma,[\Gamma(\tau)\sigma^\dagger,\rho]])\\&+[\Gamma\sigma^\dagger,\Gamma(\tau)\sigma\rho]+[\Gamma\sigma,\Gamma(\tau)\sigma^\dagger\rho]-[\Gamma\sigma,[\Gamma(\tau)\sigma^\dagger,\rho]]\}\end{aligned}$$

and where we use $\langle aa^\dagger\rangle=\langle a^\dagger a\rangle+1$.

The only term in $\mathcal{G}$ that makes an observable contribution to R is the first term. The reason for this is that except for the spontaneous emission terms all the other terms are proportional to $(\hbar k/mv_T)$ and $(\hbar k/mv_T)^2$, where v_T is the thermal velocity $(3kT/m)^{1/2}$. The dimensionless ratio $(\hbar k/mv_T)$ is the percentage change of momentum of the two-level system on absorption or emission of a laser photon. The ratio is less than 10^{-3} for neon.

To prove the statement about $\mathcal{G}$ in the previous paragraph we introduce the dimensionless variables

$$\bar\nu_2=\nu_2/\omega_D\,,\quad y=kx\,,\quad \bar t=\omega_D t\,,\quad \bar\Delta=\Delta/\omega_D\,,\quad \xi=v/v_T\,,$$

where a bar indicates a time or frequency made dimensionless with ω_D, the Doppler frequency. The statement about $\mathcal{G}$ is true for both quantum-mechanical and classical center-of-mass motion. However, for convenience we introduce the classical nature of the center-of-mass motion by replacing commutators of center-of-mass variables by $(-\hbar/i)$ times the Poisson brackets.

The expression for $\mathcal{G}$ in dimensionless variables and with classical center-of-mass variables is

$$\begin{aligned}\mathcal{G}=&-\tfrac{1}{2}i[\delta\omega_0\hat\sigma,\rho]-\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\frac{1}{T(y,\xi)}\{\langle a^\dagger a\rangle[\sigma,[\sigma^\dagger,\rho]]+\tfrac{1}{2}([\sigma,[\sigma^\dagger,\rho]]+[\sigma^\dagger,\sigma\rho]-[\sigma,\sigma^\dagger\rho])\}\\&+\frac{\hbar k}{mv_T}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2A+\left(\frac{\hbar k}{mv_T}\right)^2\left(\frac{\gamma\omega_0}{\omega_D}\right)^2B,\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}A=&\,i\langle a^\dagger a\rangle\int\cos\Delta\tau\exp[-\bar\nu_2\bar\tau][\{\Gamma(\bar\tau),\sigma\rho\sigma^\dagger+\sigma^\dagger\rho\sigma\}+\{\Gamma,\Gamma(\bar\tau)(\sigma^\dagger\rho\sigma+\sigma\rho\sigma^\dagger)\}-\Gamma\{\Gamma(\bar\tau),\rho\}]d\bar\tau\\&-\int\cos\bar\Delta\bar\tau\exp[-\bar\nu_2\bar\tau][\{\Gamma,\Gamma(\bar\tau)\}[\sigma^\dagger,\rho]\sigma-\{\Gamma,\Gamma(\bar\tau)\rho\}\hat\sigma-\Gamma(\tau)\{\rho,\Gamma\}\sigma^\dagger\sigma]d\bar\tau-\langle a^\dagger a\rangle\int\sin\bar\Delta\bar\tau\\&\times\exp[-\bar\nu_2\bar\tau][\Gamma(\bar\tau)\{\Gamma,\sigma^\dagger\rho\sigma-\sigma\rho\sigma^\dagger\}-\{\Gamma(\bar\tau),\Gamma(\sigma^\dagger\rho\sigma-\sigma\rho\sigma^\dagger)\}]d\bar\tau\end{aligned}$$

and

$$B=(\langle a^\dagger a\rangle+\sigma^\dagger\sigma)\int_0^\infty\exp[-\bar\nu_2\bar\tau]\cos\bar\Delta\bar\tau\{\Gamma,\{\Gamma(\bar\tau),\rho\}\}d\bar\tau\,.$$

The symbol $\{C,D\}$ is the Poisson brackets of C and D taken with respect to the center-of-mass variables. The dimensionless "relaxation time" $T(y,\xi)$ is defined as follows:

$$[T(y,\xi)]^{-1}=2\int_0^\infty d\bar{\tau}\exp[-\bar{\nu}_2\bar{\tau}]\cos\bar{\Delta}\bar{\tau}(\Gamma\Gamma(\bar{\tau}))\,. \quad (3.4)$$

The frequency-shift operator for the matter Hamiltonian is

$$\delta\bar{\omega}_0(y,\xi)=\tfrac{1}{2}\langle a^\dagger a\rangle\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\int_0^\infty d\bar{\tau}\times\exp[-\bar{\nu}_2\bar{\tau}]\sin\bar{\Delta}\bar{\tau}(\Gamma\Gamma(\bar{\tau}))\,.$$

The terms A and B represent the effect of absorption and emission on the center-of-mass motion. The operator B is a generalized Brownian motion of the center-of-mass motion due to the repeated small recoils when a two-level system absorbs or emits a photon. However, the coefficient $(hk/mv_T)^2$ makes the magnitude of B of the order of 10^{-6} times the first term in $\mathcal{G}$ and thus negligible. In summary, the reaction of the electromagnetic field on the center of mass is negligible so we drop the operators A and B. However, the action of the center-of-mass motion on the electromagnetic field variables is important and appears through $T(y,\xi)^{-1}$.

Finally the kinetic equation for $\rho(y,\xi,\bar{t})$ is

$$\frac{\partial\rho(y,\xi,\bar{t})}{\partial\bar{t}}=-\xi\frac{\partial\rho(y,\xi,\bar{t})}{\partial y}+\mathcal{L}_c\rho(y,\xi,\bar{t})-\tfrac{1}{2}i[(\bar{\omega}_0+\delta\bar{\omega}_0)\acute{\sigma},\rho(y,\xi,\bar{t})]+\mathcal{K}_{\rm int}\rho(y,\xi,\bar{t})-\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\frac{1}{T(y,\xi)}\left[\frac{\langle a^\dagger a\rangle}{2}([\sigma,[\sigma^\dagger,\rho]]+[\sigma^\dagger,[\sigma,\rho]])+\tfrac{1}{2}([\sigma,[\sigma^\dagger,\rho]]+[\sigma^\dagger,\sigma\rho]-[\sigma,\sigma^\dagger\rho])\right]. \quad (3.5)$$

The last three terms of Eq. (3.5) give rise to the spontaneous emission term.

