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Effective Interaction of He^3 Atoms in Dilute Solutions of He^3 in He^4 at Low Temperatures

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The approximate form of the effective interaction between He^3 atoms in superfluid He^4 is derived from the experimental data on spin diffusion and phase separation in dilute mixtures of He^3 in He^4. The interaction is weak, and attractive at long wavelengths. Calculations of the Fermi-liquid parameters for the normal state of He^3 in solution yield results for the effective mass and spin susceptibility in agreement with experiment. The temperature for a superfluid transition associated with the He^3 is estimated to be $\sim 2\times 10^{-6}$ °K; the maximum solubility of He^3 in He^4 is found to be $\sim 6\%$ at $T=0$. Thermodynamic and microscopic arguments are used to calculate the long-wavelength part of the effective interaction between the He^3 atoms. The contribution arising from the exchange of a virtual He^4 phonon is shown to be large and attractive, while the remaining part of the interaction is almost as large but repulsive; the calculated interaction at long wavelengths is thus weak and attractive and is in excellent agreement with that determined empirically; the physical origin of the weakness of the interaction is that He^3 is an isotopic impurity. Finally, it is estimated that the application of pressure serves to weaken the effective interaction.

I. INTRODUCTION

The experimental study of dilute solutions of He^3 in He^4 received considerable impetus from the heat-capacity measurements of Edwards *et al.*,[1] who concluded from experiments at 100 m°K and above that there should be no phase separation down to $T=0$°K for concentrations of less than about 6% He^3. Their prediction was confirmed in part by Anderson *et al.*,[2] who measured the heat capacity for a solution with a concentration of about 5% He^3, down to a temperature of roughly 10 m°K. They found that for this mixture He^3 was soluble in He^4 down to this temperature, and that the specific heat of the He^3 varied linearly with temperature over a range from 8 m°K to 35 m°K. Subsequently, Anderson *et al.*[3] extended the heat-capacity measurements to temperatures of 3 m°K, and measured as well the spin diffusion coefficient and spin susceptibility of dilute solutions of nominal concentration 1.3% and 5% of He^3. Their measurements show that in these respects the He^3 atoms in dilute solutions behave like a normal Fermi liquid, in agreement with the predictions of Landau and Pomeranchuk.[4] Moreover, since the Fermi momentum is varied by changing the concentration, it is possible to deduce from their experiments an approximation to the magnitude, sign, and momentum dependence of the quasiparticle interactions.[5]

Dilute solutions of He^3 in He^4 are of special interest for at least three reasons:

(1) They represent a new set of Fermi liquids, whose density, and hence degeneracy temperature, may be varied at will. One can thus study, in the He^3 system, the transition from classical gaseous behavior to that characteristic of a low-temperature Fermi liquid for a wide range of densities and temperatures.

(2) As we shall see, these Fermi liquids are unique in that the effective interaction between the He^3 quasiparticles is sufficiently weak that perturbation theory, in the effective potential, may be used to calculate the thermodynamic parameters and transport coefficients.

(3) The effective interaction is also attractive, so that at sufficiently low temperatures there should be a superfluid transition associated with the He^3.

In this paper we describe our derivation of the effective He^3 quasiparticle interaction from the experimental data. We then use the resulting effective interaction to calculate various Fermi-liquid quantities; these are found to be in good agreement with experiment. We also estimate the superfluid transition temperature, and find it to be in the *microdegree* range.

It had been suggested earlier[6] that there should be an attractive interaction between two He^3 atoms in solution which arises from the exchange of a He^4 phonon, and that this interaction might lead to a superfluid transition. We find that in the long-wavelength limit this part of the effective interaction may be determined exactly with the aid of thermodynamic or deformation-potential arguments, and that it is given by

$$V_0^{\mathrm{ph}} = (\partial\mu_{3\uparrow}/\partial n_4)_{n_3}(\partial n_4/\partial n_{3\downarrow})_P = -(1+\alpha)^2 m_4 s^2/n_4\,,$$

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where $\mu_{3\uparrow}$ is the chemical potential of a spin-up He^3, n_4 is the He^4 particle density, $n_{3\downarrow}$ is the density of He^3 with spin down, and P is the pressure; $\alpha\approx0.28$ is the relative increase in effective volume of the mixture resulting from the replacement of a He^4 atom by a He^3 atom. Also s is the sound velocity in pure He^4 while m_4 is the atomic mass of He^4; the quantity m_4s^2/n_4, the effective interaction between a pair of He^4 particles, sets the scale of interaction energies.

However, there is an additional "direct" part of the interaction which may be determined in the limit of very dilute mixtures by thermodynamic arguments; it is

$$V_0^{\mathrm{dir}}=(\partial\mu_{3\uparrow}/\partial n_{3\downarrow})_{n_4}=(1+2\alpha)m_4s^2/n_4.$$

The over-all effective interaction in the long-wavelength limit is then

$$V_0=-\alpha^2m_4s^2/n_4,$$

a result which is in excellent agreement with the corresponding limit of the empirically determined potential.

The physical reason for the very considerable cancellation between the direct and the phonon-induced contributions to the effective interaction $[\alpha^2/(1+\alpha)^2\cong1/20]$ lies in the fact that He^3 is an isotopic impurity. The force fields of the He^3 and the He^4 atoms are identical, but the He^3 atoms have a smaller mass and therefore occupy a slightly larger volume in the liquid than the He^4 atoms. It is primarily through this increase in volume that a He^3 atom becomes aware of the fact that another He^3 in the liquid is not a He^4. For a dilute system, the change in the total free energy resulting from the replacement of two He^4 atoms by two He^3 atoms is essentially that due to the interaction of two holes in the liquid of relative size α; thus the system energy is changed by a factor $\alpha^2\cong0.08$ times the effective interaction between two He^4 atoms. L. S. Campbell (private communication) has made some estimates on this basis. One of us has calculated this effective interaction at large separation by a variational method[7]; the details are discussed later in this paper.

A complete calculation of the net interaction from microscopic theory so as to derive its dependence on momentum or distance is difficult. Emery[8] has made some progress in this direction and has discussed the problems involved.

Throughout this paper we shall confine our attention to the very-low-temperature, or Fermi-liquid regime of the He^3 system, for which a knowledge of the effective interaction between He^3 atoms at momentum transfers less than twice the Fermi momentum suffices to determine the system's behavior. At higher temperatures, larger momentum transfers begin to play an important role; Ebner[9] has constructed a suitable extension of our effective potential to cover this domain, and finds good agreement with the experiments of Edwards *et al.*[1] at the higher temperatures.

We assume that the effective interaction between quasiparticles depends only on the momentum difference between initial and final states and is independent of concentration. The justification for this assumption is as follows. The effective interaction may be defined as a reaction or t matrix in the sense of Brueckner, Bethe, and Goldstone. As such, it includes effects of virtual transitions to states outside of the excluded Fermi sphere and would depend to some extent on the concentration and on the initial state. Our effective interaction may be regarded as the reaction matrix corresponding to a very low concentration of He^3 (of the order 1%, let us say), where concentration dependent effects are negligible. Using this interaction we calculate energies in the Hartree-Fock approximation and scattering in the Born approximation. For larger concentrations ($\sim$5%) we treat incorrectly density-dependent terms in the perturbation-theoretic expansion, terms which correspond to virtual transitions forbidden by the Pauli principle. However, since the effective interaction is weak, and the concentration is still quite low, the errors introduced by neglect of the concentration dependence are small.

We describe in Sec. II our deduction of an effective interaction potential from experiment, and in Sec. III apply it to a determination of various liquid properties. In Sec. IV we derive the exact matrix element for a slowly moving He^3 quasiparticle to emit or absorb a long-wavelength He^4 phonon, and use this result to determine the phonon-induced attractive interaction between He^3 atoms in solution. We give a thermodynamic derivation of the direct contribution to the effective interaction in Sec. V, concluding with microscopic considerations on the magnitude of the effective interaction and a discussion of its pressure dependence in Sec. VI.

II. DERIVATION OF THE EFFECTIVE QUASIPARTICLE INTERACTION

Let us consider first the motion of a single He^3 atom producing a quasiparticle of momentum $\mathbf{p}$ in liquid He^4 at very low temperatures ($T\lesssim100$ m°K). The quasiparticle's energy $E(\mathbf{p})$ may be written as

$$E(\mathbf{p})=E_0+p^2/2m, \tag{1}$$

where E_0 is the energy by which the impurity is bound in the liquid and m is the effective mass of the impurity. The latter is appreciably greater than the mass m_3 of a bare He^3 atom, as a result of the interaction of the He^3 atom with the He^4 background. One may picture the He^3 atom as moving slowly, surrounded by a cloud of virtual He^4 excitations—phonons and rotons; it is this polarization cloud which is responsible for the added mass. (At these temperatures, the impurity lacks

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[8] V. J. Emery, Phys. Rev. **148**, 138 (1966).

[9] C. Ebner, following paper, Phys. Rev. **156**, 222 (1967).

sufficient energy to excite a phonon or roton in the He^4; therefore only virtual processes need to be taken into account.) We shall treat E_0 and m as unknown parameters, to be determined from experiment. We may expect that E_0 will be on the order of the binding energy of a single He^4 atom in the liquid, while m is known from experiment[3] to be on the order of $2.4m_3$.

If we now increase the concentration of He^3 atoms in the solution, we may expect their motion to be modified by their mutual interaction. As we have mentioned earlier, because of the presence of the medium, this interaction will differ from that of two bare He^3 atoms; the van der Waal's attraction and short-range repulsion will be altered and there is, as well, a significant contribution arising from the exchange of virtual He^4 phonons between the impurities. Here we shall make no attempt to calculate from first principles the net effective interaction between two He^3 atoms; rather, we derive it from experiment under the assumptions that it is spin- and velocity-independent, independent of concentration, *and weak*. We thus assume that the effective interaction between two atoms separated by a distance r may be written as

$$V(\mathbf{r})=\int V_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}\frac{d\mathbf{k}}{(2\pi)^3} \tag{2}$$

and use the phase separation curve for $T\gtrsim 200$ m°K, and the spin diffusion experiments in the millidegree region[3] to determine $V_{\mathbf{k}}$. Once $V_{\mathbf{k}}$ is determined for the range of momentum transfers of interest in the low-temperature regions, we may use perturbation theory to calculate the thermodynamic and transport properties of the He^3 atoms in the He^3-He^4 mixtures.

