PHYSICAL REVIEW VOLUME 156, NUMBER 1 5 APRIL 1967

Effects of He^3-He^3 Interactions in Dilute Solutions of He^3 in He^4

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(Received 10 November 1966)

Using the classical gas expression for the diffusion coefficient and experimental measurements of the spin diffusion coefficient of a 5% solution of He^3 in He^4 at temperatures up to 1°K, we have extended the effective interaction between He^3 quasiparticles, as derived by Bardeen, Baym, and Pines, to larger values of the momentum. This potential is then used in first-order perturbation theory to calculate the high-temperature magnetic susceptibility of dilute solutions and the variation in the He^3 chemical potential with concentration. From the latter, and from measurement of the phase-separation curve at finite temperature, the maximum solubility of He^3 in He^4 is determined down to $T=0$. Agreement with experiment is good.

I. EFFECTIVE INTERACTION

The spin diffusion coefficient D of dilute solutions of He^3 in He^4 was first measured by Garwin and Reich[1] at temperatures down to 0.5°K. Although this work was performed under applied pressure, the results are qualitatively the same as the recent work of Anderson *et al.*[2] and of Edwards[3] in the temperature region below 1°K and above those temperatures at which Fermi statistics become important, or $T \gtrsim 2T_F$. For a 5% solution, the latter condition is roughly $T \gtrsim 0.6$°K.

The main features are the following: Above about 1°K, the spin diffusion coefficients of solutions of different concentration become equal, indicating that the predominant limiting mechanism is the scattering of He^3 from thermal excitations of the background He^4. This problem has been discussed by Khalatnikov.[4] At lower temperatures, but above $\sim 2T_F$, D is very nearly proportional to n_3^{-1}, where n_3 is the He^3 number density in the solution. This proportionality indicates that the rate of diffusion is determined by He^3-He^3 scattering, which is to say, by the effective He^3-He^3 interaction. Boltzmann statistics are obeyed in this temperature region. Finally, for these temperatures, D is approximately proportional to T^{-1}, implying that the interaction is very different from the popular hard-sphere model, which predicts[5] $D \sim T^{1/2}$.

For $T < 1$°K, and such that Boltzmann statistics are valid, we may use the classical gas expression for diffusion[5]:

$$D = \frac{3}{8}\frac{\kappa T}{m^* n_3}\frac{1}{\Omega^{(1,1)}}, \tag{1}$$

where

$$\Omega^{(1,1)} = \left(\frac{4\pi\kappa T}{m^*}\right)^{1/2} \int_0^\infty d\gamma\, e^{-\gamma^2}\gamma^5 \times \int_0^\pi d\theta\, \sin\theta(1-\cos\theta)\frac{d\sigma(\gamma,\theta)}{d\Omega}. \tag{2}$$

m^* is the effective mass of a He^3 quasiparticle; γ is the reduced initial relative speed and is related to the momentum k of each particle in the center-of-mass frame by

$$k = \gamma(m^*\kappa T)^{1/2}; \tag{3}$$

θ is the center-of-mass–frame scattering angle; and $d\sigma/d\Omega$ is the differential scattering cross section in the center-of-mass frame. In the present calculation, we take $m^* = 2.46 m_3$; m_3 is the mass of a He^3 atom, and this value of m^* is determined by specific-heat measurements[2] on a 5% solution at very low temperature.

The present calculation is formally the same as that of the spin diffusion coefficient of He^3 gas at temperatures such that Boltzmann statistics may be used. The latter has been treated in the literature by Emery.[6] As the spin diffusion rate is determined by the mutual scattering of opposite spin quasiparticles, which are distinguishable, one need not symmetrize the scattering amplitude. Furthermore, because the He^3-He^3 effective interaction is weak, the first Born approximation may be used to calculate this amplitude.[7] We make the simplifying assumption, as in Bardeen, Baym, and Pines (BBP), that the effective interaction is a function of the distance between He^3 atoms alone, and so we have, in the center-of-mass frame,

$$d\sigma/d\Omega = (m^*/4\pi\hbar^2)^2 |V(q)|^2, \tag{4}$$

where q is the magnitude of the momentum transferred in the scattering process. In terms of the variables appearing in Eqs. (2) and (3),

$$q = 2k\sin\tfrac{1}{2}\theta = 2\gamma(m^*\kappa T)^{1/2}\sin\tfrac{1}{2}\theta. \tag{5}$$

$V(q)$ is the Fourier transform of the effective interac-

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* National Science Foundation Predoctoral Fellow.

† Work supported in part by National Science Foundation Grant No. NSF GP 4937.

[1] R. L. Garwin and H. A. Reich, Phys. Rev. **115**, 1478 (1959).

[2] A. Anderson, D. O. Edwards, R. Roach, R. Sarwinski, and J. C. Wheatley, Phys. Rev. Letters **17**, 367 (1966).

[3] D. O. Edwards (private communication).

[4] I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965).

[5] J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, *Molecular Theory of Gases and Liquids* (John Wiley & Sons, Inc., New York, 1954), Chap. 8.

[6] V. J. Emery, Phys. Rev. **133**, A661 (1964).