The solution of Eq. (3.5) for $\rho(y,\xi,\bar{t})$ contains all the properties of the matter distribution but, as mentioned in the previous section, the only properties of the matter density matrix we need are $\rho_{++}(y,\xi,\bar{t})$ and $\rho_{--}(y,\xi,\bar{t})$ which we denote by $n_+(y,\xi,\bar{t})$ and $n_-(y,\xi,\bar{t})$, respectively. The probability of observing an atom in its excited state at position y, with velocity ξ, at time t is $Nn_+(y,\xi,\bar{t})$.

When we multiply Eq. (3.5) by $n_{\rm op}{}^+=\sigma^\dagger\sigma$ and take the trace over the internal atomic variables, we obtain

$$\frac{\partial n_+(y,\xi,\bar{t})}{\partial\bar{t}}=-\xi\frac{\partial n_+(y,\xi,\bar{t})}{\partial y}+\mathcal{L}_c n_+(y,\xi,\bar{t})-\left(\frac{\delta}{\delta\bar{t}}n_+(y,\xi,\bar{t})\right)_{\rm int}-\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\frac{1}{T(y,\xi)}\times[\langle a^\dagger a\rangle[n_+(y,\xi,\bar{t})-n_-(y,\xi,\bar{t})]+n_+(y,\xi,\bar{t})]\,, \quad (3.6)$$

where

$$((\delta/\delta\bar{t})n_+(y,\xi,\bar{t}))_{\rm int}={\rm Tr}_\sigma\sigma^\dagger\sigma\mathcal{K}_{\rm int}\rho\,.$$

The term ${\rm Tr}_\sigma\sigma^\dagger\sigma\mathcal{K}_{\rm int}\rho$ contains the pump and matter reservoir interactions.

The equation for n_- is

$$\frac{\partial}{\partial\bar{t}}n_-=-\xi\frac{\partial}{\partial y}n_-+\mathcal{L}_c n_--\left(\frac{\delta}{\delta\bar{t}}n_-\right)_{\rm int}+\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\frac{1}{T(y,\xi)}\times[\langle a^\dagger a\rangle[n_+-n_-]+n_+]\,. \quad (3.7)$$

The equations (3.6), (3.7), and (2.8) form a complete nonlinear equation for the radiation density matrix R. In order to see this more clearly we take the classical limit of the center-of-mass motion in Eq. (2.8). After separating real and imaginary parts in Eq. (2.8), we obtain the following equation of motion for R:

$$\frac{\partial R}{\partial\bar{t}}=-i[(\bar{\Omega}+\delta\bar{\Omega})a^\dagger a,R]+\mathcal{K}_r R-\frac{N}{2}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\int dyd\xi\frac{1}{T(y,\xi)}\times\Big[n_+(y,\xi\bar{t})([a,Ra^\dagger]-[a^\dagger,Ra]+[a,[a^\dagger,R]])+n_-(y,\xi,\bar{t})([a^\dagger,Ra]-[a,Ra^\dagger]+[a^\dagger,[a,R]])\Big]\,, \quad (3.8)$$

where

$$\delta\bar{\Omega}=\frac{N}{2}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\int_0^\infty d\bar{\tau}\exp[-\bar{\nu}_2\bar{\tau}]\sin\bar{\Delta}\bar{\tau}\int dyd\xi\Gamma\Gamma(\bar{\tau})\times(n_+(y,\xi,\bar{t})-n_-(y,\xi,\bar{t}))\,.$$

When we substitute the solution of Eqs. (3.6) and (3.7) in Eq. (3.8), the nonlinear dependence appears through the dependence of $\langle a^\dagger a\rangle$ on R.

Before finding the equation of motion for $\langle a^\dagger a\rangle$ in the next section it is appropriate at this point to observe that although Eqs. (3.6), (3.7), and (3.8) appear rather formidable they represent simple microscopic processes. The underlying microscopic events are the first-order Born-approximation absorption and emission of a quantum of radiation by a two-level system with the effective lifetime $\nu_2{}^{-1}$.

IV. THE EQUATION OF MOTION FOR $\langle a^\dagger a\rangle$

We need to know the time-dependent electromagnetic energy $\hbar\Omega\langle a^\dagger a\rangle$ to complete our solution of the kinetic equation for R. We obtain the following equation of motion for $\langle a^\dagger a\rangle$ by multiplying Eq. (3.8) by $a^\dagger a$ and taking the trace

$$\frac{\partial}{\partial\bar{t}}\langle a^\dagger a\rangle=N\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\langle a^\dagger a\rangle\int\frac{dyd\xi}{T(y,\xi)}[n_+(y,\xi,\bar{t})-n_-(y,\xi,\bar{t})]+N\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\int\frac{dyd\xi}{T(y,\xi)}n_+(y,\xi,\bar{t})-\bar{\nu}_r\langle a^\dagger a\rangle\,, \quad (4.1)$$

where we use ${\rm Tr}_a a^\dagger a\mathcal{K}_r R=\nu_r\langle a^\dagger a\rangle$.

Equations (3.6), (3.7), and (4.1) constitute a closed set of equations for n_+, n_- and $\langle a^\dagger a\rangle$. When we eliminate n_+ and n_- from Eq. (4.1) by means of Eqs. (3.6) and (3.7), we have the desired equation for $\langle a^\dagger a\rangle$ alone. For convenience we neglect the spontaneous emission terms in the ratio of $\langle a^\dagger a\rangle^{-1}$. The spontaneous emission terms do not measurably affect the steady-state values[2] and their main function is to provide a trigger for the start of laser action. Also, we consider the spatially homogeneous case which, in practice, means we can allow spatial variations long compared with the wavelength of laser light by treating the spatial dependence as a parameter. We specify the effect of the combined pump-dissipation operator in Eqs. (3.6) and (3.7) by requiring that the pump create a Maxwellian distribution of laser atoms at a rate ν_1, where ν_1 is the relaxation time of the two-level population inversion in the absence of laser radiation.