Phase separation measurements permit one to estimate V_0, the spatial average of $V(\mathbf{r})$. Along the phase separation line, the chemical potential $\mu_3(x)$ of a He^3 atom dissolved in the solution must equal $\mu_3{}^{\text{pure}}$, that of a He^3 atom in the separated (and assumed pure) He^3 phase. We may therefore write

$$\begin{aligned}\mu_3(x)=\mu_f(T_s,x)+\mu_3{}'(x)=\mu_3{}^{\text{pure}}\\ =-L_3-\int_{T=0}^{T_s}dT\,S_3{}^{\text{pure}},\end{aligned} \tag{3}$$

where $\mu_f(T,x)$ is the chemical potential for a noninteracting Fermi gas of mass m and density

$$n_3=nx\cong n_4x. \tag{4}$$

n_3, n_4 are the number densities of He^3 and He^4 in the solution, $n=n_3+n_4$ is the total particle density, and x, the He^3 concentration, has been assumed to be small; L_3 and $S_3{}^{\text{pure}}$ are the latent heat of vaporization (at $T=0$) and entropy per atom of pure He^3, while $T_s(x)$ is the phase-separation temperature.

In the fully degenerate region, and with the assumption of weak interaction, we have

$$\mu_3{}'(x)=E_0+n_3V_0-\int_{|\mathbf{k}+\mathbf{k}_f|<k_f}V_{\mathbf{k}}\frac{d\mathbf{k}}{(2\pi)^3}, \tag{5}$$

where k_f is the Fermi momentum of the He^3, and $\mathbf{k}_f$ is an arbitrary vector on the Fermi surface; the second term on the right corresponds to a Hartree correction to μ_3 while the third represents the exchange correction, as calculated in the Hartree-Fock approximation. The phase-separation measurements of Edwards *et al.*[1] have been carried out at concentrations ranging from 8 to 15%, for which the corresponding values of T_s range from 200 m°K to 400 m°K. In this temperature range, the exchange correction in (5) is comparatively small, and may be neglected in a first approximation. Along the phase separation curve, one then has, from (3) and (5),

$$-[E_0+L_3+n_4V_0x]=\mu_f(T_s,x)+\int_{T=0}^{T_s(x)}dT\,S_3{}^{\text{pure}}. \tag{6}$$

If, following Edwards *et al.*,[1] one assumes an approximate value for m, the right side of (6) is determined, and the slope of the resulting curve yields V_0 directly. Edwards *et al.*[1] have plotted the right-hand side of (6) as a function of x, using $m\cong 2.5m_3$; their results lie very nearly on a straight line, and yield a slope such that

$$n_4V_0\cong -0.1m_4s^2, \tag{7}$$

where m_4 is the mass of a He^4 atom, s is the velocity of first sound in He^4 at very low temperatures (~238 m/sec), and n_4 is the density of pure He^4.

On the basis of (7), we may conclude that the average effective potential is attractive, and weak. That it is a weak potential follows from a comparison of the Hartree contribution to the chemical potential with μ_f; we have, using (7) (and taking $m\approx 2.5m_3$),

$$n_4|V_0|x/\mu_f\approx x^{1/3}. \tag{8}$$

Since the average interaction is less than $|V_0|$, this is an overestimate of the error resulting from neglect of the concentration dependence of the effective interaction. For the 5% solution, $x^{1/3}$ is about 0.37.

The spin-diffusion experiments of Anderson *et al.*[3] provide information on the momentum dependence of $V_{\mathbf{k}}$ and enable us to make a somewhat more accurate determination of V_0. For a weakly coupled degenerate Fermi liquid, the spin-diffusion coefficient D is given by[10]

$$DT^2=\frac{2\hbar T_f{}^2(\chi_{\text{ideal}}/\chi)}{\pi^3m^*[N(0)]^2\langle V^2\rangle_d}, \tag{9}$$

where m^* is the He^3 effective mass at concentration x, as measured in a heat-capacity experiment [m

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[10] D. Hone, Phys. Rev. **125**, 1494 (1962).

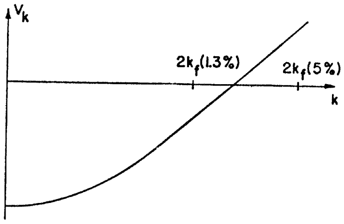

FIG. 1. Effective potential V_k as a function of k. Twice the Fermi momentum of the 1.3 and 5% solutions are indicated. The curve crosses the axis at $2k_f$ for a 2.36% solution.

$=m^*(x=0)]$,

$$N(0)=m^*k_f/2\pi^2\hbar^3 \tag{10}$$

is the density of states at the Fermi surface for one spin orientation, k_f is the Fermi momentum, and

$$KT_f=k_f^2/2m^*. \tag{11}$$

χ is the measured He^3 spin susceptibility, χ_{ideal} is that calculated for an ideal gas with effective mass m^*, while

$$\langle V^2\rangle_d=\frac{3}{8k_f^3}\int_0^{2k_f} k^2|V_{\mathbf{k}}|^2dk \tag{12}$$

is the moment of $V_{\mathbf{k}}$ measured by spin diffusion.[11] Anderson *et al.*[3] have measured χ, m^*, and DT^2 for solutions with He^3 concentrations of 1.3% and 5%; from their measurements, we find, using (9)

$$\begin{aligned} N(0)[\langle V^2\rangle_d]^{1/2}&=0.085, \quad x=0.013\\ &=0.0845, \quad x=0.05. \end{aligned} \tag{13}$$

From (13) we may at once conclude that the effective interaction is momentum-dependent; were it not, one could expect to see a corresponding increase in $N(0)\times[\langle V^2\rangle_d]^{1/2}$, since $N(0)$ increases by a factor of 1.61 on going from the 1.3% to the 5% solution. We further note that according to (7)

$$N(0)|V_0|\cong 0.30 \quad \text{at} \quad x=0.05,$$

a result some four times larger than that found experimentally; this, too, points to a momentum-dependent interaction.

We conclude that the interaction clearly must fall off with increasing $\mathbf{k}$; since one expects V_k to be rather flat in the vicinity of $k=0$, a simple two-parameter form for $V_{\mathbf{k}}$ is

$$V_k=-|V_0|\cos(\beta k/\hbar). \tag{14}$$

This gives an interaction that goes to zero when $\beta k/\hbar=\pi/2$, and V_k becomes positive for larger k. The maximum k we are concerned with is only a little larger than the crossing point, as illustrated in Fig. 1. Any function with these general features would be satisfactory; the form (14) is taken for mathematical convenience. The important parameters are V_0, which sets the scale of the interaction, and the value of k for which V_k goes through zero. With this form for V_k, we have

$$N(0)[\langle V^2\rangle_d]^{1/2}=N(0)|V_0|[G(\zeta)]^{1/2}, \tag{15}$$

where $\zeta=2\beta k_f$ and

$$G(\zeta)=\tfrac{1}{2}+(3/8\zeta^3)[(2\zeta^2-1)\sin 2\zeta+2\zeta\cos 2\zeta]. \tag{16}$$

We choose V_0 and β (or ζ) so as to fit the spin-diffusion results at the two concentrations, as given by (13);

$$n_4|V_0|=0.0754m_4s^2, \quad \beta=3.16\ \text{Å} \tag{17}$$

so that

$$\zeta=5.47x^{1/3}.$$

These parameters were used in the plot of Fig. 1.

The value of V_0 determined in this manner depends to some extent on the choice (14) for V_k. Other choices of reasonable shape might give V_0's differing by 10 to 15%. We note that the value of V_0 is of the same order as that determined from the phase separation curves; it is, however, roughly 25% smaller. The difference between these results is the influence of exchange effects, as discussed by Ebner.[9]

III. FERMI-LIQUID PROPERTIES

We now consider various properties of the solution that are influenced by the effective interaction between the He^3 atoms, as specified by (14) and (17). Calculations of quasiparticle properties, such as the spin susceptibility and specific heat, are most conveniently carried out in the framework of the Fermi-liquid theory of Landau.[12] In the weak-coupling (or Hartree-Fock) approximation, the interaction between two quasiparticles of momentum $\mathbf{p}$ and $\mathbf{p}'$, spin σ and σ' is given by[13]

$$f_{\mathbf{pp}'}{}^{\sigma\sigma'}=V_0-V_{\mathbf{p}-\mathbf{p}'}\delta_{\sigma,\sigma'}. \tag{18}$$

It is convenient to introduce the spin-symmetric and spin-antisymmetric parts of $f_{\mathbf{pp}'}{}^{\sigma\sigma'}$ according to

$$\begin{aligned} f_{\mathbf{pp}'}{}^{s}&=(f_{\mathbf{pp}'}{}^{\uparrow\uparrow}+f_{\mathbf{pp}'}{}^{\uparrow\downarrow})/2=V_0-V_{\mathbf{p}-\mathbf{p}'}/2,\\ f_{\mathbf{pp}'}{}^{a}&=(f_{\mathbf{pp}'}{}^{\uparrow\uparrow}-f_{\mathbf{pp}'}{}^{\uparrow\downarrow})/2=-V_{\mathbf{p}-\mathbf{p}'}/2. \end{aligned} \tag{19}$$

Since the f's depend only on the angle, ξ, between $\mathbf{p}$ and $\mathbf{p}'$ when both momenta lie on the Fermi surface, we expand them in a series of Legendre polynomials,

$$f_{\mathbf{pp}'}{}^{s(a)}=\textstyle\sum_l f_l{}^{s(a)}P_l(\cos\xi). \tag{20}$$

Finally, we introduce the dimensionless measure of the strength of the quasiparticle interaction

$$F_l{}^{s(a)}=2N(0)f_l{}^{s(a)}. \tag{21}$$

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[11] To the best of our knowledge, the simple result (12), applicable to weakly interacting systems, has not previously appeared in the literature.

[12] L. D. Landau, Zh. Eksperim. i Teor. Phys. **30**, 1058 (1956); **32**, 59 (1957) [English transls.: Soviet Phys.—JETP **3**, 920 (1956); **5**, 101 (1957)].

[13] D. Pines and P. Nozières, *Theory of Quantum Liquids* (W. A. Benjamin, Inc., New York, 1966), Vol. I, p. 277.