[7] J. Bardeen, G. Baym, and D. Pines, Phys. Rev. Letters, **17**, 372 (1966).

tion,

$$V(q)=\int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}}V(r). \tag{6}$$

Combining (1) through (5) and setting $\sin\frac{1}{2}\theta=x$,

$$D=\frac{3\pi^2}{8}\left(\frac{\kappa T}{m^*\pi}\right)^{1/2}\left(\frac{\hbar^2}{m^*}\right)^2\frac{1}{n_3}\times\left\{\int_0^\infty e^{-\gamma^2}\gamma^5 d\gamma\int_0^1 x^3dx\,|V[2\gamma x(m^*\kappa T)^{1/2}]|^2\right\}^{-1}. \tag{7}$$

One of the integrations may be done after a coordinate transformation, leading to

$$D=\frac{3\pi^2}{4}\left(\frac{\kappa T}{m^*\pi}\right)^{1/2}\left(\frac{\hbar^2}{m^*}\right)^2\frac{1}{n_3}\times\left\{\int_0^\infty e^{-\gamma^2}\gamma^3 d\gamma\,|V[2\gamma(m^*\kappa T)^{1/2}]|^2\right\}^{-1}. \tag{8}$$

We now use this expression for analysis of the measurements of Anderson *et al.*[2] for a 5% solution, for $0.5°\text{K}\leq T\leq 1°\text{K}$. In addition, we demand that $V(q)$ approximate the potential of BBP for $k/\hbar\lesssim 6\times10^7\ \text{cm}^{-1}$. This latter potential is

$$V_{\text{BBP}}(q)=V_0\cos(\beta q/\hbar), \tag{9}$$

with $\beta=3.16$ Å and $V_0=-0.075m_4s^2/n_4$, where m_4 is the mass of a He^4 atom, s is the velocity of first sound in He^4 at $T=0$, and n_4 is the number density of pure He^4 at $T=0$.

We have chosen to work with the function

$$V(q)=|V_0|(\alpha_1+\alpha_2\exp[-C_2(q/2k_0)^2]+\alpha_3\exp[-C_3(q/2k_0)^2]), \tag{10}$$

where V_0 is as in BBP and $k_0/\hbar=0.497$ Å^{-1} is the wave number for which V_{BBP} is zero. The reasons for choosing this functional form are that it is easily made to display the desired shape and that the integrals which arise in various stages of the theory presented below can be done in closed form. Using (10) and (8) we obtain

$$D=\frac{3\pi^2}{4}\left(\frac{\kappa}{m^*\pi}\right)^{1/2}\left(\frac{\hbar^2}{m^*V_0}\right)^2\frac{1}{n_3}\left(\frac{T^{1/2}}{M(T)}\right), \tag{11}$$

where

$$M(T)=\frac{\alpha_1^2}{2}+\frac{\alpha_2^2}{2}\left(1+\frac{2m^*\kappa TC_2}{k_0^2}\right)^{-2}+\frac{\alpha_3^2}{2}\left(1+\frac{2m^*\kappa TC_3}{k_0^2}\right)^{-2}+\alpha_1\alpha_2\left(1+\frac{m^*\kappa TC_2}{k_0^2}\right)^{-2}+\alpha_1\alpha_3\left(1+\frac{m^*\kappa TC_3}{k_0^2}\right)^{-2}+\alpha_2\alpha_3\left(1+\frac{m^*\kappa T(C_2+C_3)}{k_0^2}\right)^{-2}. \tag{12}$$

After some numerical work we chose as parameters

$$\alpha_1=10;\ \alpha_2=-1.48;\ \alpha_3=-9.60;\quad C_2=4.0752;\ C_3=0.0564. \tag{13}$$

These parameters are not by any means unique, nor, for that matter, is there any physical significance in the choice of the function (10) as $V(q)$. The general shape of the potential is the quantity of interest, and that is what should be viewed as having been determined here, aside from inaccuracies imposed by the approximate nature of the theory and by experimental uncertainty. The greatest source of error probably lies in the use of an approximate solution to the transport equation. The neglect of degeneracy effects in the statistics and experimental inaccuracies contribute relatively less error. Considering all of these sources, we feel that $V(q)$ should contain some 10–15% inaccuracy overall. Figure 1 compares the present $V(q)$ with $V_{\text{BBP}}(q)$. Figure 2 shows the experimental D at 5% concentration and our attempt to match it.

$V(q)$ cannot be expected to be valid beyond $q\simeq 3k_0$. This is clear from the integral appearing in (8), which is proportional to

$$\int_0^\infty q^3dq\,|V(q)|^2\exp(-q^2/4m^*\kappa T). \tag{14}$$

The weight function multiplying $|V(q)|^2$ is a function of T and is peaked at $q/k_0\simeq 2$ at the largest T we consider, or 1°K. For $q/k_0>3$, there is essentially no contribution to the integral. We also note that the potential deviates somewhat from $V_{\text{BBP}}(q)$ at very small q, being 8% larger in magnitude at $q=0$. This unfortunate consequence of the choice of parameters made above has no effect on observables such as D or the magnetic susceptibility χ, as large values of q are much more heavily weighted. On the other hand, for the determination of the He^3 chemical potential and the solubility curve, it becomes somewhat more important. Further comments on this are found in Sec. III.

To conclude this section, we use our $V(q)$ in (8) to determine D at a concentration other than 5%. This is not completely trivial, as D is not precisely proportional to n_3^{-1}, the reason being that m^* depends slightly on concentration. Anderson *et al.*[2] also measured D for a 1.3% solution, for which m^* is found from specific-heat measurements[2] to be $2.38m_3$. Figure 3 compares the experimental D with the present theory. Agreement is well within experimental error and inaccuracies imposed by the approximate nature of the theory. From this agreement we surmise that the change in $V(q)$ with concentration cannot be very great, at least for concentrations below 5%.