When we subtract Eq. (3.7) from Eq. (3.6), assume spatial homogeneity, neglect spontaneous emission terms, and specify the pump, we obtain

$$\left[\frac{\partial}{\partial\bar{t}}-\mathfrak{L}_c+2\frac{\langle a^\dagger a\rangle}{T(\xi)}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2+\bar{\nu}_1\right][n_+(\xi,\bar{t})-n_-(\xi,\bar{t})]$$
$$=\bar{\nu}_1[n_+^0(\xi)-n_-^0(\xi)]=[\bar{\nu}_1/(2\pi)^{1/2}]\exp(-\xi^2/2). \quad (4.2)$$

The solution of Eq. (4.2) is

$$n_+(\xi,\bar{t})-n_-(\xi,\bar{t})=(2\pi)^{-1/2}\int_0^{\bar{t}}d\tau\exp\left\{-\left[(\bar{\nu}_1-\mathfrak{L}_c)(\bar{t}-\tau)\right.\right.$$
$$\left.\left.+2\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\frac{1}{T(\xi)}\int_\tau^{\bar{t}}\langle a^\dagger a\rangle_{t'}dt'\right]\right\}\bar{\nu}_1\exp(-\xi^2/2)$$
$$+\text{a solution of the homogeneous part of Eq. (4.2).} \quad (4.3)$$

The t' subscript on $\langle a^\dagger a\rangle_{t'}$ represents the average value of the energy evaluated at time t'. The solution of the homogeneous equation decays in ν_1^{-1} sec which is of the order of 10^{-7}–10^{-8} sec in a typical gas laser. The order of factors in Eqs. (4.2) and (4.3) is important since, in general, the collision kernel $\mathfrak{L}_c$ does not commute with $T(\xi)$.

The steady-state solution of Eq. (4.2) is

$$(n_+-n_-)_s=\left[\bar{\nu}_1-\mathfrak{L}_c+2\frac{\langle a^\dagger a\rangle_s}{T(\xi)}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\right]^{-1}\frac{\bar{\nu}_1}{(2\pi)^{1/2}}$$
$$\times\exp\left(\frac{-\xi^2}{2}\right), \quad (4.4)$$

where a subscript s indicates steady state.

When we substitute Eq. (4.3) in Eq. (4.1), we obtain

$$\frac{\partial}{\partial\bar{t}}\langle a^\dagger a\rangle=N\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\langle a^\dagger a\rangle\frac{\bar{\nu}_1}{(2\pi)^{1/2}}\int d\xi\int_0^{\bar{t}}dt'\frac{1}{T(\xi)}$$
$$\times\exp\left\{-\left[(\bar{\nu}_1-\mathfrak{L}_c)(\bar{t}-t')+\frac{2}{T(\xi)}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\int_{t'}^{\bar{t}}\langle a^\dagger a\rangle_{t''}dt''\right]\right\}$$
$$\times\exp\left(\frac{-\xi^2}{2}\right)-\nu_r\langle a^\dagger a\rangle. \quad (4.5)$$

The solution of Eq. (4.5) goes monotonically over to the following steady-state solution:

$$\bar{\nu}_r=\frac{N}{(2\pi)^{1/2}}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\bar{\nu}_1\int\frac{d\xi}{T(\xi)}$$
$$\times\left[\frac{1}{\bar{\nu}_1-\mathfrak{L}_c+2\langle a^\dagger a\rangle_s(\gamma\omega_0)^2T(\xi)^{-1}\omega_D^{-2}}\right]$$
$$\times\exp(-\xi^2/2). \quad (4.6)$$

In order to evaluate the velocity integrals in Eqs. (4.5) and (4.6) we need to know the velocity-dependent relaxation time $T(\xi)$ defined by Eq. (3.4). For definiteness we assume rectangular geometry, so Eq. (3.4) becomes

$$\frac{1}{T(\xi)}=\int_0^\infty d\bar{\tau}\exp(-\bar{\nu}_2\bar{\tau})\cos\bar{\Delta}\bar{\tau}\sin kx\sin kx(\bar{\tau}),$$

where $x(\bar{\tau})$ is the solution of the equation of motion for the center of mass in the presence of collisions. We use the Lebowitz-Shimony[9] definition of the equation of motion for any function of the center-of-mass coordinates and momenta in the presence of collisions with reservoir atoms (in the He–Ne laser the reservoir atoms are the He). Their definition for an arbitrary function D is

$$dD/d\bar{t}=\{D,H_{\text{c.m.}}\}+\mathfrak{L}_cD$$
$$=\{D,H_{\text{c.m.}}\}+\int L(xp|x'p')D(x'p')dx'dp'$$
$$-D(xp)\int L(x'p'|xp)dx'dp', \quad (4.7)$$

where $L(xp|x'p')$ is the probability per unit time of a collision which takes the center of mass from $(x'p')$ to (xp). We assume the collisions are instantaneous so that $L(xp|x'p')$ may be written as $\delta(x-x')l(p|p')$. Equation (4.7) for the position coordinate is

$$\frac{dx}{d\bar{t}}=\frac{p}{m}(\bar{t});\quad x(\bar{t})=x+\int_0^{\bar{t}}V(t')dt',$$

[9] J. Lebowitz and A. Shimony, Phys. Rev. 128, 1945 (1962).

where V is the velocity of the center of mass. To determine the time dependence of $V(t)$, we need to solve the collision problem. Thus collisions enter our theory in two ways: First, they enter through $\mathcal{L}_c$ in the equation for ρ, and second, they enter through $T(\xi)$.

When we do the integrations in the definition of $T(\xi)$, Eq. (3.4), for the case of no collisions we obtain

$$\frac{1}{T(\xi)}=\frac{\bar{\nu}_2}{2}\left[\frac{1}{(\xi+\bar{\Delta})^2+\bar{\nu}_2{}^2}+\frac{1}{(\xi-\bar{\Delta})^2+\bar{\nu}_2{}^2}\right]\sin^2 y,$$

where we use $x(\bar{t})=x+V\bar{t}$.

We now have the kinetic equation for the density matrix R. The first step is to solve Eq. (4.5) for $\langle a^\dagger a\rangle$ and then to substitute the solution for $\langle a^\dagger a\rangle$ into Eq. (4.3). The final step is to substitute the resultant expression for (n_+-n_-) into Eq. (2.8) for R. The resultant equation for R is linear with the known function $\langle a^\dagger a\rangle$ as a coefficient in the equation. That is, when we solve the ordinary differential (3.4) for the c-number function $\langle a^\dagger a\rangle$, we reduce the nonlinear operator equation of motion for R to a linear-operator equation.