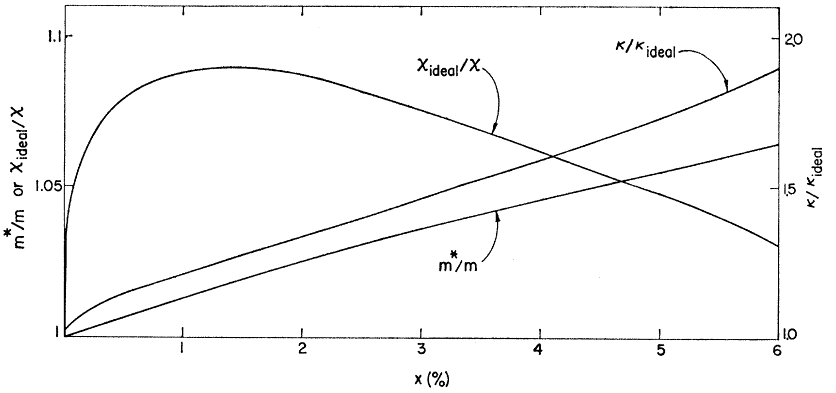

FIG. 2. Calculated values, as a function of concentration x, of m^*/m, the ratio of the He³ effective mass to its value at $x=0$; χ_{ideal}/χ, the inverse ratio of the spin susceptibility to that of an ideal gas with effective mass m^* at the same density; and $\kappa/\kappa_{\text{ideal}}$, the ratio of the He³ "compressibility" to that of the ideal gas.

The "specific-heat" effective mass m^* and spin susceptibility χ then take the form[13]

$$\frac{m^*}{m}=1+\tfrac{1}{3}F_1{}^s=1-\frac{N(0)}{2k_f{}^2}\int_0^{2k_f}V_k k\left(1-\frac{k^2}{2k_f{}^2}\right)dk, \tag{22}$$

$$\frac{\chi_{\text{ideal}}}{\chi}=1+F_0{}^a=1-\frac{N(0)}{2k_f{}^2}\int_0^{2k_f}V_k k dk, \tag{23}$$

on making use of Eqs. (19), (20), and (21). If we now substitute our expression for the effective potential, (14), we find

$$m^*/m=1+N(0)|V_0|M(\zeta), \tag{24}$$

$$\chi_{\text{ideal}}/\chi=1+N(0)|V_0|R(\zeta), \tag{25}$$

where

$$M(\zeta)=(2/\zeta^2)[\cos\zeta+\zeta\sin\zeta-1]-(4/\zeta^4)\times[(3\zeta^2-6)\cos\zeta+(\zeta^3-6\zeta)\sin\zeta-6], \tag{26}$$

$$R(\zeta)=(2/\zeta^2)[\cos\zeta+\zeta\sin\zeta-1], \tag{27}$$

and ζ is specified by Eq. (17).

Equation (24) for the effective mass m^* when used in conjunction with the experimental results enables us to calculate m, the mass of an isolated He³ atom immersed in He⁴. From (13), (15), and (17), we have

$$\begin{aligned} N(0)V_0&=0.146; \quad x=0.013\\ N(0)V_0&=0.235; \quad x=0.05. \end{aligned} \tag{28}$$

If we now choose to fit the specific-heat experiment at 1.3% $[(m^*/m_3)=2.38\pm0.04]$, we find upon combining (24) and (28)

$$m/m_3=2.34. \tag{29}$$

Having determined m, it is straightforward to calculate χ and m^* at arbitrary concentrations; in Fig. 2 we give a plot of (m^*/m) and $(\chi_{\text{ideal}}/\chi)$ as a function of concentration. Our calculations are compared with experiment in Table I. We see that agreement with experiment is good.

A check on the consistency of our calculations of m and $F_1{}^s$ is provided by calculating the ratio of the "bare" state densities at 1.3% and 5% concentrations. Let us define:

$$N^0(0)=mk_f/2\pi^2\hbar^3;$$

we then have

$$\frac{N^0(0)_{5\%}}{N^0(0)_{1.3\%}}=\left(\frac{0.050}{0.013}\right)^{1/3}=1.560.$$

On the other hand, from (22) and (24)

$$\begin{aligned} N^0(0)V_0M(\zeta)&=\tfrac{1}{3}F_1{}^s/(1+\tfrac{1}{3}F_1{}^s)\\ &=\tfrac{1}{3}F_1{}^s-(\tfrac{1}{3}F_1{}^s)^2+\cdots. \end{aligned} \tag{30}$$

The values of $\frac{1}{3}F_1{}^s$ for the two concentrations may be determined (without prior knowledge of m) directly from the relation $\frac{1}{3}F_1{}^s=N(0)|V_0|M(\zeta)$. Using these values ($\frac{1}{3}F_1{}^s=0.01740$ at 1.3%, $\frac{1}{3}F_1{}^s=0.0551$ at 5%), and our calculated values of $N(\zeta)$, we find

$$\frac{N^0(0)_{5\%}}{N^0(0)_{1.3\%}}=\frac{0.2220}{0.1430}=1.552,$$

in good agreement with the theoretical value of 1.560 quoted above.

We also give in Table I the calculated values of

$$F_0{}^s=-2N(0)|V_0|+F_0{}^a, \tag{31}$$

TABLE I. Calculated parameters for 1.3% and 5% concentration solutions of He³ in He⁴, compared with experiment. The potential has been chosen to fit the theoretical values of DT^2 with experiment.

	1.3%		5%	
	Theory	Experiment	Theory	Experiment
DT^2 (cm² K°²/sec)	(17.2×10^{-6})	17.2×10^{-6}	(90×10^{-6})	90×10^{-6}
$F_1{}^s/3$ $(=F_1/3)$	0.0174		0.055	
m^*/m_3	2.38	2.38 ± 0.04	2.47	2.46 ± 0.04
$m/m_3=2.34$				
$F_0{}^a$ $(=Z_0/4)$	0.090		0.048	
χ_{ideal}/χ	1.090	1.09 ± 0.03	1.048	1.08 ± 0.03
$F_0{}^s$	-0.202		-0.423	
$\kappa_{\text{ideal}}/\kappa$	0.798		0.577	
$F_0{}^s+F_1{}^s$	-0.150		-0.258	

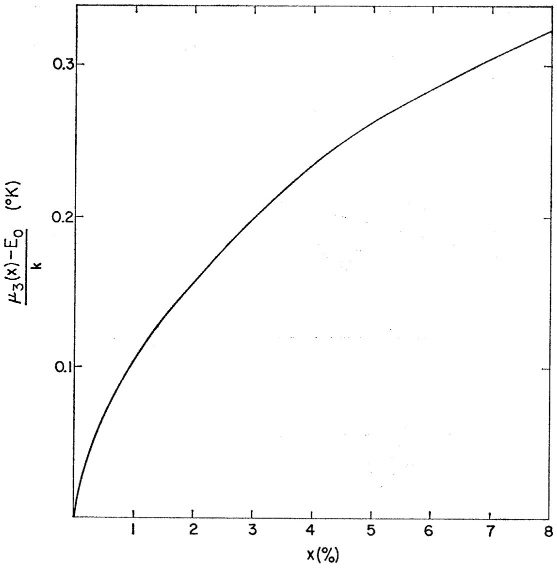

FIG. 3. Plot of $\mu_3(x)-E_0$ in °K versus x. The value of x, ≈ 0.06, for which $\mu_3(x)-E_0=-L_3-E_0\approx 0.29$°K, is the maximum solubility of He^3 in He^4 at zero temperature.

a parameter which determines the "He^3 compressibility," and plays an important role in the theory of zero sound. The compressibility κ of the He^3 system, defined from the change in energy with concentration in the solution, is given by

$$\kappa_{\text{ideal}}/\kappa=1+F_0{}^s; \tag{32}$$

we plot this ratio in Fig. 2 as a function of concentration. In the weak-coupling limit, the criterion for the existence of zero sound (under the assumption that only F_0 and F_1 are important) is[14]

$$F_0{}^s+F_1{}^s/(1+\tfrac{1}{3}F_1{}^s)\cong F_0{}^s+F_1{}^s>0. \tag{33}$$

We see from Table I that Fermi-liquid effects are actually most pronounced on the He^3 compressibility, which can be determined from first- and second-sound experiments. Unfortunately, the large negative values of $F_0{}^s$ preclude the observation of zero sound in He^3-He^4 mixtures.

We note in passing that with our choice of effective interaction, (14) and (17), the higher harmonics of $f_{\mathbf{p}\mathbf{p}'}$ play little role in determining the various properties of the He^3 solutions. The reason is that the relevant integrals depend to an increasing extent on values of V_k at larger momentum transfers, where V_k is seen to be small.

We can also use our effective potential to estimate the concentration at which phase separation will occur at $T=0$. To do this we need to calculate the exchange contribution to the chemical potential,

$$\mu_{\text{exch}}=-\int_{|\mathbf{k}+\mathbf{k}_f|<k_f} V_k\frac{d\mathbf{k}}{(2\pi)^3}$$
$$=-\frac{1}{4\pi^2}\int_0^{2k_f} k^2\left(1-\frac{k}{2k_f}\right)V_{\mathbf{k}}dk. \tag{34}$$

With our choice of potential, (17), we find

$$\mu_{\text{exch}}=n_4x|V_0|P(\zeta), \tag{35}$$

where

$$P(\zeta)=\frac{24}{\zeta^3}\sin\zeta+\left(\frac{36}{\zeta^4}-\frac{6}{\zeta^2}\right)\cos\zeta-\frac{36}{\zeta^4}. \tag{36}$$

At $T=0$, the concentration at which phase separation will take place is, according to (3) and (5), determined by

$$-L_3-E_0=\mu_3(x)-E_0=\mu_f(x)-n_4|V_0|x[1-P(\zeta)]. \tag{37}$$

A plot of $\mu_3(x)-E_0$ as a function of x is given in Fig. 3. Determination of the maximum concentration at which He^3 is soluble in He^4 then depends on a knowledge of L_3+E_0. According to the calculations of Ebner,[9]

$$L_3+E_0\cong-0.29°\text{K}, \tag{38}$$

from which we conclude the maximum solubility of He^3 in He^4 is at a concentration of about[15] 6%. This agrees with the recently measured value 6.0±0.3%.[16]

Finally, we estimate the temperature T_c at which the He^3 subsystem may be expected to exhibit a transition to a superfluid state. (Below T_c *both* the He^3 and the He^4 are superfluid; above T_c, the He^3 subsystem behaves as normal fluid.) We note first that because v_f, the velocity of a quasiparticle on the Fermi surface, is small compared to s, the velocity of first sound in He^4 ($v_f/s\approx 0.3x^{1/3}$), retardation effects are negligible; T_c may therefore be calculated from the weak-coupling Bardeen-Cooper-Schrieffer (BCS) theory.[17] In order to allow for possibilities other than s-state pairing, it is convenient to expand the effective interaction between quasiparticles on the Fermi surface as

$$V_{\mathbf{p}-\mathbf{p}'}=\textstyle\sum_l V(l)P_l(\cos\xi)(2l+1), \tag{39}$$

where ξ is the angle between $\mathbf{p}$ and $\mathbf{p}'$. The transition temperature is then given by[18]

$$T_c\cong T_f\exp[1/N(0)V(l)], \tag{40}$$

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[14] A. A. Abrikosov and I. M. Khalatnikov, *Reports on Progress in Physics* (The Physical Society, London, 1959), Vol. 22, p. 329.