II. HIGH-TEMPERATURE MAGNETIC SUSCEPTIBILITY

The relative magnetic susceptibility of mixtures of various concentrations has been measured both at low

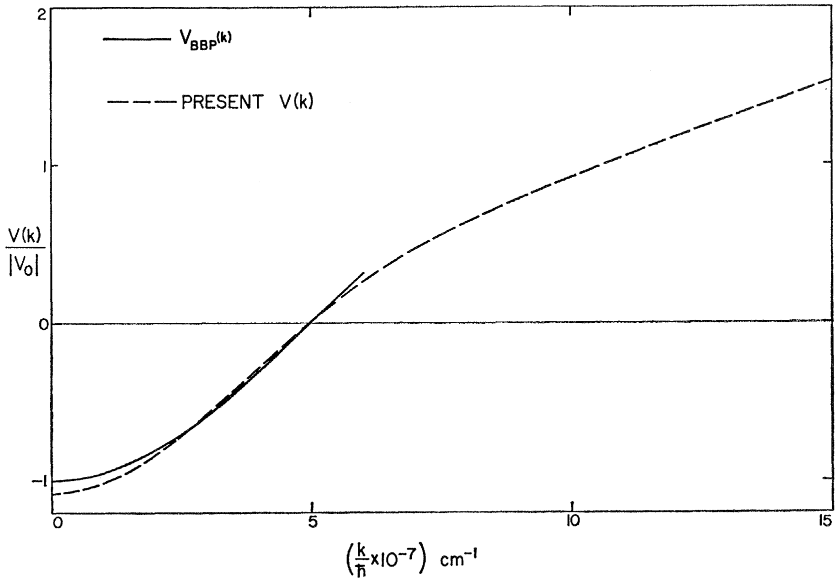

FIG. 1. Graph of the effective potential $V(k)/|V_0|$ as a function of $k/\hbar$ (cm^{-1}).

temperatures much less than[2] T_F and at relatively high temperatures such that the He^3 is almost nondegenerate.[8] The departure from ideal Fermi gas behavior comes about mainly through the effective He^3-He^3 interaction, as He^4 has no spin and interacts with "spin-up" and "spin-down" He^3 atoms in an equivalent manner. Because the effective interaction is weak, we may make a perturbative calculation of χ, the magnetic susceptibility, in powers of the interaction strength divided by T, at high T, or in powers of the interaction strength divided by T_F, the Fermi temperature, at low T. This may be done via the formal development of the partition function, taking the appropriate derivatives to obtain χ. The same result may be obtained in a more transparent manner by consideration of the quasiparticle distribution in momentum space in the presence of a

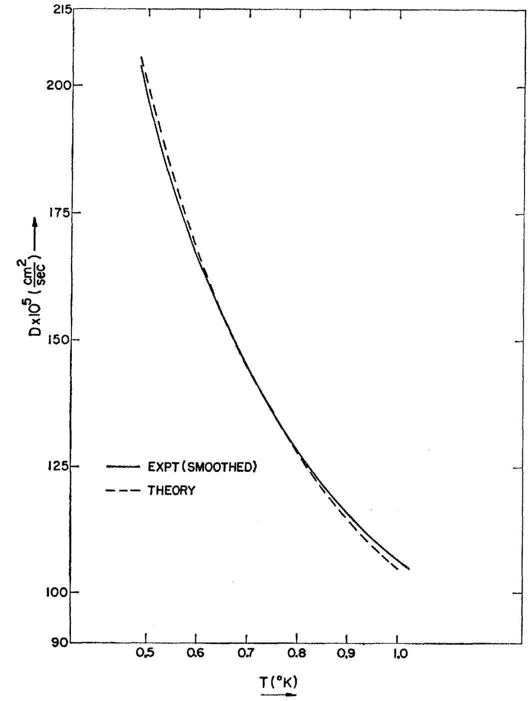

FIG. 2. Graph of the spin diffusion coefficient D(cm^2/sec) as a function of T(°K) at 5% concentration.

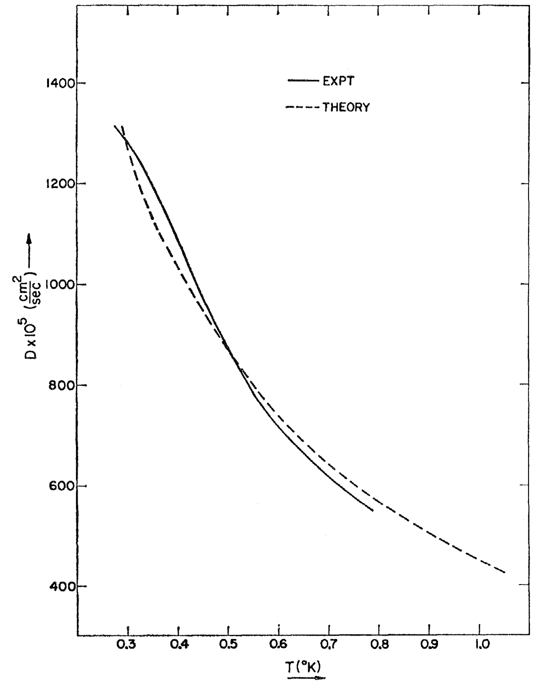

FIG. 3. Graph of the spin diffusion coefficient D(cm^2/sec) as a function of T(°K) at 1.3% concentration.

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[8] D. L. Husa, D. O. Edwards, and J. R. Gaines, Phys. Letters **21**, 28 (1966).

magnetic field H. The second derivation is given here.