The time-dependent Eq. (4.5) and the stationary-state Eq. (4.6) for $\langle a^\dagger a\rangle$ contain all positive powers of γ^2. We observe that we start with kinetic equations for R and ρ valid to order γ^2, but that the elimination of the matter variables leads to an equation for $\langle a^\dagger a\rangle$ which contains all positive powers of γ^2. If we solve Eq. (4.6) exactly, does this mean we have a solution of the original problem to all orders in γ^2? Unfortunately, the answer is no. The next step in an exact solution is to derive kinetic equations for R and ρ to order γ^4. This means that we must derive kinetic equations where the microscopic event is not just a first-order Born approximation but also a second-order Born approximation between a single atom and the radiation field and the virtual exchange of a photon between two different atoms. We then get a new equation for $\langle a^\dagger a\rangle$ which contains terms such as $\langle a^\dagger a a^\dagger a\rangle$. We will carry out the derivation of the kinetic equations to order γ^4 in a future publication. The term proportional to γ^{2n} in the solution of Eq. (4.6) is the result of the iteration of the first-order Born approximation n times. A completely rigorous solution of the full problem would mean that the γ^{2n} term in the solution of $\langle a^\dagger a\rangle$ would contain all processes up to the nth-order Born approximation and would contain the average value of n $a^\dagger$'s and n a's such as $\langle a^\dagger a\cdots a^\dagger a\rangle$.

In the next section we find some stationary-state solutions to our equations.

V. THE STATIONARY STATE

The threshold number of atoms required to start laser action is

$$N_T=\left(\frac{\omega_D}{\gamma\omega_0}\right)^2\bar{\nu}_r(2\pi)^{1/2}\times\left[\bar{\nu}_1\int_{-\infty}^{\infty}\frac{d\xi}{T(\xi)}\left(\frac{1}{\bar{\nu}_1-\mathcal{L}_c}\right)\exp\left(\frac{-\xi^2}{2}\right)\right]^{-1}, \tag{5.1}$$

where we put $\langle a^\dagger a\rangle_s$ equal to zero in Eq. (4.6). In a gas laser the magnitude of $\mathcal{L}_c$ is proportional to the collision frequency $\bar{\nu}_c$ which is usually much smaller than $\bar{\nu}_1$ so we neglect $\mathcal{L}_c$ in the remainder of this paper. As a result we can explicitly carry out some of the integrals without a detailed analysis of collisions. When we substitute Eq. (4.8) in Eq. (5.1), we obtain

$$N_T=\left(\frac{\omega_D}{\gamma\omega_0}\right)^2\frac{\bar{\nu}_r}{I(\bar{\nu}_2,\bar{\Delta})},\qquad I(\bar{\nu}_2,\bar{\Delta})=\int_0^\infty dx\exp[-(\bar{\nu}_2x+x^2/2)]\cos\bar{\Delta}x. \tag{5.2}$$

The stationary-state equation for $\langle a^\dagger a\rangle_s$ is rather complicated for arbitrary detuning $\bar{\Delta}$. However, for $\bar{\Delta}=0$ and $\bar{\nu}_2\ll 1$ we can solve for $\langle a^\dagger a\rangle$ for all pump power N. When we substitute Eq. (5.1) in Eq. (4.6), we obtain

$$\frac{N_T}{N}=\frac{\int_0^\infty dx\exp[-(\bar{\nu}_2{}'x+x^2/2)]}{\int_0^\infty dx\exp[-(\nu_2x+x^2/2)][1+2\langle a^\dagger a\rangle_s(\gamma\omega_0)^2(\nu_1\nu_2)^{-1}]^{1/2}}, \tag{5.3}$$

where

$$\bar{\nu}_2{}'=\bar{\nu}_2\left[1+2\frac{\langle a^\dagger a\rangle_s(\gamma\omega_0)^2}{\nu_1\nu_2}\right]^{1/2}.$$

We rewrite Eq. (5.3) in the following more suggestive form:

$$\langle a^\dagger a\rangle_s=\frac{\nu_1\nu_2}{2(\gamma\omega_0)^2}\left\{\left(\frac{NI(\bar{\nu}_2{}',0)}{N_TI(\bar{\nu}_2,0)}\right)^2-1\right\}\xrightarrow[\bar{\nu}_2\to 0]{}\frac{\nu_1\nu_2}{2(\gamma\omega_0)^2}\left\{\left(\frac{N}{N_T}\right)^2-1\right\}, \tag{5.4}$$

which shows that for $\bar{\nu}_2 \ll 1$ the steady-state electromagnetic density depends quadratically on N. If $\bar{\nu}_2$ is not small, then Eq. (5.4) is a transcendental equation for $\langle a^\dagger a\rangle_s$ because $\bar{\nu}_2'$ depends on $\langle a^\dagger a\rangle_s$.

When we divide the integral appearing in Eq. (2.8) for R into its real and imaginary parts, we find that the imaginary part is a commutator of the energy operator $a^\dagger a$ with R. We rewrite Eq. (3.8) to explicitly display the cavity frequency shift

$$\frac{\partial R}{\partial \bar{t}} = -i[(\bar{\Omega}+\delta\bar{\Omega})a^\dagger a, R] - \frac{N}{2}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2 \int \frac{d\xi}{T(\xi)} \times \Big[(n_+(\xi,\bar{t}) - n_-(\xi,\bar{t}))([a,Ra^\dagger]-[a^\dagger,Ra]) + (n_+(\xi,\bar{t})+n_-(\xi,\bar{t}))[a^\dagger,[a,R]]\Big] + \mathcal{K}_r R, \quad (5.5)$$

where

$$\delta\bar{\Omega} = \frac{N}{2}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2 \int d\xi \int_0^\infty d\bar{\tau}\, \exp(-\bar{\nu}_2\bar{\tau}) \sin\bar{\Delta}\bar{\tau} \cos\xi\bar{\tau} \times [n_+(\xi,\bar{t}) - n_-(\xi,\bar{t})].$$

The steady-state frequency shift we obtain when we substitute Eq. (4.4) in the expression for $\delta\bar{\Omega}$ is very complicated for arbitrary N and $\bar{\Delta}$. In the Appendix we show that to first order in $\bar{\Delta}$ and for $\bar{\nu}_2 \ll 1$ the frequency shift for arbitrary pump power is given by Eq. (A5).

Near threshold we obtain the Lamb dip of Ref. 1. A study of Eq. (4.6) shows that the dip eventually disappears for $N \gg N_T$. The precise value of N where the dip disappears is the solution of a very complicated transcendental equation.