[15] Using a hard-sphere model and the pseudopotential method to first order, and assuming equal molar volumes for the two isotopic components, E. G. D. Cohen and J. M. J. van Leeuwen [Physica **26**, 1171 (1960); **27**, 1157 (1961)] calculated a maximum $T=0$ solubility of He^3 in He^4 of 15%.

[16] E. M. Ifft, M. M. Skertic, and D. O. Edwards, Bull. Am. Phys. Soc. **12**, 96 (1967).

[17] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957); **108**, 1175 (1957).

[18] R. Balian, *Lectures on the Many-Body Problem*, edited by E. Caianiello (Academic Press Inc., New York, 1964), Vol. 2.

where $V(l)$ is the most attractive spherical harmonic of the potential V; the pairing first takes place in states of angular momentum l. The Fermi temperature T_f plays the role of a cutoff frequency. One finds from (19), (20), and (21), that

$$\begin{aligned} N(0)V(l) &= -F_l{}^a/(2l+1) \\ &= -F_l{}^s/(2l+1)\,, \quad l>0. \end{aligned} \tag{41}$$

We need consider only s- and p-state pairing, the higher harmonics of V being negligible for our choice of a potential. We have

$$\begin{aligned} N(0)V(0) &= -F_0{}^a, \quad s\text{-state pairing} \\ N(0)V(1) &= -\tfrac{1}{3}F_1{}^s, \quad p\text{-state pairing}. \end{aligned} \tag{42}$$

At $x=1.3\%$, s-state pairing is most favorable; for the interaction specified by (17), we find

$$T_c \cong 2\times10^{-6}\ {}^\circ\mathrm{K}, \quad x=0.013. \tag{43}$$

At $x=5\%$, p-state pairing is slightly more favorable, but T_c plunges to 5×10^{-9} °K. It turns out that $x=1.6\%$ is the optimal concentration for a superfluid transition, with $T_c \cong 2.2\times10^{-6}$ °K; as one goes to larger concentrations, $N(0)$ increases as $x^{1/3}$, but the average $V(0)$ of the potential decreases in magnitude more rapidly than that.

The crucial point in estimating T_c is that the spin-diffusion experiments place so small an upper bound on the magnitude of the scattering in opposite spin states that even a more refined potential would not predict a transition temperature significantly larger than a few microdegrees.[19] Furthermore, as we shall see in Sec. V, putting pressure on the system will most likely only further decrease T_c for s-state pairing.

In conclusion, we note that one need not rely on the theory for an estimate of T_c for s-state pairing, should sufficiently accurate experimental measurements of χ become available. For on comparing (42), (22), and (23), we see that

$$N(0)V(0) = (1-\chi_{\mathrm{ideal}}/\chi).$$

However, it is likely that our theoretical estimates based on a fit to the spin-diffusion experiments are more reliable than those obtained from the present measurements of χ. [At 1.3%, the experimental value of $\chi_{\mathrm{ideal}}/\chi = 1.09\pm0.03$ leads to transition temperatures between 8×10^{-9} °K and 3.5×10^{-5} °K; for the 5% solution, (43) leads to an estimated T_c which lies between 7×10^{-10} °K and 3.5×10^{-5} °K.]

IV. PHONON-INDUCED INTERACTION BETWEEN He^3 QUASIPARTICLES

We turn now to the question of the origin of the effective interaction between He^3 atoms in solution in He^4. We first give an elementary derivation of the form of that part of the interaction due to the exchange of virtual phonons[20]; we then use thermodynamic arguments to fix the strength of the coupling of a He^3 quasiparticle to a long-wavelength phonon, and thence the size of the phonon-exchange contribution. Let us emphasize that the phonon-induced interaction is only part of the effective interaction between He^3 atoms; we consider the remaining contributions in the following section.

We assume at first an effective interaction, U between the He^3 atoms at positions $\mathbf{r}_i$ and the He^4 background at positions $\mathbf{r}_j$ of the form

$$H_{\mathrm{int}} = \sum_{ij} U(\mathbf{r}_i-\mathbf{r}_j) = \int \frac{d\mathbf{q}}{(2\pi)^3} U_{\mathbf{q}}\rho_{\mathbf{q}}{}^\dagger \gamma_{\mathbf{q}}\,, \tag{44}$$

where

$$\rho_{\mathbf{q}}{}^\dagger = \sum\nolimits_j e^{i\mathbf{q}\cdot\mathbf{r}_j} \tag{45}$$

is the Fourier transform of the He^4 density operator, and

$$\gamma_{\mathbf{q}} = \sum\nolimits_i e^{-i\mathbf{q}\cdot\mathbf{r}_i} \tag{46}$$

similarly describes a He^3 density fluctuation. $U_{\mathbf{q}}$ is the Fourier transform of $U(\mathbf{r}_i-\mathbf{r}_j)$ and it may be regarded microscopically as the complete matrix element (or vertex) for a He^3 density fluctuation of wave number $\mathbf{q}$ to create a He^4 density fluctuation of the same wave number. In general the density fluctuations of the He^4 possess matrix elements between states that differ by a single phonon or roton of wave number $\mathbf{q}$, or by various combinations of these elementary excitations with total wave number $\mathbf{q}$. However, by sum-rule arguments one can show that for $q \lesssim 1$ Å^{-1} (and these are the momenta of interest in the preceding section), single phonon states predominate[21]; the coupling (44) in this momentum region is therefore primarily between He^3 atoms and single He^4 phonons. [It is quite possible though that multi-excitation terms are important in determining the momentum dependence of the effective interaction (14).]

At the very low temperatures with which we are concerned a given He^3 atom in the solution lacks sufficient energy to excite a real phonon in the He^4. As the atom travels through the solution it does, however, induce fluctuations in the He^4 density, or equivalently, it excites virtual phonons. To the extent that the He^4 system responds linearly to the motion of the atom, the component of the induced He^4 density fluctuation of wave number $\mathbf{q}$ is given by

$$\rho_{\mathbf{q}}{}^{\mathrm{ind}} = \chi(q,\omega) U_{\mathbf{q}} \gamma_{\mathbf{q}}{}^{(a)}\,, \tag{47}$$

where $\gamma_{\mathbf{q}}{}^{(a)}$ is the $\mathbf{q}$th component of the He^3 density fluctuation associated with the motion of the particular

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[19] A more detailed solution of the appropriate integral equation for T_c leads to values of this same order of magnitude [K. H. Michel (private communication)].

[20] Our derivation is closely patterned on that given in Ref. 13, pp. 243–251, for the effective interaction between electrons in a metal.

[21] A. Miller, P. Nozières, and D. Pines, Phys. Rev. **127**, 1452 (1962).

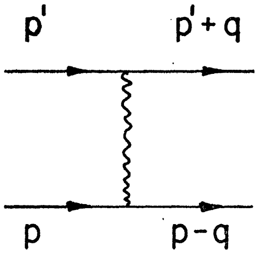

FIG. 4. Schematic representation of the phonon-induced interaction between He^3 atoms.

He^3 atom; $\chi(\mathbf{q},\omega)$ is the retarded density-density correlation function of the He^4 system,[22] and ω is the frequency at which the atom in question perturbs the system. A second atom in the system will be scattered by the density fluctuation (47), via the interaction (44), and this leads to an effective interaction between the two atoms, as shown in Fig. 4. If we let $\gamma_{\mathbf{q}}^{(b)}$ be the He^3 density fluctuation associated with the motion of this second atom, then the resulting interaction between the two He^3 atoms is

$$|U_{\mathbf{q}}|^2\chi(\mathbf{q},\omega)(\gamma_{\mathbf{q}}^{a})^{\dagger}\gamma_{\mathbf{q}}^{b}. \tag{48}$$

The matrix element for this phonon-induced interaction is thus

$$V_{\mathbf{q}}^{\mathrm{ph}}=|U_{\mathbf{q}}|^2\chi(\mathbf{q},\omega). \tag{49}$$

In the limit of zero concentration, χ in Eq. (49) is the density response function of pure He^4. At wavelengths such that $qr_0\lesssim 1$, where r_0 is the average spacing of He^4 atoms, χ in pure He^4 is given by[23]

$$\chi(q,\omega)=\frac{n_4q^2/m_4}{\omega^2-s^2q^2}\quad(qr_0\lesssim 1). \tag{50}$$

Furthermore, we expect that at these wavelengths, the coupling between the He^3 and He^4 density fluctuations differs little from its long-wavelength limit; thus we write

$$U_{\mathbf{q}}=U_{3-4}\quad(qr_0\lesssim 1). \tag{51}$$

We have then

$$V_q^{\mathrm{ph}}=\frac{n_4q^2/m_4}{\omega^2-s^2q^2}|U_{3-4}|^2. \tag{52}$$

Moreover, as we emphasized, the frequencies associated with the motion of He^3 atoms, on the order of qv_f, are small compared with those characteristic of the He^4; thus we may neglect the ω^2 in the denominator of (52), and obtain the static interaction resulting from the exchange of a long-wavelength virtual phonon between the two He^3 atoms:

$$V_q^{\mathrm{ph}}=-|U_{3-4}|^2n_4/m_4s^2\quad(qr_0\lesssim 1). \tag{53}$$

It remains for us to determine U_{3-4}, the matrix element for the effective coupling of a He^3 atom to a long wavelength He^4 density fluctuation. This matrix element may be expected to be quite different from that determined by the "bare" van der Waal's interaction between isolated He^3 and He^4 atoms; that interaction is very strong and must consequently be altered by both multiple scattering effects and the influence of the surrounding medium. Still, we can determine U_{3-4} in terms of measured parameters exactly, because on the one hand the He^3 subsystem is dilute and also, on the other, we are dealing with essentially a long-wavelength static phenomenon, for which thermodynamic arguments suffice.