The energy of a He^3 quasiparticle in the presence of a magnetic field is given by

$$\epsilon_\sigma(p)=E_0+\frac{p^2}{2m^*}-\mu_B\sigma H+\frac{N_3}{\Omega}V(0)$$
$$-\int\frac{d^3q}{(2\pi\hbar)^3}V(\mathbf{p}-\mathbf{q})n_\sigma(q) \quad (15)$$

to first order in $V(q)$. Here, Ω is the volume, N_3 is the number of atoms of He^3, and μ_B is proportional to the nuclear magneton. $\sigma=\pm1$ depending on whether the spin is parallel or antiparallel to the applied field H, and $n_\sigma(q)$ is the quasiparticle distribution function. The various terms in $\epsilon_\sigma(q)$ may be identified as the binding energy of a single He^3 atom in the solution, kinetic energy, magnetic-field interaction energy, direct-(Hartree) interaction energy, and exchange energy. The Hartree term will not appear in the result of the present calculation as it does not distinguish between pairs of quasiparticles of parallel spin and pairs of opposite spin and so will not affect the susceptibility to first order in $V(0)$. E_0 will also have no effect upon the susceptibility. It is convenient to remove this term from the calculation by defining a modified He^3 chemical potential μ_3'':

$$\mu_3''=\mu_3-E_0. \quad (16)$$

We first consider the very high-temperature case, $T\gg T_F$, for which Boltzmann statistics apply. Denoting spin parallel to H by ↑,

$$n_\uparrow(p)=\exp[-\beta(\epsilon(p)-\mu_3)]$$
$$=\exp\left[-\beta\left(\frac{p^2}{2m^*}+n_3V(0)\right.\right.$$
$$\left.\left.-\int\frac{d^2q}{(2\pi\hbar)^3}V(\mathbf{p}-\mathbf{q})n_\uparrow{}^0(q)-\mu_3''\right)\right]\exp[\beta\mu_BH]$$
$$\times\exp\left[\beta\int\frac{d^3q}{(2\pi\hbar)^3}V(\mathbf{p}-\mathbf{q})(n_\uparrow(q)-n_\uparrow{}^0(q))\right], \quad (17)$$

where $n_\uparrow{}^0(q)$ is $n_\uparrow(q)$ for $H=0$. We expand the last two exponentials to first order in V and in H and then integrate over all momenta to obtain the number density of spin-up quasiparticles to first order in V and H:

$$n_{3\uparrow}=(1+\beta\mu_BH)\left\{n_{3\uparrow}{}^0+\beta\int\frac{d^3pd^3q}{(2\pi\hbar)^6}\right.$$
$$\times\exp\left[-\beta\left(\frac{p^2}{2m^*}-\mu_3''\right)\right]$$
$$\left.\times V(\mathbf{p}-\mathbf{q})(n_\uparrow(q)-n_\uparrow{}^0(q))\right\}. \quad (18)$$

$n_{3\uparrow}{}^0$ is the number density of spin-up quasiparticles for $H=0$. To this same order of validity, we may use the relations

$$n_\uparrow(q)-n_\uparrow{}^0(q)=(n_{3\uparrow}-n_{3\uparrow}{}^0)$$
$$\times e^{-\beta q^2/2m^*}\left(\int\frac{d^3p}{(2\pi\hbar)^3}e^{-\beta p^2/2m^*}\right)^{-1} \quad (19)$$

and

$$e^{\beta\mu_3''}=\frac{n_3}{2}\left(\int\frac{d^3q}{(2\pi\hbar)^3}e^{-\beta q^2/2m^*}\right)^{-1}, \quad (20)$$

to write (18) as

$$n_{3\uparrow}=(1+\beta\mu_BH)[n_{3\uparrow}{}^0+\beta n_3(n_{3\uparrow}-n_{3\uparrow}{}^0)\langle V\rangle/2], \quad (21)$$

where

$$\langle V\rangle\equiv\frac{\int d^3pd^3q\ V(\mathbf{p}-\mathbf{q})\exp[-\beta(p^2+q^2)/2m^*]}{\int d^3pd^3q\ \exp[-\beta(p^2+q^2)/2m^*]}. \quad (22)$$

Similarly, one may show that for spin-down quasiparticles,

$$n_{3\downarrow}=(1-\beta\mu_BH)[n_{3\downarrow}{}^0+\beta n_3(n_{3\downarrow}-n_{3\downarrow}{}^0)\langle V\rangle/2]. \quad (23)$$

In the system under consideration, which is paramagnetic, $n_{3\uparrow}{}^0=n_{3\downarrow}{}^0=n_3/2$. Using this fact, and subtracting (23) from (21), we find the magnetization per unit volume to be

$$\frac{M}{\Omega}=\mu_B(n_{3\uparrow}-n_{3\downarrow})=\frac{\beta\mu_B{}^2Hn_3}{1-\beta\frac{1}{2}n_3\langle V\rangle}, \quad (24)$$

from which

$$\chi=\frac{N_3\mu_B{}^2/\kappa}{T-n_3\langle V\rangle/2\kappa}. \quad (25)$$

We repeat that the criteria for the validity of this expression are that Boltzmann statistics apply and that

$$|n_3V(q)/2\kappa T|\ll1.$$

This latter condition turns out to be approximately $x\ll T$°K where x is the mole fraction of He^3 in the solution. The apparent ferromagnetic (or antiferromagnetic, as $\langle V\rangle<0$ for $T<0.34$°K) behavior which appears here is completely spurious because the transition temperature is so small that neither of the above criteria is satisfied.