VI. COMPARISON WITH THE LAMB THEORY

Reference 1 actually contains two different theories. The explicit results of the first theory depend on the condition that the system be near threshold. The second theory is a plausible generalization of a previous theory[10] which did not contain center-of-mass motion. The generalization is discussed in Secs. 16–20 of Ref. 1.

We first show that our results reduce to Lamb's near-threshold theory when Lamb's "saturation parameter" is small compared to 1. The definition of the "saturation parameter" is $(\gamma^2\langle a^\dagger a\rangle\omega_0^2/\nu_1\nu_2)$. We reexpress the "saturation parameter" in terms of the parameters of the radiation matter system plus reservoir with the help of Eq. (5.4)

$$1 \gg \frac{\gamma^2\langle a^\dagger a\rangle_s\omega_0^2}{\nu_1\nu_2} = \frac{1}{2}\left\{\left(\frac{N}{N_T}\right)^2 - 1\right\} \approx \frac{N}{N_T} - 1. \quad (6.1)$$

Thus, the smallness of Lamb's "saturation parameter" requires the system to be near threshold. The only requirement on N in our theory is that $(\gamma\omega_0/\omega_D)^2 N \ll 1$. For the He–Ne laser this condition is equivalent to $N < 2.5\times10^3 N_T$. Consequently, the Lamb near threshold theory is a special case of our results and in fact is the first term in the infinite series expansion of the denominators of Secs. IV and V.

When we substitute Eq. (5.2) in Eq. (4.6), set $\bar{\Delta}=0$, and rearrange terms, we obtain

$$\frac{N_T}{N} = \frac{1}{I(\nu_2,0)}(2\pi)^{-1/2} \times \int_{-\infty}^{\infty} \frac{d\xi \exp(-\xi^2/2)}{\xi^2 + \bar{\nu}_2^2[1+2\langle a^\dagger a\rangle_s\gamma^2\omega_0^2(\nu_1\nu_2)^{-1}]}. \quad (6.2)$$

When the integral is expanded to first order in $(\gamma^2\omega_0^2\langle a^\dagger a\rangle/\nu_1\nu_2)$, the result is equivalent to Lamb's Eq. (87). For arbitrary $\bar{\Delta}$, Eq. (4.6) to first order in the "saturation parameter" reduces to Eq. (96) of Reference 1. A more detailed analysis of the time-dependent equation for $\langle a^\dagger a\rangle$ shows that to first order in the "saturation parameter," Eq. (4.5) is the same as Eq. (81) of Ref. 1. Since near threshold our results reduce to those of Lamb, we obtain the same Lamb dip.[11] When we expand Eq. (5.6) about threshold, our result reduces to the line shift calculated by Lamb in his Eq. (89).

Thus, in the limit $N \rightarrow N_T$ our theory reproduces Lamb's near-threshold dynamics and his stationary state exactly. In some cases it is possible to calculate properties for all N in closed form. However, the corrections to the Lamb theory for all values of $\bar{\Delta}$ and $\bar{\nu}_2$ can easily be obtained by expanding Eq. (4.6) to second or higher powers of $(\gamma^2\omega_0^2\langle a^\dagger a\rangle_s/\nu_1\nu_2)$. Calculation of higher-order integrals in Eq. (4.6) is much easier than attempting to generalize Lamb's method to order γ^6.

The agreement with Lamb's theory is at first surprising because Lamb's theory is based on the electromagnetic fields $\langle a\rangle$ and $\langle a^\dagger\rangle$, not on the electromagnetic energy density $\langle a^\dagger a\rangle$. The important point is that the electromagnetic density matrix R (whether diagonal or not) depends on the matter through only the diagonal matter matrix elements $n_+(\xi,t)$ and $n_-(\xi,t)$, which in turn depend on the radiation only through $\langle a^\dagger a\rangle$. As a result, except for the small spontaneous emission term, the steady-state equations for $\langle a^\dagger a\rangle$ and $\langle a\rangle$ are the same as for $\langle a^\dagger a\rangle$. We prove that the stationary state (neglecting the spontaneous emission term) for $\langle a^\dagger\rangle$ and $\langle a\rangle$ is the same as for $\langle a^\dagger a\rangle$ by multiplying Eq. (5.5) by $a^\dagger$ and taking the trace of the resultant equation

$$\frac{d}{d\bar{t}}\langle a^\dagger\rangle = \frac{N}{2}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2 \langle a^\dagger\rangle \int_{-\infty}^{\infty} \frac{d\xi}{T(\xi)} \times \{n_+(\xi,\bar{t}) - n_-(\xi,\bar{t})\} - \frac{\bar{\nu}_r}{2}\langle a^\dagger\rangle. \quad (6.3)$$

[10] W. E. Lamb, *Quantum Mechanical Amplifiers in Lectures in Theoretical Physics*, edited by W. E. Britten and B. W. Downs (Interscience Publishers, Inc., New York, 1960).

[11] Thus the statement in Ref. 2 that the Lamb dip requires kinetic equations rigorous to order γ^4 applies only to the case of laser models that do not have center-of-mass internal-variable correlations. The theory of Ref. 1 and the present theory obtain the Lamb dip with kinetic equations correct to order γ^2.

The equation of motion for $\langle a\rangle$ is the same as for $\langle a^\dagger\rangle$. We see that Eq. (6.3) differs from Eq. (4.1) for $\langle a^\dagger a\rangle$ in two ways. There is no spontaneous emission term and there is a factor of $\frac{1}{2}$ in each term on the right-hand side of Eq. (6.3). This factor of $\frac{1}{2}$ is just what we expect for the difference between the amplitude and intensity. To obtain the steady state we set the time derivative equal to zero in Eq. (6.3) and the resultant equation is the same as the steady state Eq. (4.1) minus the spontaneous-emission term. In Appendix B we show that the average field goes to zero due to spontaneous emission.