Equation (47) describes the response of the He^4 to He^3 density fluctuations at constant He^4 chemical potential, μ_4. If we set $\omega=0$ and then let q approach zero, Eq. (47) then describes the change δn_4 in the He^4 density caused by a uniform infinitesimal change δn_3 in the He^3 density, at constant μ_4. In this limit we have then

$$\left(\frac{\partial n_4}{\partial n_3}\right)_{\mu_4}=\lim_{q\to 0}\chi(q,0)U_{3-4}; \tag{54}$$

thus from (50),

$$U_{3-4}=-(\partial n_4/\partial n_3)_{\mu_4}m_4s^2/n_4. \tag{55}$$

Since

$$m_4s^2/n_4=(\partial\mu_4/\partial n_4)_{n_3} \tag{56}$$

in the limit $x\to 0$, Eq. (55) is simply the statement that

$$U_{3-4}=-(\partial n_4/\partial n_3)_{\mu_4}(\partial\mu_4/\partial n_4)_{n_3}=(\partial\mu_4/\partial n_3)_{n_4}. \tag{57}$$

Equivalently, since $\mu_4=(\partial F/\partial n_4)_{n_3}$ and $\mu_3=(\partial F/\partial n_3)_{n_4}$, where F is the free energy per unit volume, we have

$$U_{3-4}=(\partial\mu_3/\partial n_4)_{n_3}. \tag{58}$$

The derivative $(\partial n_4/\partial n_3)_{\mu_4}$ occurring in (55) is easily related to measured parameters by noting that as $x\to 0$, keeping μ_4 constant is the same as keeping the pressure constant. This follows from the Gibbs-Duhem relation

$$dP=SdT+n_4d\mu_4+n_3d\mu_3; \tag{59}$$

for constant T and μ_4, $dP=n_3d\mu_3$, so that as $x\to 0$, P is also constant. Now the molar volume v of the solution may, for small concentrations, be written as

$$v=v_{04}(P)+v_1(P)x. \tag{60}$$

The total number density of the system is $n=A/v$, where A is Avogadro's number, while

$$n_3=xn,\quad n_4=(1-x)n. \tag{61}$$

In the limit $x\to 0$, then,

$$\left(\frac{\partial n_4}{\partial n_3}\right)_P=\frac{(\partial n_4/\partial x)_P}{(\partial n_3/\partial x)_P}=-(1+v_1/v_{04}). \tag{62}$$

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[22] For a discussion of the properties of the density-density response function, see Ref. 13, Chap. 2.

[23] J. Gavoret and P. Nozières, Ann. Phys. (N. Y.) 28, 349 (1964); P. Hohenberg and P. C. Martin, *ibid.* 34, 291 (1965); D. Pines, in *Proceedings of the Ninth International Conference on Low-Temperature Physics, Columbus, Ohio, 1964*, edited by J. G. Daunt, D. O. Edwards, F. D. Milford, and M. Yaqub (Plenum Press, Inc., New York, 1965), Vol. A, p. 61.

Thus letting

$$\alpha = v_1/v_{04}, \tag{63}$$

we have

$$U_{3-4} = (1+\alpha) m_4 s^2/n_4, \tag{64}$$

and from (53) we find our desired result for the long-wavelength limit of the phonon-induced interaction,

$$V_q^{\mathrm{ph}} = -(1+\alpha)^2 m_4 s^2/n_4 \quad (qr_0 \lesssim 1). \tag{65}$$

The results above, (64), (58), and (56), enable us to derive a useful expression for α in terms of thermodynamic derivatives:

$$\alpha = \frac{n_4}{m_4 s^2} \frac{\partial}{\partial n_4} (\mu_3 - \mu_4); \tag{66}$$

the derivative is evaluated at constant n_3 in the limit $x \rightarrow 0$. We shall use this expression in Sec. VI to obtain a theoretical estimate of α.

The molar volume parameters may be estimated directly from the measurements of Kerr[24] in the region below the λ point where the He^3 is nondegenerate. He finds, at $T = 1.2$°K,

$$v_{04} = 27.52 \text{ cc/mole}, \quad v_1 = 7.6 \text{ cc/mole}, \tag{67}$$

so that

$$\alpha = v_1/v_{04} \simeq 0.28. \tag{68}$$

Actually this ratio appears to be reasonably independent of temperature below the λ point. That v_1 is positive and of this size is a direct consequence of the additional zero-point motion of a He^3 atom beyond that of the He^4 atom it displaces.

We see from (64) that the long-wavelength interaction between a He^3 and a He^4 atom is repulsive in the sense that raising the He^4 density increases the He^3 quasiparticle energy (and vice versa). On the other hand, the effective interaction between two He^3 atoms resulting from this coupling is attractive; as is the case with phonon exchange between electrons at the Fermi surface in metals, this is because the interaction arises from the exchange of virtual quanta at frequencies low compared with the frequency characteristic of the quanta themselves.

In Eq. (44) we have taken into account only the coupling of a He^3 to a He^4 phonon caused by the density changes accompanying the phonon. There is a further coupling arising from the fact that in the presence of a phonon, the He^4 has a nonzero local velocity; for long wavelengths this coupling can be calculated from Galilean invariance arguments. We consider the $x \rightarrow 0$ limit only.

Suppose that the He^4 has a uniform velocity $\mathbf{v}$. Then in the coordinate frame moving with the He^4, the energy of a He^3 quasiparticle of momentum $\mathbf{p}$ is given by $E_p = p^2/2m + E_0$. In the laboratory frame the momentum of this quasiparticle is $\mathbf{p} + m_3\mathbf{v}$ and its energy is

$$E_{\mathbf{p}+m_3\mathbf{v}}(\mathbf{v}) = E_{\mathbf{p}} + \mathbf{p}\cdot\mathbf{v} + m_3 v^2/2; \tag{69}$$

$E_{\mathbf{p}}(\mathbf{v})$ denotes the energy of a quasiparticle of momentum $\mathbf{p}$ in the presence of a He^4 background flowing with velocity $\mathbf{v}$. Thus from (69)

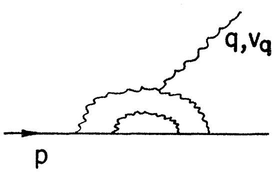
$$E_{\mathbf{p}}(\mathbf{v}) = E_{\mathbf{p}-m_3\mathbf{v}} + \mathbf{p}\cdot\mathbf{v} - m_3 v^2/2, \tag{70}$$

and to first order in $\mathbf{v}$ the change in the quasiparticle energy due to the He^4 flow is

$$\delta E_{\mathbf{p}} = -\nabla_{\mathbf{p}} E_{\mathbf{p}} \cdot m_3 \mathbf{v} + \mathbf{p}\cdot\mathbf{v} = (\delta m/m)\,\mathbf{p}\cdot\mathbf{v}, \tag{71}$$

where $\delta m = m - m_3$. Now the point is that a long-wavelength sound wave looks to a quasiparticle like a uniform motion of the He^4 with a (local) velocity $\mathbf{v}$. Therefore (71), with $\mathbf{v}$ being the local velocity associated with a phonon, is an additional coupling of a He^3 quasiparticle to the phonons.[25]

FIG. 5. Schematic representation of the $\mathbf{p}\cdot\mathbf{v}(\delta m/m)$ coupling between a phonon and a moving quasiparticle.



Microscopically this coupling is due to the interaction of a phonon with the He^4 screening cloud that is attached to the He^3 atom,[26] as shown in Fig. 5. As a consequence the strength of (71) is proportional to δm, the mass density associated with this cloud. We plan in a later paper to discuss in some detail the dynamical consequences of this interaction, e.g., its contribution to the effective mass m of a single He^3 atom in He^4. For the present we shall only calculate its contribution to the phonon induced part of the interaction between two quasiparticles, in the limit $x \rightarrow 0$.

The argument is simply a generalization of that given in the beginning of this section. The total coupling of a long-wavelength He^3 density fluctuation of momentum $\mathbf{q}$ to the He^4 background is, from (44) and (71),

$$H_{\mathrm{eff}}(\mathbf{q}) = U_{3-4}\rho_{\mathbf{q}}^{\dagger}\gamma_{\mathbf{q}} + \frac{1}{2}\frac{\delta m}{m} \sum_i \mathbf{p}_i \cdot \mathbf{v}_q e^{i\mathbf{q}\cdot\mathbf{r}_i} + \text{H.c.}, \quad (qr_0 \lesssim 1) \tag{72}$$

where $\mathbf{v}_{\mathbf{q}}$ is the Fourier transform of the He^4 velocity field.

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[24] E. C. Kerr, *Low-Temperature Physics and Chemistry*, edited by J. R. Dillinger (University of Wisconsin Press, Madison, Wisconsin, 1958), p. 160.

[25] We note that the magnitude (and sign) of our $\mathbf{p}\cdot\mathbf{v}$ coupling differs from that considered by I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965), p. 167.

[26] The coupling between the He^3 and He^4 is also modified slightly by the dependence of the effective mass of a He^3 quasiparticle on the He^4 density. Extrapolating the data of D. J. Sandiford and H. A. Fairbank [*Proceedings of the Seventh International Conference on Low-Temperature Physics, 1960* (University of Toronto Press, Toronto, 1961)] to $T=0$ we find: $|\delta m/\delta n_4| \sim m_4/n_4$. Thus this contribution is of order $(v_f/s)^2 \sim 0.01$, smaller than (54), but the sign is uncertain.

We consider now the density fluctuation induced in the He^4 background when a quasiparticle of momentum $\mathbf{p}$ is scattered to a state $\mathbf{p}-\mathbf{q}$; it is

$$\rho_{\mathbf{q}}{}^{\mathrm{ind}}=[\chi(q,\omega)U_{3-4}+\mathbf{\Phi}(\mathbf{q},\omega)\cdot(\mathbf{p}-\mathbf{q}/2)\delta m/m]\gamma_{\mathbf{p},\mathbf{q}}{}^{(a)}, \tag{73}$$

where $\gamma_{\mathbf{p},\mathbf{q}}{}^{(a)}$ is the He^3 density fluctuation associated with the quasiparticle in its scattering. The first term in (72) is the density fluctuation due to the coupling of the He^3 to the He^4 density. [This coupling is the same for bare He^3 atoms and He^3 quasiparticles, since the number density of quasiparticles equals the density of atoms.] The second term in (73) represents the coupling of the He^3 quasiparticle to the He^4 through the interaction (71); $\mathbf{\Phi}(\mathbf{q},\omega)$ is the density-velocity correlation function of pure He^4, while $\mathbf{p}-\mathbf{q}/2$ is the average of the initial and final momentum of the quasiparticle. The particle in scattering also produces a superfluid velocity in the He^4 given by

$$\mathbf{v}_{\mathbf{q}}{}^{\mathrm{ind}}=[\mathbf{\Phi}(\mathbf{q},\omega)U_{3-4}+\mathbf{\Psi}(\mathbf{q},\omega)\cdot(\mathbf{p}-\mathbf{q}/2)\delta m/m]\gamma_{\mathbf{p},\mathbf{q}}{}^{(a)}, \tag{74}$$

where $\mathbf{\Psi}(\mathbf{q},\omega)$ is the velocity-velocity correlation function of the He^4. For small $\mathbf{q}$ and ω, and as $x\rightarrow 0$, we have[23]

$$\begin{aligned}\mathbf{\Phi}(\mathbf{q},\omega)&=\frac{\omega\mathbf{q}/m_4}{\omega^2-s^2q^2},\\ \mathbf{\Psi}(\mathbf{q},\omega)&=\frac{s^2\mathbf{q}\mathbf{q}/m_4n_4}{\omega^2-s^2q^2}.\end{aligned} \tag{75}$$

We note in passing that in the static limit, only the density-density interaction contributes to (73); thus our thermodynamic argument to determine U_{3-4} is not affected by the coupling (71).