Indeed, the correction that we have derived to the ideal gas susceptibility is so small that even for $T/T_F\simeq2$, the deviation from Boltzmann statistics is equally important and must be included. We may make this alteration by starting from Fermi statistics and developing a series expansion of the magnetization in powers of $\exp(\beta\mu_3'')$. In turn, $\exp(\beta\mu_3'')$ may be written as a series expansion in powers of $(T_F/T)^{3/2}$ where T_F is defined by

$$\kappa T_F=p_F{}^2/2m^*=(3\pi^2n_3)^{2/3}\hbar^2/2m^*. \quad (26)$$

The derivation is otherwise formally the same as that given above. Indeed, Boltzmann statistics are equivalent to the lowest-order term in the proposed expansion.

We start from

$$n_\uparrow(p)=\left\{\exp\left[\beta\left(\frac{p^2}{2m^*}-\mu_B H-\int\frac{d^3q}{(2\pi\hbar)^3}\times V(\mathbf{p}-\mathbf{q})n_\uparrow(q)+n_3V(0)-\mu_3''\right)\right]+1\right\}^{-1}. \quad (27)$$

The expression for $n_\downarrow(p)$ is given by the alterations

$$H \text{ becomes} -H \text{ and } (\uparrow) \text{ becomes } (\downarrow). \quad (27')$$

Expanding the difference $n_\uparrow(p)-n_\downarrow(p)$ to first order in H and in V and integrating over all momenta gives

$$n_{3\uparrow}-n_{3\downarrow}=\frac{\mu_BH}{\kappa T}\left\{2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{g(p)}{[1+g(p)]^2}+\frac{2}{\kappa T}\int\frac{d^3pd^3q}{(2\pi\hbar)^6}\frac{g(p)g(q)}{[1+g(p)]^2[1+g(q)]^2}\left(V(\mathbf{p}-\mathbf{q})-2\frac{[1+g(q)][1-g(p)]}{1+g(p)}[V(0)-\tfrac{1}{2}V(\mathbf{p}-\mathbf{q})]\right)\right\}, \quad (28)$$

where

$$g(k)\equiv\exp[-\beta(k^2/2m^*-\mu_3'')]. \quad (29)$$

$g(k)$ is a small number for all k at sufficiently large T, as $\mu_3''\rightarrow-\infty$ for $T\rightarrow\infty$. We expand in powers of g, keeping through the third order. The result is

$$\frac{\Omega\mu_B(n_{3\uparrow}-n_{3\downarrow})}{H}=\chi=\frac{\mu_B{}^2\Omega}{\kappa T}\left\{2\int\frac{d^3p}{(2\pi\hbar)^3}\times(g(p)-2[g(p)]^2+3[g(p)]^3)+\frac{2}{\kappa T}\int\frac{d^3pd^3q}{(2\pi\hbar)^6}\times[V(0)(-2g(q)g(p)+10g(q)[g(p)]^2)-\tfrac{1}{2}V(\mathbf{p}-\mathbf{q})(-4g(q)g(p)+18g(q)[g(p)]^2)]\right\}. \quad (30)$$

In order to remove the chemical potential from (30), we solve the following expression for $\exp(\beta\mu_3'')$ as a power series in $(T_F/T)^{3/2}$:

$$n_3=\frac{(2m^*\kappa T_F)^{3/2}}{3\pi^2}=\int\frac{d^3p}{(2\pi\hbar)^3}(n_\uparrow(p)+n_\downarrow(p)), \quad (31)$$

where $n_\uparrow(p)$ is given by (27) and $n_\downarrow(p)$ by the modifications (27′). As usual, only first-order terms in H and in V are retained. After some algebra we obtain

$$\exp(\beta\mu_3'')=\frac{4}{3\sqrt{\pi}}\left(\frac{T_F}{T}\right)^{3/2}\left\{1+\frac{\sqrt{2}}{3\sqrt{\pi}}\left(\frac{T_F}{T}\right)^{3/2}+\frac{4}{9\pi}\left(1-\frac{4}{3\sqrt{3}}\right)\left(\frac{T_F}{T}\right)^3+\frac{n_3V(0)}{\kappa T}\left(1+\frac{\sqrt{2}}{3\sqrt{\pi}}\left(\frac{T_F}{T}\right)^{3/2}\right)-\frac{1}{2}\frac{n_3\langle V\rangle}{\kappa T}\left(1+\left(\frac{T_F}{T}\right)^{3/2}\frac{4\sqrt{2}}{3\sqrt{\pi}}\left[1-\frac{3}{4}\frac{\langle V\rangle'}{\langle V\rangle}\right]\right)\right\}, \quad (32)$$

where $\langle V\rangle$ is defined by (22) and

$$\langle V\rangle'\equiv\frac{\int d^3pd^3q\ V(\mathbf{p}-\mathbf{q})\exp[-\beta(p^2+2q^2)/2m^*]}{\int d^3pd^3q\ \exp[-\beta(p^2+2q^2)/2m^*]}. \quad (33)$$

Combining (32) and (30), and rearranging the expansion in such a way as to write χ^{-1} as a power series, we arrive at the final result:

$$\frac{1}{\chi}=\frac{1}{\chi_0}\left[1+\frac{\sqrt{2}}{3\sqrt{\pi}}\left(\frac{T_F}{T}\right)^{3/2}-\frac{2}{3\pi}\left(\frac{16}{9\sqrt{3}}-1\right)\left(\frac{T_F}{T}\right)^3-\frac{n_3\langle V\rangle}{2\kappa T}f\right], \quad (34)$$

where

$$f=1+\frac{3}{(2\pi)^{1/2}}\left(\frac{T_F}{T}\right)^{3/2}\left(1-\frac{\langle V\rangle'}{\langle V\rangle}\right) \quad (35)$$

and

$$\chi_0=\frac{N_3\mu_B{}^2/\kappa}{T}. \quad (36)$$

We define T^* by

$$\frac{T^*}{T}\equiv1+\frac{\sqrt{2}}{3\sqrt{\pi}}\left(\frac{T_F}{T}\right)^{3/2}-\frac{2}{3\pi}\left(\frac{16}{9\sqrt{3}}-1\right)\left(\frac{T_F}{T}\right)^3. \quad (37)$$