When we assume $\langle a^\dagger a\rangle_{t''}$ in Eq. (4.3) is "slowly varying," we can do the integrals, and for $\bar{t}>\bar{\nu}_1^{-1}$ we obtain

$$n_+(\xi,t)-n_-(\xi,t)=\left\{1+\frac{2\langle a^\dagger a\rangle_t}{t(\xi)}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\right\}^{-1}\frac{1}{\sqrt{\pi}}\times\exp(-\xi^2/2)\,,\quad(6.4)$$

where we have neglected collisions. The substitution of Eq. (6.4) in (6.3) yields

$$\frac{d}{dt}\langle a^\dagger\rangle=\frac{N}{2}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\frac{\langle a^\dagger\rangle}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{d\xi}{T(\xi)}\times\frac{\exp(-\xi^2/2)}{[1+2\langle a^\dagger a\rangle_t(\gamma\omega_0)^2T^{-1}(\xi)\omega_D^{-2}\bar{\nu}_1^{-1}]}-\bar{\nu}_r\langle a^\dagger\rangle.\quad(6.5)$$

It is interesting to observe that if we replace $\langle a^\dagger a\rangle$ by $\langle a^\dagger\rangle\langle a\rangle$ in Eq. (6.5), we obtain Eq. (184) of Ref. 1.

VII. GAS LASERS AND RATE EQUATIONS

In I we showed that, if initially there is no correlation between the center-of-mass variables and the internal atomic variables, the second moments of R and ρ satisfy the following rate equations:

$$\frac{d\langle a^\dagger a\rangle}{d\bar{\tau}}=2N\left(\frac{\gamma\omega_0}{\omega_D}\right)^2C_R\{n_+(\langle a^\dagger a\rangle+1)-(1-n_+)\langle a^\dagger a\rangle\}-\frac{\bar{\nu}_r}{2}\langle a^\dagger a\rangle\,,\quad(7.1\text{a})$$

$$\frac{dn_+}{d\bar{t}}=-2\left(\frac{\gamma\omega_0}{\omega_D}\right)^2C_R\{n_+(\langle a^\dagger a\rangle+1)-(1-n_+)\langle a^\dagger a\rangle\}-\bar{\nu}_1(n_+-1)\,,\quad(7.1\text{b})$$

where n_+ is not a function of velocity, but only of time. The symbol C_R is a function of $\bar{\Delta}$ and $\bar{\nu}_2$ and is obtained from an integration over velocity space.

We wish to determine whether or not the second moments of R and $\rho(\xi,t)$ of the present paper satisfy rate equations. By inspection it is clear that Eq. (4.1) and the velocity integral of Eq. (4.2) do not satisfy the simple rate Eqs. (7.1a) and (7.1b). However, perhaps the second moments of R and $\rho(\xi,t)$ satisfy a more complicated type of rate equation. We investigate this possibility by rewriting Eq. (4.1) for the spatial homogeneous case in the following suggestive form:

$$\frac{\partial\langle a^\dagger a\rangle}{\partial\bar{t}}=N\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\left\{(\langle a^\dagger a\rangle+1)\left\langle\frac{n_+(\xi,\bar{t})}{T(\xi)}\right\rangle-\left(\left\langle\frac{\exp(-\xi^2/2)}{T(\xi)}\right\rangle-\left\langle\frac{n_+(\xi,\bar{t})}{T(\xi)}\right\rangle\right)\langle a^\dagger a\rangle\right\}-\bar{\nu}_r\langle a^\dagger a\rangle\,,\quad(7.2)$$

where

$$\left\langle\frac{n_+(\xi,\bar{t})}{T(\xi)}\right\rangle=\int_{-\infty}^{\infty}\frac{n_+(\xi,\bar{t})}{T(\xi)}d\xi.$$

Consequently, $\langle a^\dagger a\rangle$ satisfies a rate equation with $\langle n_+(\xi,\bar{t})/T(\xi)\rangle$ replacing $n_+(\bar{t})$. If $\langle n_+(\xi,\bar{t})/T(\xi)\rangle$ satisfies a similar equation, then the modified second moments of R and $\rho(\xi,\bar{t})$ with zeroth-order correlations also satisfy rate equations.

We find the equation of motion for $\langle n_+(\xi,\bar{t})/T(\xi)\rangle$ by multiplying Eq. (4.2) by $T(\xi)^{-1}$ and averaging over velocity. The result is

$$\frac{\partial}{\partial\bar{t}}\left\langle\frac{n_+(\xi,\bar{t})}{T(\xi)}\right\rangle=-\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\times\left\{(\langle a^\dagger a\rangle+1)\left\langle\frac{n_+(\xi,\bar{t})}{T^2(\xi)}\right\rangle-\langle a^\dagger a\rangle\left\langle\frac{n_-(\xi,\bar{t})}{T^2(\xi)}\right\rangle\right\}-\bar{\nu}_1\left(\left\langle\frac{n_+(\xi,\bar{t})}{T(\xi)}\right\rangle-\left\langle\frac{\exp(-\xi^2/2)}{T(\xi)}\right\rangle\right).\quad(7.3)$$

The right-hand side of Eq. (7.3) contains $\langle n_+(\xi,\bar{t})/T^2(\xi)\rangle$ instead of $\langle n_+(\xi,\bar{t})/T(\xi)\rangle$. Consequently, the modified second moments of R and $\rho(\xi,\bar{t})$ do not satisfy rate equations if there are zeroth-order correlations between the center-of-mass and internal-atomic variables. The equations (7.2) and (7.3) do not even form a closed set of equations because, in order to evaluate $\langle n_+(\xi,\bar{t})/T^2(\xi)\rangle$, we need $\langle n_+(\xi,\bar{t})/T^3(\xi)\rangle$ and so on. Thus, the He–Ne laser which does have zeroth-order correlations cannot be described by rate equations.

There are several differences between the solution of the equations of the present paper and rate equations. One of the more important differences is the steady-state N dependence. In I we showed the steady-state electromagnetic energy density of the rate equations is proportional to $(N-N_T)$, where our steady-state Eq. (5.4) is proportional to $[(N/N_T)^2-1]$.

VIII. CONCLUSIONS

We showed that the lowest-order kinetic equation with zeroth-order correlations between internal atomic and center-of-mass variables is more general than existing gas laser theories in five ways. First, we derived the self-contained nonlinear equation of motion for the full radiation density matrix. Second, our derivation is fully quantum mechanical. Third, our derivation holds for all N such that $N(\gamma\omega_0/\omega_D)^2 \ll 1$, which in a He–Ne laser allows N to be as great as a hundred times threshold. Fourth, we reduced the treatment of collisions to quadrature. Fifth, our derivation can be generalized to higher order in γ in a straightforward manner.