We also note that if we define a superfluid *mass* density by the conservation law

$$\omega\rho_{s,\mathbf{q}}=m_4n_4\mathbf{q}\cdot\mathbf{v}_{\mathbf{q}}{}^{\mathrm{ind}}, \tag{76}$$

then from (73), (74), and (75) we find that the "normal" *mass* density of the He^4 is given by

$$\begin{aligned}\rho_{n,\mathbf{q}}&\equiv m_4\rho_{\mathbf{q}}{}^{\mathrm{ind}}-\rho_{s,\mathbf{q}}\\ &=\delta m\mathbf{q}\cdot(\mathbf{p}-\mathbf{q}/2)\gamma_{\mathbf{p},\mathbf{q}}{}^{(a)}/m\omega.\end{aligned} \tag{77}$$

On the energy shell,

$$\omega=\frac{(\mathbf{p}-\mathbf{q})^2}{2m}-\frac{p^2}{2m}=\frac{\mathbf{q}\cdot(\mathbf{p}-\frac{1}{2}\mathbf{q})}{m}, \tag{78}$$

so that

$$\rho_{n,\mathbf{q}}=\delta m\gamma_{\mathbf{p},\mathbf{q}}{}^{(a)}. \tag{79}$$

We thus see that the quasiparticle carries with it a mass δm of normal He^4 in its screening cloud of virtual phonons. This is just the result required to make its total contribution to the normal mass density be

$$(\delta m+m_3)\gamma_{\mathbf{p},\mathbf{q}}{}^{(a)}=m\gamma_{\mathbf{p},\mathbf{q}}{}^{(a)}, \tag{80}$$

in agreement with the theory of Landau and Pomeranchuk.[4]

The matrix element for the scattering of two quasiparticles of initial momenta $\mathbf{p}$ and $\mathbf{p}'$ to final momenta $\mathbf{p}-\mathbf{q}$ and $\mathbf{p}'+\mathbf{q}$ due to their exchanging a virtual phonon of momentum $\mathbf{q}$ is found, as before, by substituting (73) and (74) into the effective interaction, (72), which couples the second quasiparticle to the He^4 background. We find for scattering on the energy shell, where

$$\omega=\mathbf{q}\cdot(\mathbf{p}-\tfrac{1}{2}\mathbf{q})/m=\mathbf{q}\cdot(\mathbf{p}'+\tfrac{1}{2}\mathbf{q})m, \tag{81}$$

that the effective scattering matrix element is simply

$$V_{\mathbf{q}}{}^{\mathrm{ph}}=-(1+\alpha)^2m_4s^2/n_4+V_{\mathbf{q}}{}^{d},\quad(qr_0\lesssim 1) \tag{82}$$

where

$$V_{\mathbf{q}}{}^{d}=\frac{m_4s^2\omega^2}{n_4(\omega^2-s^2q^2)}[1+\alpha+\delta m/m_4]^2. \tag{83}$$

In writing (82) and (83) we have used the forms (50) and (75) for the correlation functions of pure He^4.

The first term in (82) is the interaction (65) we have previously derived. The second term is effectively a dipolar interaction between the quasiparticles which depends on their momenta. As before we neglect the ω^2 in the denominator of (83). Then for $q\ll p/m$ and p'/m,

$$V_{\mathbf{q}}{}^{d}=-\frac{\mathbf{p}\cdot\hat{q}}{m}\frac{\mathbf{p}'\cdot\hat{q}}{m}(1+\alpha+\delta m/m_4)^2m_4/n_4. \tag{84}$$

This interaction is the exact quantum analog of the classical dipolar interaction of two hard spheres at large separation in an incompressible fluid.[27] It is of order $v_f{}^2/s^2\sim 0.01$ smaller than the first part of the interaction (65).

Let us now compare the phonon-induced interaction (65) with the long-wavelength part of the empirical effective interaction V_0, as specified by (17). On making use of (68) we see that the phonon-induced interaction is some 20 times stronger than the empirically determined V_0; there must necessarily be a very considerable cancellation between the phonon-induced interaction and the remaining contributions to the effective interaction between the He^3 atoms. In the following section we develop thermodynamic arguments which indicate the source of this cancellation and enable us to derive a net effective interaction, V_0, which is in good agreement with that determined empirically; we defer more microscopic considerations until Sec. VI.

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[27] For a classical hard sphere of radius b, $1+v_1/v_{04}=4\pi b^3/3$ and $\delta m/m_4=2\pi b^3n_4/3$. Then $V_q{}^{\mathrm{dip}}=-(2\pi b^3)^2m_4n_4(\mathbf{p}\cdot\mathbf{q}/m)(\mathbf{p}'\cdot\mathbf{q}/m)$. This is the Fourier transform of the classical dipolar interaction, using the *canonical momenta* as variables. Expressed in terms of the *velocities* of the particles, the interaction has the opposite sign.

V. THERMODYNAMIC CONSIDERATIONS ON THE EFFECTIVE INTERACTION

We begin our discussion by relating the long-wavelength limit of the effective interaction V_0 to appropriate thermodynamic derivatives. From this we shall see that there are two contributions to V_0, one coming from the phonon-induced interaction, the other representing a direct interaction which occurs at constant He^4 density. In the limit of very dilute solutions it proves possible to evaluate this second term accurately; we shall see that the essential step in its evaluation is the explicit recognition of the fact that He^3 and He^4 are isotopes and therefore the bare interatomic potential between the He^3 and He^4 atoms is the same as that between He^4 atoms or between He^3 atoms. When this is taken into account, we find that

$$V_0 = -\alpha^2 m_4 s^2 / n_4 , \qquad (85)$$

in agreement with (17).

As the He^3 atoms move through the liquid they displace the He^4 in their way. The He^4, driven essentially by local changes in their chemical potential μ_4, move in such a way as to keep μ_4, or equivalently, the local pressure constant. Thus the effective interaction between two He^3's at long wavelengths is simply related to the net change in energy of the system when two He^3's are substituted for two He^4's at constant pressure.

The free energy required to add one He^3 to the system at constant pressure is μ_3, the He^3 chemical potential. Let us write, as before in Eq. (3),

$$\mu_3 = \mu_f + \mu' ,$$

where μ_f is the chemical potential of a noninteracting Fermi gas of effective mass m and density n_3. The effective interaction (2) describes directly the amplitude for the scattering of two quasiparticles of opposite spin. [The scattering amplitude of two He^3's of the same spin includes a further exchange term.] Thus V_0 is the derivative of μ' for a He^3 of one spin orientation (up, say) with respect to a uniform change, at constant μ_4, of the density of He^3 with opposite spin (down):

$$V_0 = (\partial \mu_\uparrow' / \partial n_{3\downarrow})_{\mu_4} ; \qquad (86)$$

this derivative, as well as subsequent ones in this section, is evaluated at constant T and $n_{3\uparrow}$, in the limit $x \rightarrow 0$. We note that our identification of V_0 is analogous to the Landau definition of the effective interaction between quasiparticles in a pure Fermi liquid. Since $\mu_{f\uparrow}$ is independent of $n_{3\downarrow}$, we can equally well write

$$V_0 = (\partial \mu_{3\uparrow} / \partial n_{3\downarrow})_{\mu_4} . \qquad (87)$$

(Note that this step would not be permissible if we had taken derivatives with respect to $n_{3\uparrow}$, since $\mu_{3\uparrow}$ depends on $n_{3\uparrow}$.)

We can establish the connection with our considerations of the previous section by regarding $\mu_{3\uparrow}$ as a function of $n_{3\downarrow}$ and n_4. Thus we can write (87) as

$$V_0 = \left(\frac{\partial \mu_{3\uparrow}}{\partial n_{3\downarrow}}\right)_{n_4} + \left(\frac{\partial \mu_{3\uparrow}}{\partial n_4}\right)_{n_3} \left(\frac{\partial n_4}{\partial n_{3\downarrow}}\right)_{\mu_4} . \qquad (88)$$

The first term on the right side of (88) is the contribution to the effective interaction that is present when no variation in the He^4 background density takes place. The second term on the right is the phonon-induced interaction between the He^3 atoms; it involves a density change in the He^4 background corresponding to the emission and reabsorption of a virtual phonon. Thus $(\partial n_4/\partial n_3)_{\mu_4}$, which equals $(\partial n_4/\partial n_3)_P$ in the limit of zero concentration, is the density fluctuation induced in the He^4 background by a unit change in n_3 at constant pressure, and is, according to (62), given by $-(1+\alpha)$; then $(\partial \mu_{3\uparrow}/\partial n_4)_{n_3}$ measures the response of a second He^3 atom to that density variation, and is, from (58) and (64), given by $-(1+\alpha) m_4 s^2/n_4$. Their product, $-(1+\alpha)^2 m_4 s^2/n_4$, is the phonon-induced interaction (65).