T^* is the magnetic temperature of an ideal Fermi gas at high temperature; that is,

$$\chi_0=\frac{N_3\mu_B{}^2/\kappa}{T^*} \quad (38)$$

is a valid approximation to the magnetic susceptibility of an ideal Fermi gas for $T\gtrsim T_F$. Indeed, comparison of this expression with the numerical work of McDougall and Stoner[9] shows that (38) is valid to better than 1% down to $T/T_F=0.6$, which is sufficient for our purposes. Using (37) in (34),

$$\frac{1}{\chi}=\frac{\kappa}{N_3\mu_B{}^2}\left(T^*-\frac{n_3\langle V\rangle}{2\kappa}f\right). \quad (34')$$

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[9] J. McDougall and Edmund C. Stoner, Phil. Trans. Roy. Soc. London 237, 67 (1938).

In terms of the effective potential (10),

$$\langle V\rangle = |V_0|\left\{\alpha_1+\alpha_2\left(1+\frac{m^*\kappa TC_2}{k_0^2}\right)^{-3/2}\right. \\ \left.+\alpha_3\left(1+\frac{m^*\kappa TC_3}{k_0^2}\right)^{-3/2}\right\}, \qquad (39)$$

$$\langle V\rangle' = |V_0|\left\{\alpha_1+\alpha_2\left(1+\frac{3m^*\kappa TC_2}{4k_0^2}\right)^{-3/2}\right. \\ \left.+\alpha_3\left(1+\frac{3m^*\kappa TC_3}{4k_0^2}\right)^{-3/2}\right\}. \qquad (40)$$

In Figs. (4) and (5) we have plotted $C\chi^{-1}$ versus T^*/T_F for concentrations of 5, 10, and 15%, and for temperatures $>0.6T_F$. Here C is the Curie constant $N_3\mu_B{}^2/\kappa$. For reference purposes, plots of $C\chi_0{}^{-1}$ versus T^*/T_F are also given at these concentrations. We may compare our results with the measurements of Husa *et al.*[8] These authors plot χ^{-1} in arbitrary units versus $C/T_F\chi_0$, which is T^*/T_F. They find linear experimental curves at all concentrations investigated. Because they do not give the absolute magnitude of χ^{-1}, we can compare our results only with the horizontal intercepts of the extrapolated experimental curves and, of course, with the observed linear behavior. The experimental intercepts are approximately zero for He³ concentrations less than ~10%; for larger concentrations up to 15% they are small and positive, being on the order of $\frac{1}{10}$.

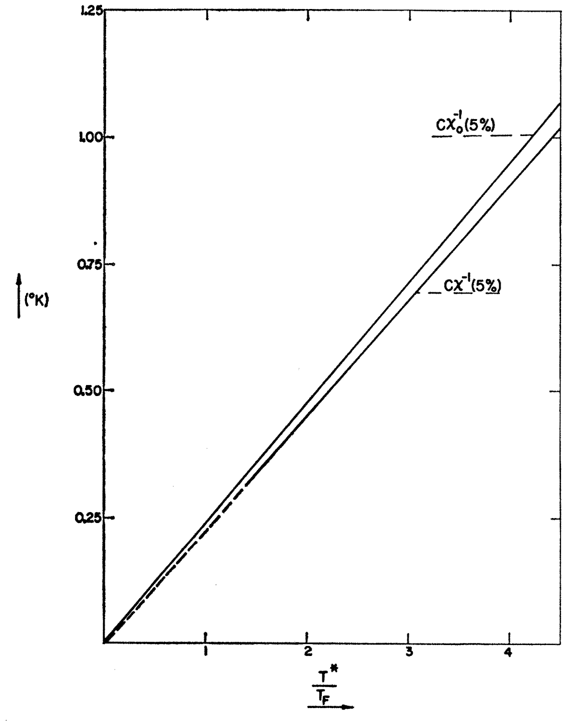

FIG. 4. Graph of $C\chi^{-1}$(°K) as a function of T^*/T_F at 5% concentration.

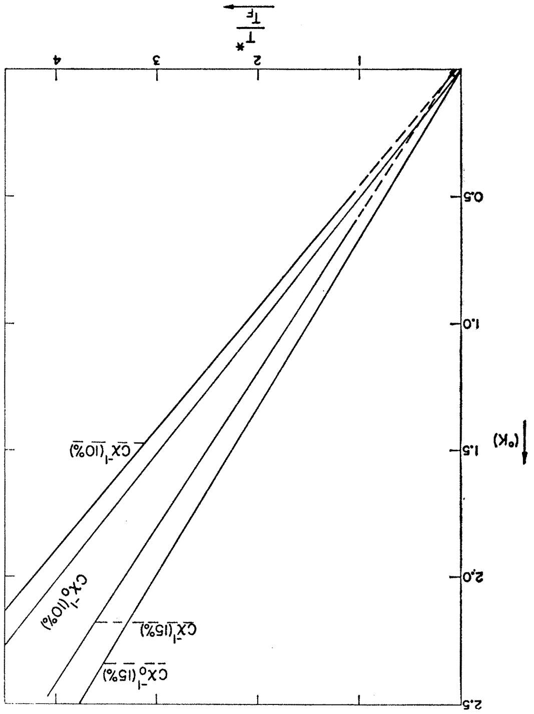

FIG. 5. Graph of $C\chi^{-1}$(°K) as a function of T^*/T_F at 10 and 15% concentration.