In this paper we showed that the results of the preceding paragraph follow from treating the fundamental microscopic absorption and emission processes in the first-order Born approximation. If we wish to derive kinetic equations to order γ^4, then we must include fourth-order microscopic processes. There are two new microscopic processes to order γ^4. The first, which is proportional to N, is a second-order Born approximation of the interaction of a single atom with radiation. The second process, which is proportional to N^2, comes from the exchange of a virtual photon between two different atoms. The second-order Born approximation is approximately proportional to the square of the electromagnetic energy density, while the two-atom process is proportional to the first power of the electromagnetic energy density. Since the electromagnetic energy density goes as $[(N/N_T)^2-1]$, near threshold the dominant process is the two-atom process, while far above threshold the second-order Born approximation is the dominant process. In a future publication we will derive the results stated in this paragraph.

Our method of treating collisions reduces the collision problem to one of evaluating integrals. The model of a velocity dependent relaxation time can be evaluated in closed form at least for $\bar{\Delta}=0$. The more general case of the linear Boltzmann equation or the generalized Fokker-Planck equation can be evaluated by expanding the denominator of Eq. (4.6) in powers of the collision frequency divided by the Doppler frequency; this ratio is a small number for gas lasers.

In a recent publication[12] Smith has experimentally verified an expression equivalent to our Eq. (5.3) for pump power 2–3 times threshold.

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APPENDIX A

We derive the expression for $\delta\bar{\Omega}$ given in Sec. V. The definition of $\delta\bar{\Omega}$ is

$$\delta\bar{\Omega}=\frac{N}{2}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\bar{\tau}\exp(-\bar{\nu}_2\bar{\tau})\sin\Delta\tau\cos\xi\tau \times[n_+(\xi)-n_-(\xi)]_s. \quad \text{(A1)}$$

For convenience we split off the lowest-order term in the phase shift and obtain

$$\delta\bar{\Omega}=\frac{\bar{\nu}_r}{2}\frac{\int_0^{\infty}\exp(-\bar{\nu}_2\bar{\tau})\sin\bar{\Delta}\bar{\tau}\exp(-\bar{\tau}^2/2)d\bar{\tau}}{\int_0^{\infty}\exp(-\bar{\nu}_2\bar{\tau})\cos\bar{\Delta}\bar{\tau}\exp(-\bar{\tau}^2/2)d\bar{\tau}} +\frac{N}{2}\left(\frac{\gamma\omega_0}{\omega_D}\right)^2\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\bar{\tau}\exp(-\bar{\nu}_2\bar{\tau})\sin\bar{\Delta}\bar{\tau} \times\cos\xi\bar{\tau}\left[n_+(\xi)-n_-(\xi)-\frac{\exp(-\xi^2/2)}{(2\pi)^{1/2}}\right]. \quad \text{(A2)}$$

In the limit as $\bar{\Delta}$ approaches zero, Eq. (A2) becomes

$$\delta\bar{\Omega}=\delta\bar{\Omega}_1-\bar{\Delta}\frac{N}{2}\left(\frac{\gamma\omega_0}{\omega}\right)^2\int_0^{\infty}\exp(-\bar{\nu}_2\bar{\tau})f(\bar{\tau})\bar{\tau}d\bar{\tau}, \quad \text{(A3)}$$

where

$$\delta\bar{\Omega}_1=\frac{\bar{\Delta}\bar{\nu}_r}{2}\frac{\int_0^{\infty}d\bar{\tau}\,\bar{\tau}\exp(-\bar{\nu}_2\bar{\tau})\exp(-\bar{\tau}^2/2)}{\int_0^{\infty}d\bar{\tau}\exp(-\bar{\nu}_2\bar{\tau})\exp(-\bar{\tau}^2/2)},$$

$$f(\tau)\equiv\frac{[(\bar{\nu}_2')^2-(\bar{\nu}_2)^2]}{(2\pi)^{1/2}}\int_0^{\infty}d\xi\frac{\cos\xi\bar{\tau}\exp(-\xi^2/2)}{[\xi^2+(\bar{\nu}_2')^2]},$$

$$\bar{\nu}_2'=\bar{\nu}_2[1+2\langle a^\dagger a\rangle_s\gamma^2\omega_0^2(\nu_1\nu_2)^{-1}]^{1/2}.$$

The integral in the definition of $f(\bar{\tau})$ is

$$(2\pi)^{-1/2}\int_0^{\infty}d\xi\frac{\cos\xi\bar{\tau}\exp(-\xi^2/2)}{[\xi^2+(\bar{\nu}_2')^2]} =\tfrac{1}{4}(\tfrac{1}{2}\pi)^{1/2}\frac{1}{\bar{\nu}_2'}\exp[\tfrac{1}{2}(\bar{\nu}_2')^2] \times\{\exp(-\bar{\nu}_2\bar{\tau})\,\mathrm{Erfc}[(\surd\tfrac{1}{2})\bar{\nu}_2'-(\surd\tfrac{1}{2})\bar{\tau}] +\exp(\bar{\nu}_2\bar{\tau})\,\mathrm{Erfc}[(\surd\tfrac{1}{2})\bar{\nu}_2'+(\surd\tfrac{1}{2})\bar{\tau}]\} \approx\tfrac{1}{4}(\tfrac{1}{2}\pi)^{1/2}\exp(-\bar{\nu}_2'\bar{\tau}), \quad \text{(A4)}$$

where the approximate value in Eq. (A4) follows in the limit as $\bar{\nu}_2$ approaches zero. When we substitute Eq. (A4) into Eq. (A3) and do the remaining integration, we obtain

$$\delta\bar{\Omega}=\frac{\bar{\nu}_r\bar{\Delta}}{2}\left[1-\frac{1}{2\bar{\nu}_2}\left\{\frac{(N/N_T)-1}{(N/N_T)+1}\right\}\right]. \quad \text{(A5)}$$

The expression (A5) to lowest order in (N/N_T) is equivalent to Lamb's Eq. (94).

[12] P. W. Smith, IEEE J. Quant. Electron. 2, 62 (1966).