We note that with the use of a "chain rule" for partial derivatives, the above expression for the phonon-induced interaction may be cast in still more familiar form, viz.:

$$\left(\frac{\partial \mu_{3\uparrow}}{\partial n_4}\right)_{n_3} \left(\frac{\partial n_4}{\partial n_{3\downarrow}}\right)_{\mu_4} = -\left(\frac{\partial \mu_{3\uparrow}}{\partial n_4}\right)_{n_3} \left(\frac{\partial n_4}{\partial \mu_4}\right)_{n_3} \left(\frac{\partial \mu_4}{\partial n_{3\downarrow}}\right)_{n_4}$$

$$= -\left(\frac{\partial \mu_{3\uparrow}}{\partial n_4}\right)_{n_3} \left(\frac{\partial n_4}{\partial \mu_4}\right)_{n_3} \left(\frac{\partial \mu_{3\downarrow}}{\partial n_4}\right)_{n_3} ; \qquad (89)$$

in writing the last form we have used the identity

$$U_{3-4} = \left(\frac{\partial \mu_4}{\partial n_{3\downarrow}}\right)_{n_4} = \left(\frac{\partial \mu_{3\downarrow}}{\partial n_4}\right)_{n_3} . \qquad (90)$$

The final form on the right side of (89) possesses the usual structure of a phonon-induced interaction, that is, a vertex for the emission of a virtual density fluctuation (or phonon), $(\partial \mu_{3\uparrow}/\partial n_4)_{n_3}$, a term describing the propagation of that density fluctuation $(\partial n_4/\partial \mu_4) = n_4/m_4 s^2$, and a vertex part, $(\partial \mu_{3\downarrow}/\partial n_4)_{n_3}$, for the absorption of the density fluctuation by the opposite spin He^3 atom (see Fig. 4).

We turn now to the calculation of the direct (or nonphonon–induced) contribution to the interaction energy, $(\partial \mu_{3\uparrow}/\partial n_{3\downarrow})_{n_4}$; this term measures the shift in the chemical potential of a spin-up He^3 atom due to a unit change, at constant background He^4 density, of the density of down-spin He^3 atoms. To evaluate this term we recognize the fact that the He^3 and He^4 atoms are isotopes, and thus we are led to inquire what changes in the physical properties of the solution result from the replacement of He^4 atoms by He^3 atoms. The direct interaction energy may be examined from such a point of view by rewriting it as follows:

$$\left(\frac{\partial\mu_{3\uparrow}}{\partial n_{3\downarrow}}\right)_{n_4}=\left(\frac{\partial\mu_4}{\partial n_4}\right)_{n_3}+\left(\frac{\partial(\mu_{3\downarrow}-\mu_4)}{\partial n_4}\right)_{n_3}+\left(\frac{\partial(\mu_{3\uparrow}-\mu_4)}{\partial n_{3\downarrow}}\right)_{n_4}. \qquad (91)$$

[To obtain (91), we have simply added and subtracted $(\partial\mu_4/\partial n_4)_{n_3}$, and $(\partial\mu_{3\downarrow}/\partial n_4)_{n_3}$, and made use of the identity (90).] In the form (91), the direct interaction has a simple physical interpretation. The two interacting He^3's, one with spin up and one with spin down, have displaced two He^4 atoms in the solution. The first term on the right is the interaction energy, in the sense of Landau, of these two He^4 atoms prior to being displaced. This energy, given by Eq. (56),

$$\left(\frac{\partial\mu_4}{\partial n_4}\right)_{n_3}=\frac{m_4s^2}{n_4},$$

sets the scale of interaction energies in the system. The second term is then the extra energy (beyond $\mu_{3\downarrow}-\mu_4$ evaluated at equilibrium) required to replace one of the He^4 by the down-spin He^3, in the presence of the other He^4. From Eqs. (56), (58), and (64) we see that this energy is

$$\left(\frac{\partial(\mu_{3\downarrow}-\mu_4)}{\partial n_4}\right)_{n_3}=\alpha\frac{m_4s^2}{n_4}. \qquad (92)$$

The third term on the right is the extra energy (beyond $\mu_{3\uparrow}-\mu_4$) required to replace, in the presence of the spin-down He^3, the other He^4 by the spin-up He^3.

We shall now give a physical argument that this last term in (91) has the same value as the second, i.e.,

$$\left(\frac{\partial(\mu_{3\uparrow}-\mu_4)}{\partial n_{3\downarrow}}\right)_{n_4}=\left(\frac{\partial(\mu_{3\uparrow}-\mu_4)}{\partial n_4}\right)_{n_{3\downarrow}}=\alpha m_4s^2/n_4. \qquad (93)$$

This says that the extra energy required to replace one He^4 by an up-spin He^3, in the presence of a down-spin He^3, is also $\alpha m_4s^2/n_4$. Assuming this to be so, for the moment, we find on combining Eqs. (56), (91), (92), and (93) that the direct part of the effective interaction at long wavelengths is given by

$$V_0^{\mathrm{dir}}=(1+2\alpha)m_4s^2/n_4; \qquad (94)$$

adding the phonon-induced interaction, $-(1+\alpha)^2m_4s^2/n_4$, to this yields the result (85) for the net long-wavelength effective interaction between the He^3 atoms.

Equation (93) is equivalent to the assertion that $\mu_{3\uparrow}-\mu_4$, the energy required to replace a He^4 by an up-spin He^3, depends, for fixed $n_{3\uparrow}$, only on the total density n, that is,

$$\left(\frac{\partial(\mu_{3\uparrow}-\mu_4)}{\partial n_{3\downarrow}}\right)_{n,n_{3\uparrow}}=0. \qquad (95)$$

The crucial point is that because He^3 and He^4 are isotopes, the force fields produced by He^3 and He^4 atoms are identical; the average force fields felt by He^3 and He^4 in the solution are the same. Furthermore varying $n_{3\downarrow}$ at fixed n and $n_{3\uparrow}$ produces no change in the average force field (since it corresponds to replacing a He^4 atom by a spin-down He^3 atom). On the other hand, because He^3 has a lighter mass than He^4, a He^3 atom has a larger zero point motion than a He^4, and it thus tends to occupy a larger volume than a He^4. This volume difference is measured by the quantity v_1 in Eq. (60); from (68) we see that a He^3 occupies a 28% larger volume than a He^4. The difference between $\mu_{3\uparrow}$ and μ_4 arises essentially because of this extra zero-point energy which must be supplied to the He^3. However, this extra kinetic energy depends only on the average volume per particle at the site where the replacement of a He^4 by a He^3 takes place. Thus for fixed $n_{3\uparrow}$ we expect $\mu_{3\uparrow}-\mu_4$ to depend only on n, the average particle density, whence follows (95). In the last section we present a more microscopic version of this argument.

We may look upon the effective interaction from a different point of view by asking the question: how do He^3 atoms in the solution become dynamically aware of each other? Since the force fields produced by He^3 atoms and He^4 atoms are identical a He^3 atom cannot tell whether the potential it feels is produced by a He^3 or He^4 atom. It is only through the ways that the He^3 differ from their He^4 environment that an effective interaction is produced; only the differences in He^3-He^4 mass and statistics enable the He^3 atoms to identify each other.

Even though He^3 are fermions and He^4 are bosons, statistics play a minor role in determining the effective interaction. This is because the effective interaction is determined by the behavior of two opposite spin He^3 in the solution. The scattering of two He^3 particles of the same spin includes a further exchange term. But He^3 atoms of opposite spin do not obey the exclusion principle, and thus statistics alone offer a He^3 atom of one spin orientation no way of distinguishing a He^3 atom of opposite spin from a He^4 atom in the solution.

The important source of the interaction is thus the He^3-He^4 mass difference, which has the consequence, as we have already noted, that a He^3 atom occupies a volume in the solution $(1+\alpha)$ times that occupied by a He^4 atom. Thus, in a sense, the effective interaction between two He^3 is equivalent to that between two "holes" of relative volume α in the liquid. Their interaction is expected to be of relative order α^2. That the interaction is attractive does not, however, follow from so general a consideration.

We would conclude from this argument and from (85) that if $\alpha=0$, i.e., if the He^3 had the same mass as the He^4, and so played the role of "marked" He^4 atoms, then aside from possible minor statistical corrections, such marked He^4 atoms would have a vanishingly small effective interaction. To see how this comes about, we note that the phonon-induced contribution to the marked atom interaction energy is, according to (65), $-m_4s^2/n_4$, while the direct contribution to the interaction energy is the same as that for pure He^4 atoms, m_4s^2/n_4; the two contributions cancel completely.

In the limit of vanishingly small He^3 concentration, i.e., $x\rightarrow 0$, our result [Eq. (85)]

$$V_0=-\alpha^2 m_4s^2/n_4$$

is an exact description of the physical effects we have considered (one-phonon exchange plus associated direct terms of this same order). We expect that other physical effects, such as two-phonon exchange, will lead to corrections in V_0 of higher order in α^2. Taking the experimental value, $\alpha=0.28$, we find

$$V_0=-0.078m_4s^2/n_4. \tag{96}$$

This result is in striking agreement with the empirically determined value of Eq. (17)

$$V_0=-0.0754m_4s^2/n_4.$$

We note for future reference the *exact* $x\rightarrow 0$ result for V_0, which does not assume the validity of (95):

$$V_0=-\alpha^2 m_4s^2/n_4+[\partial(\mu_{3\uparrow}-\mu_4)/\partial n_{3\downarrow}]_{n,n_{3\uparrow}}. \tag{97}$$

VI. MICROSCOPIC CONSIDERATIONS ON THE EFFECTIVE INTERACTION

We give, in this section, a microscopic calculation, based on the variational method, of the ground-state energy ϵ of the He^3-He^4 mixture. We use the resulting expression to determine μ_3 and μ_4, and find that their difference depends only on the over-all system density, in agreement with the thermodynamic considerations of the previous section. We obtain an explicit expression for α, one which depends only on the properties of pure He^4 in its ground state, and use it to estimate α; the resulting estimate is in good accord with experiment. In conclusion we estimate, in two different ways, the pressure dependence of V_0, and conclude that V_0 decreases in magnitude with applied pressure.

The exact Hamiltonian of the mixture is

$$H=\sum_{\mathrm{He}^4}\frac{p_i^2}{2m_4}+\tfrac{1}{2}\sum_{i,j}u(\mathbf{r}_i-\mathbf{r}_j)+\sum_{\mathrm{He}^3}\frac{p_j^2}{2m_3}. \tag{98}$$

u is the bare interatomic potential and the sum in the middle term is over all pairs of particles. Since $m_3=3m_4/4$ we can equivalently write H as

$$H=\sum\frac{p_i^2}{2m_4}+\tfrac{1}{2}\sum_{ij}u(\mathbf{r}_i-\mathbf{r}_j)+\sum_{\mathrm{He}^3}\frac{p_i^2}{6m_4}, \tag{99}$$

where the sum in the first term is now over *both* the He^3 and the He^4 particles. The first two terms in (99) are formally the Hamiltonian for $N=N_3+N_4$ He^4 atoms; the last term represents the additional kinetic energy of the He^3 atoms due to their lighter mass. This last term, the additional zero-point energy of the He^3, which is relatively small, is essentially the perturbation that leads to an effective interaction. It is in the form of a "one-body operator," a sum of contributions from each He^3 atom individually.