This intercept is a rough measure of the effective interaction strength and sign. A positive intercept is taken to imply that a predominantly repulsive interaction is being sampled. We comment, however, that because $V(q)$ is a widely varying function of q, the situation can be quite different. In particular, a zero intercept need not necessarily mean that the interaction is very weak. A more sensitive measure of the interaction is the deviation of χ^{-1} from $\chi_0{}^{-1}$, as a function of temperature, as this quantity is proportional to $n_3\langle V\rangle f$. When χ^{-1} is smaller than $\chi_0{}^{-1}$, the part of the interaction being sampled is predominantly repulsive. At the concentrations and temperatures considered here, this is the case, as we see from the theoretical plots in Figs. (4) and (5). Furthermore, the theoretical curves for the interacting system are very nearly straight lines, as are the experimental curves of Husa *et al.*,[8] and we also find that only for the relatively large concentrations of 10 and 15% do the extrapolated theoretical curves have horizontal intercepts appreciably different from zero. Nevertheless, the deviation of $T^*-n_3\langle V\rangle/2\kappa f$ from T^* is significant even at concentrations as small as 5%.

Finally, we should point out that at lower temperatures and at low concentrations $\lesssim 6\%$, the attractive part of $V(q)$, which lies at small q, will become more important, causing χ^{-1} to become larger than $\chi_0{}^{-1}$. This is consistent with the very low-temperature measurements of Anderson *et al.*[2] and with the theoretical considerations of BBP.

III. CHEMICAL POTENTIAL AND SOLUBILITY CURVE

We define $\mu_3'(n_3)$ by

$$\mu_3'(n_3)\equiv\mu_3(n_3,T)-\mu_f(m^*,n_3,T)\,, \tag{41}$$

that is, by the difference between the chemical potential of the He^3 in the solution and that of an ideal Fermi gas having the same effective mass and number density as in the solution. As pointed out in BBP, the variation of μ_3' with concentration gives a direct evaluation of $V(0)$ plus an exchange correction. From (32) we may obtain an expression for μ_3' to first order in V:

$$\mu_3'=\mu_3''+E_0-\mu_f\,,$$

or

$$\mu_3'(n_3)=E_0+n_3V(0)-\tfrac{1}{2}n_3\langle V\rangle\left[1+\left(\frac{T_F}{T}\right)^{3/2}\frac{\sqrt{2}}{\sqrt{\pi}}\left(1-\frac{\langle V\rangle'}{\langle V\rangle}\right)\right]. \tag{42}$$

If we can determine E_0, then we have complete knowledge of the He^3 chemical potential. We may do this through the use of a point on the phase-separation curve. On this curve, the chemical potential in the solution (He^4-rich phase) must be equal to that of the He^3-rich phase, as the two are in equilibrium. Following Edwards *et al.*,[10] the latter phase is assumed to be pure He^3, in which case the chemical potential at temperature T_s may be written as

$$\mu_3^{\text{pure}}(T_s)=-L_3^0-\int_0^{T_s}S_3^{\text{pure}}(T)dT. \tag{43}$$

L_3^0 is the latent heat of vaporization per atom of pure He^3 at $T=0$ and $S_3^{\text{pure}}(T)$ is the entropy per atom of the pure liquid. T_s is the temperature at which phase separation occurs. Combining (42) and (43),

$$-E_0-L_3^0-n_3V(0)+\tfrac{1}{2}n_3\langle V\rangle\times\left[1+\left(\frac{T_F}{T_s}\right)^{3/2}\frac{\sqrt{2}}{\sqrt{\pi}}\left(1-\frac{\langle V\rangle'}{\langle V\rangle}\right)\right]=\mu_f(m^*,n_3,T_s)+\int_0^{T_s}S_3^{\text{pure}}(T)dT. \tag{44}$$

TABLE I. Values of $(-E_0-L_3^0)/\kappa$ in °K as determined from phase-separation measurements on solutions of different He^3 mole fraction x.

x	$(-E_0-L_3^0)/\kappa$ in °K
0.081	0.274
0.099	0.285
0.121	0.271
0.151	0.274

Edwards *et al.*[10] give measurements of T_s at various concentrations and then calculate the right-hand side of (44). The value of this quantity for one concentration is sufficient information for us to determine $-E_0-L_3^0$, and hence the entire phase-separation curve, using experimental measurements of He^3 specific heat[11] for $S_3^{\text{pure}}(T)$. In order to carry out this program, however, we must treat the exchange term more carefully. As it appears in (44), it is valid for $T/T_F\gtrsim1$. This condition is not satisfied; rather, $T_s/T_F\lesssim0.6$. We may develop a low-temperature expression for μ_3'' by expanding (31) as a power series in $(\kappa T/\mu_3'')^2$ and solving the resulting equation for μ_3''. To order $V(T/T_F)^2$, the result of this algebra is of the form

$$\mu_3''=\mu_f(m^*,n_3,T)+n_3V(0)-\tfrac{1}{2}n_3|V_0|\left\{\frac{\langle V\rangle''}{|V_0|}+G\frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right\}, \tag{45}$$

where

$$\langle V\rangle''\equiv\frac{3}{4\pi p_F^3}\int_{q<p_F}d^3q\;V(\mathbf{p}_F-\mathbf{q})\,, \tag{46}$$

and where G is a more complicated function of the concentration, generally having value ~1. Using this expression for μ_3'', (44) becomes

$$-E_0-L_3^0-n_3V(0)+\tfrac{1}{2}n_3|V_0|\left\{\frac{\langle V\rangle''}{|V_0|}+G\frac{\pi^2}{12}\left(\frac{T_s}{T_F}\right)^2\right\}=\mu_f(m^*,n_3,T_s)+\int_0^{T_s}S_3^{\text{pure}}(T)dT. \tag{47}$$