APPENDIX B

We show that spontaneous emission causes the average electric field to vanish as $t\to\infty$. The equations satisfied by $\langle a^\dagger\rangle$ and $\langle a\rangle$ are

$$\frac{d}{dt}\langle a^\dagger\rangle = N\frac{(\gamma\omega_0)^2}{2\omega_D}\langle a^\dagger\rangle\int_{-\infty}^{\infty}\frac{d\xi}{T(\xi)}\times\{n_+(\xi,t)-n_-(\xi,t)\}-\frac{\nu_r}{2}\langle a^\dagger\rangle\,, \quad \text{(B1a)}$$

$$\frac{d\langle a\rangle}{dt} = N\frac{(\gamma\omega_0)^2}{2\omega_D}\langle a\rangle\int_{-\infty}^{\infty}\frac{d\xi}{T(\xi)}\times\{n_+(\xi,t)-n_-(\xi,t)\}-\frac{\nu_r}{2}\langle a\rangle\,. \quad \text{(B1b)}$$

When the definitions

$$\langle a^\dagger\rangle = fe^{i\varphi} \quad\text{and}\quad \langle a\rangle = fe^{-i\varphi}$$

are substituted into Eqs. (B1a) and (B1b) we obtain

$$\dot{f}=\tfrac{1}{2}\mathfrak{A}f \quad\text{and}\quad \dot{\varphi}=0\,, \quad \text{(B2)}$$

where

$$\mathfrak{A}\equiv N\frac{(\gamma\omega_0)^2}{\omega_D}\int_{-\infty}^{\infty}\frac{d\xi}{T(\xi)}\{n_+(\xi,t)-n_-(\xi,t)\}-\nu_r\,.$$

The equation of motion for the electromagnetic energy is

$$\frac{d\langle a^\dagger a\rangle}{dt} = \mathfrak{A}\langle a^\dagger a\rangle + N\frac{(\gamma\omega_0)^2}{\omega_D}\int\frac{d\xi}{T(\xi)}n_+(\xi,t)\,, \quad \text{(B3)}$$

where the last term in Eq. (B3) is due to spontaneous emission. Combining Eqs. (B2) and (B3), we obtain

$$\frac{df}{dt} = \tfrac{1}{2}f\left\{\frac{1}{\langle a^\dagger a\rangle}\frac{d}{dt}\langle a^\dagger a\rangle - \frac{N}{\langle a^\dagger a\rangle}\frac{(\gamma\omega_0)^2}{\omega_D}\int\frac{d\xi}{T(\xi)}n_+(\xi,t)\right\}. \quad \text{(B4)}$$

The solution of Eq. (B4) is

$$f(t)=f(0)\left[\frac{\langle a^\dagger a\rangle}{\langle a^\dagger a\rangle_0}\right]^{1/2}\times\exp\left\{-\frac{N}{2}\frac{(\gamma\omega_0)^2}{\omega_D}\int_0^t dt'\int_{-\infty}^{\infty}\frac{d\xi n_+(\xi,t')}{T(\xi)\langle a^\dagger a\rangle_{t'}}\right\}, \quad \text{(B5)}$$

where $\langle a^\dagger a\rangle_0$ and $f(0)$ are the initial energy density and field amplitude. In order to have a nonvanishing electric field amplitude $f(t)$ at time t, it is necessary to have a nonvanishing field initially. It is not necessary for $\langle a^\dagger a\rangle_0$ to be nonzero. In a time t_r the quantities $\langle a^\dagger a\rangle$, n_+, and n_- reach their steady-state values.

For $t>t_r$ the solution of Eq. (B5) is

$$f(t)=Ce^{-\alpha t}\,,$$

where

$$\alpha\equiv\frac{N(\gamma\omega_0)^2}{2\omega_D\langle a^\dagger a\rangle_s}\int_{-\infty}^{\infty}\frac{d\xi}{T(\xi)}n_+{}^s(\xi)\,,$$

$$C\equiv f(0)\left[\frac{\langle a^\dagger a\rangle_s}{\langle a^\dagger a\rangle_0}\right]^{1/2}\times\exp\left\{-\frac{N(\gamma\omega_0)^2}{2\omega_D}\int_{-\infty}^{\infty}d\xi\int_0^{t_r}\frac{n_+(\xi,t)}{T(\xi)\langle a^\dagger a\rangle_{t'}}dt'\right\}\times\exp(-\alpha t_r)\,,$$

and s indicates steady state. Thus at $t\to\infty$ the spontaneous emission term causes the average electric field to vanish.

In the absence of collisions and for $\bar{\Delta}=0$, the solution for $n_+{}^s(\xi)$ is

$$n_+{}^s(\xi)=2^{-1}(2\pi)^{-1/2}\exp(-\xi^2/2)\times\{1+[1+2\langle a^\dagger a\rangle_s T^{-1}(\xi)(\gamma\omega_0)^2\omega_D{}^{-1}\bar{\nu}_1]^{-1}\}\,. \quad \text{(B6)}$$

When Eq. (B6) is substituted in the definition of α we obtain

$$\alpha=\frac{N(\gamma\omega_0)^2}{4\omega_D\langle a^\dagger a\rangle_s}[I(\bar{\nu}_2,0)+[1+2\langle a^\dagger a\rangle_s(\gamma\omega_0)^2(\nu_1\nu_2)^{-1}]^{-1/2}I_2(\bar{\nu}_2{}',0)]\,, \quad \text{(B7)}$$

where

$$I_2(y,0)\equiv\int_0^{\infty}dx\,e^{-yx}\exp(-x^2/2)\,,$$

$$\bar{\nu}_2{}'\equiv\bar{\nu}_2[1+2\langle a^\dagger a\rangle_s(\gamma\omega_0)^2(\nu_1\nu_2)^{-1}]^{1/2}\,.$$

When we substitute Eq. (5.3) in Eq. (B7) we obtain

$$\alpha=\frac{N}{2\omega_D}\frac{(\gamma\omega_0)^2}{\langle a^\dagger a\rangle_s}\frac{1}{2}I(\bar{\nu}_2)\left[1+\frac{N_T}{N}\right]=\frac{\nu_r}{4\langle a^\dagger a\rangle_s}\left[\frac{N}{N_T}+1\right], \quad \text{(B8)}$$

where the second equality follows from the definition of N_T given in Eq. (5.2). When we use the definition

$$N\to N_a+N_b\,,\quad N_T\to N_a-N_b\,,\quad \langle a^\dagger a\rangle_s\to\langle n\rangle\,,$$

and

$$\nu_r\to\nu/Q\,,$$

the expression for α becomes

$$\alpha=\frac{1}{4}\frac{\nu}{Q}\frac{1}{\langle n\rangle}\left[\frac{N_a+N_b}{N_a-N_b}+1\right],$$

which has the same form as Eq. (9) of Ref. 13. Thus the decay constant for the average electric field has the same form whether or not we include the center-of-mass motion. Note, however, $\langle a^\dagger a\rangle_s$ is a different function of the parameters of the system when the center-of-mass motion is included.

[13] M. Scully and W. E. Lamb, Phys. Rev. Letters **16**, 853 (1966).