Using (47) and the values of Edwards *et al.*[10] for the right-hand side of (47), we may determine $-E_0-L_3^0$. We may find a value from each of their measurements of $T_s(n_3)$, or $T_s(x)$, where x is the mole fraction of He^3. These values are given in Table I. We adopt the mean value

$$(-E_0-L_3^0)/\kappa=0.276°\text{K}. \tag{48}$$

From this value and (47), we are able to determine the entire small-concentration phase-separation curve. In particular, the $T=0$ maximum solubility is given by solving

$$-E_0-L_3^0-n_3V(0)+\tfrac{1}{2}n_3\langle V\rangle''=\mu_f(m^*,n_3,T=0). \tag{49}$$

Using $m^*=2.34m_3$, we find a maximum solubility of ~6.0%.[12] The entire solubility curve, as calculated from

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[10] D. O. Edwards, D. F. Brewer, P. Seligman, M. Skertic, and M. Yaqub, Phys. Rev. Letters **15**, 773 (1965).

[11] W. R. Abel, A. C. Anderson, W. C. Black, and J. C. Wheatley, Phys. Rev. **147**, 111 (1966).

[12] The maximum $T=0$ solubility of He^3 in He^4 has been determined experimentally as 6.0±0.2%. D. O. Edwards (private communication).

(47), is shown in Fig. 6. We emphasize that just one point on the curve is sufficient information for us to determine the remainder of it. Nevertheless, we have used four points in the hope that the mean value of $(-E_0-L_3^0)/\kappa$ is more reliable than any value obtained from a single point. That the four values obtained all lie within $\sim 3\%$ of the mean value is a measure of the accuracy of the theory.

Using $V(0)=-V_0$, which is the result of BBP, one finds $(-E_0-L_3^0)/\kappa \simeq 0.29_5$°K, a larger value than obtained above. The solubility curve is affected to a lesser extent, with the maximum solubility at $T=0$ of about 6%.

IV. SUMMARY AND DISCUSSION

Using experimental values of the spin diffusion coefficient[2] for a 5% solution of He^3 in He^4 at temperatures up to 1°K, we have extended the effective He^3-He^3 potential of Bardeen, Baym, and Pines[7] to larger values of the momentum. From this $V(q)$ we have calculated in turn the spin diffusion coefficient of a 1.3% solution, finding good agreement with experiment,[2] which implies that $V(q)$ cannot be a very strong function of concentration, at least for concentrations below about 5%. We have also calculated the high-temperature magnetic susceptibility of solutions of concentration up to 15%, as well as the He^3 chemical potential as a function of concentration and temperature. From the latter, and from the measurements of Edwards *et al.*,[10] we have determined the phase-separation curve of dilute solutions. In all cases, agreement with experiment[8,10] is good. Nevertheless, because of the approximate nature of the theory and experimental uncertainty in the values of the spin diffusion coefficient, we hestitate to claim that the extended $V(k)$ is accurate to better than 15%. In particular, no effort has been made to determine the change in $V(q)$ with temperature; such an effect is almost certainly present. The nature of the change of $V(q)$ with concentration is also an open question. In addition, there still remains the considerable task of a complete determination of $V(q)$ from a first principles calculation, although some progress has been made in this direction.[7,13–15]

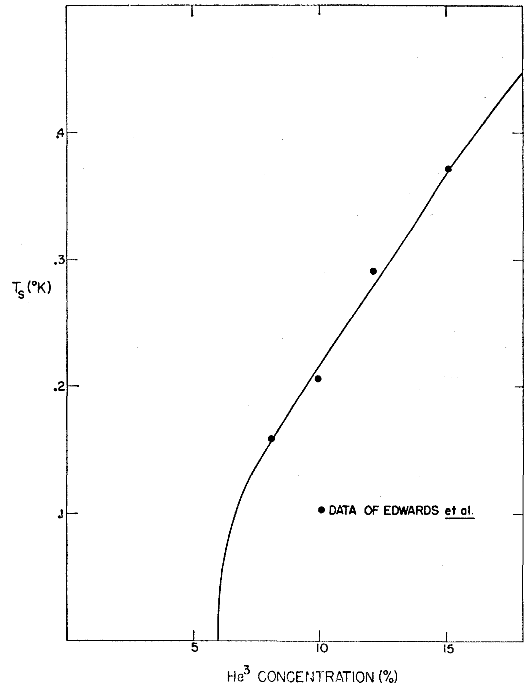

FIG. 6. Graph of the phase-separation temperature T_s(°K) as a function of the concentration of He^3 (%). Data points are those of Edwards *et al.*, Ref. 10.

ACKNOWLEDGMENTS

I would like to thank Professor Gordon Baym for suggesting this problem to me and for helpful discussions and worthwhile criticism throughout the course of the work. I would also like to thank Professor John Bardeen for a critical reading of the manuscript and Professor David Edwards for initial encouragement to consider the problem and for communicating to me unpublished measurements of the spin diffusion coefficient.

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[13] V. J. Emery, Phys. Rev. **148**, 138 (1966).

[14] J. Bardeen, G. Baym, and D. Pines, preceding paper, Phys. Rev. **156**, 207 (1967).

[15] G. Baym, Phys. Rev. Letters **17**, 952 (1966).