We estimate ϵ by taking as a trial wave function the true ground-state wave function of pure He^4 at the same particle density n as in the mixture. This trial function does not take into account the exclusion principle for He^3 atoms of the same spin. This is not of importance though for calculating either α or the interaction between opposite spin particles, since to calculate these quantities we need consider a system with at most one or two He^3 present. We note that in this low concentration limit our trial function would be the exact ground state were the masses of He^3 and He^4 equal. With this trial function

$$\epsilon(n)=\epsilon_0(n)+n_3\epsilon_1(n), \tag{100}$$

where $\epsilon_0(n)$, the expectation value of the first two terms in (99) divided by the volume, is the ground-state energy per unit volume of pure He^4 at density n; also

$$\epsilon_1(n)=\langle p^2/6m_4\rangle \tag{101}$$

is one third the average kinetic energy per particle in pure He^4 at density n and $T=0$.

Equation (100) for the energy says that the energy of the mixture is that of pure He^4 at the same density, plus a correction term, linear in n_3 at fixed n, representing the additional zero point energy of the He^3. The absence of terms in (100) of higher order in n_3 is primarily a consequence of the identity of the interatomic forces between isotopes. [We remark that one expects correction terms to (100) due to exchange effects, but these should not be significant in determining the effective interaction between opposite spin particles.] The linear form of Eq. (100) is reminiscent of the interpolation formula proposed by Prigogine and co-workers[28] for the total energy of a mixture at concentration x; at $T=0$ their formula is

$$\begin{aligned}\epsilon(n)&=(1-x)\epsilon_0(n)+x\epsilon_3(n)\\&=\epsilon_0(n)+n_3[\epsilon_3(n)-\epsilon_0(n)]/n,\end{aligned} \tag{102}$$

where $\epsilon_3(n)$ is the energy density of *pure* He^3 at density

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[28] I. Prigogine, *The Molecular Theory of Solutions* (North-Holland Publishing Company, Amsterdam, 1958); I. Prigogine, R. Bingen, and A. Bellemans, Physica **20**, 633 (1954).

n. However, as the molar volume data indicate, the coefficient of n_3 in (102) is not sufficiently accurate at low He^3 concentrations.

We calculate μ_4 and μ_3 directly from (102), according to

$$\mu_4=\left(\frac{\partial\epsilon}{\partial n_4}\right)_{n_3}=\frac{\partial\epsilon_0}{\partial n}+n_3\frac{\partial\epsilon_1}{\partial n} \qquad (103)$$

and

$$\mu_3=(\partial\epsilon/\partial n_3)_{n_4}=\mu_4+\epsilon_1(n). \qquad (104)$$

We see that our trial function has the property that the difference in chemical potentials, $\mu_3-\mu_4$, is just the additional zero-point energy $\epsilon_1(n)$, of a He^3 atom, and that furthermore this quantity depends only on the system density. The thermodynamic derivative (95) therefore vanishes, and from (97) we recover the result (85) for the net effective interaction: $V_0=-\alpha^2m_4s^2/n_4$.

The present calculation tells us nothing about the momentum dependence of the effective interaction, because when exchange effects are neglected, the total energy is only sensitive to V_0. A satisfactory calculation of this momentum dependence remains an open problem.

Let us emphasize that from a microscopic point of view, the difference in chemical potentials depending only on the density is a direct consequence of the fact that the perturbation term, the extra zero-point motion, is a sum of one-body operators. Perturbations of this form are characteristic of isotopic impurity problems; we see, in yet another way, that the interaction is of order α^2 only because He^3 is an isotope of He^4.

In the preceding section we studied the replacement of a He^4 atom by a He^3 atom with spin down, and its effect on a second He^4 atom or a He^3 atom with spin up. We argued that this replacement, if carried out at constant density, produces the same change in μ_3 and μ_4. From Eq. (103) we see explicitly that this change is

$$\frac{\partial\epsilon_1}{\partial n}=\frac{\partial}{\partial n}\left\langle\frac{p^2}{6m_4}\right\rangle.$$

This is essentially a Landau "Fermi-liquid" effect; it is the modification of the extra zero-point energy of the spin-down He^3 already present caused by the addition of an extra He^4 or spin-up He^3.

We may obtain an explicit expression for α with the aid of (66) and (104):

$$\alpha=\frac{n_4}{m_4s^2}\frac{\partial}{\partial n_4}\left\langle\frac{p^2}{6m_4}\right\rangle. \qquad (105)$$

Thus in the present calculation α is determined directly in terms of the properties of the ground state of pure He^4, in particular, by the density dependence of its average kinetic energy per particle. While it would be desirable to have a microscopic calculation of α, using detailed He^4 wave functions, so as to compare the expression (105) with experiment, we content ourselves here with a rough estimate. To find the density dependence of the average kinetic energy per particle in pure He^4, we simply regard each He^4 atom as a hard sphere of effective diameter d moving in a fixed impenetrable spherical shell formed by the atom's nearest neighbors. If a is an average nearest neighbor distance, the hard sphere moves as a point particle in a shell of radius $a-d$, and from elementary quantum mechanics the ground-state energy of the particle is

$$\langle p^2/2m_4\rangle=\pi^2\hbar^2/2m_4(a-d)^2. \qquad (106)$$

If we assume $a\sim n_4^{-1/3}$ and d independent of n_4 then

$$n_4\partial\langle p^2\rangle/\partial n_4=2\pi^2\hbar^2a/3(a-d)^3, \qquad (107)$$

so that from (105),

$$\alpha\approx(\pi\hbar/3m_4sa)^2(1-d/a)^{-3}. \qquad (108)$$

de Boer[29] has used the simple formula (106) to calculate the zero-point energy of solid He^4 and finds good numerical agreement if he chooses the empirical value $d=2.0$ Å and takes $a=2^{1/6}n_4^{-1/3}$, appropriate to an fcc lattice. We estimate (108) by using the same numerical value for d and letting $a=3.8$ Å, an average nearest-neighbor distance as determined by neutron scattering experiments[30]; then (108) gives $\alpha=0.31$, which is quite close to the experimental value[31] 0.28.

The agreement between our variational calculation for α and the experimental value provides further evidence that the predominant source of the effective interaction is the difference in zero-point motion of the He^3 and He^4.

We may also use the above model, together with Eq. (85), to estimate the pressure dependence of V_0, and thus to estimate the pressure dependence of the transition temperature T_c for s-state pairing. We shall find that V_0 decreases in magnitude with pressure, indicating that applying pressure probably would only serve to lower T_c for s-state pairing.

Using (108) we write Eq. (85) for V_0 as

$$|V_0|=\frac{(\pi\hbar/3)^4}{m_4^3s^2n_4}\frac{a^2}{(a-d)^6}. \qquad (109)$$

Assuming again that d is independent of n_4 and $a\sim n_4^{-1/3}$ we find

$$\frac{n_4}{|V_0|}\frac{\partial|V_0|}{\partial n_4}=\frac{1}{3}+\frac{2d}{a-d}-\frac{2n_4}{s}\frac{\partial s}{\partial n_4}. \qquad (110)$$

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[29] J. de Boer, *Progress in Low-Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1957), Vol. II, p. 23.

[30] D. G. Hurst and D. G. Henshaw, Phys. Rev. **91**, 1222 (1953); **100**, 994 (1955).

[31] W. L. McMillan [Phys. Rev. **138**, A442 (1965)], using his He^4 ground-state wave function, has calculated that $n\partial\langle p^2/2m_4\rangle/\partial n=3.46\times10^{-15}$ ergs/atom. Taking this value and the experimental sound velocity in Eq. (105) one finds $\alpha=0.30$. We are grateful to Dr. McMillan for communicating to us the results of his calculation.

From the measurements of Atkins and Stasior[32] we have

$$\frac{n_4}{s}\frac{\partial s}{\partial n_4}\approx 2.7.$$

Then taking the same values of a and d as above we find

$$\frac{\partial |V_0|}{\partial n_4}\approx -2.8\frac{|V_0|}{n_4}, \tag{111}$$

implying that pressure weakens the effective interaction. The value (111) is equivalent to about a 3.5% decrease in $|V_0|$ per atm. Similarly, from Eq. (108) we find the estimate

$$n_4(\partial\alpha/\partial n_4)\approx -1.13. \tag{112}$$

The pressure dependence of V_0 may also be estimated by relating it, through a thermodynamic argument, to the concentration dependence of the molar volume. We first note that from (59), the volume per particle n^{-1} can be written as

$$n^{-1}=\left(\frac{\partial}{\partial P}[x\mu_3+(1-x)\mu_4]\right)_x. \tag{113}$$

Thus

$$\left(\frac{\partial n^{-1}}{\partial x}\right)_P=\left(\frac{\partial}{\partial P}(\mu_3-\mu_4)\right)_x, \tag{114}$$

since, at fixed P,

$$x(\partial\mu_3/\partial x)_P+(1-x)(\partial\mu_4/\partial x)_P=0. \tag{115}$$

Equation (66) for α is simply Eq. (114) evaluated in the limit $x\rightarrow 0$. Now differentiating (114) with respect to x at constant pressure, and using (115) again, we find in the limit $x\rightarrow 0$ that

$$\left(\frac{\partial^2 n^{-1}}{\partial x^2}\right)_P=\left[\frac{\partial}{\partial P}\left(\frac{\partial\mu_3}{\partial x}\right)_P\right]_x. \tag{116}$$

We now relate the right side of (113) to the effective interaction by writing $\mu_3=\mu'+\mu_f$ as before. Then in the nondegenerate region

$$n_3=2\int\frac{d\mathbf{p}}{(2\pi h)^3}e^{(\mu_f-p^2/2m^*)/KT}, \tag{117}$$

so that at constant P and T,

$$\begin{aligned}\left(\frac{\partial\mu_f}{\partial x}\right)_P&=KT\left[\frac{1}{n_3}\left(\frac{\partial n_3}{\partial x}\right)_P-\frac{3}{2m^*}\left(\frac{\partial m^*}{\partial x}\right)_P\right]\\&=-KT\left[\alpha+\frac{3}{2m^*}\left(\frac{\partial m^*}{\partial x}\right)_P\right],\end{aligned} \tag{118}$$

to lowest order in x. Also $(\partial\mu'/\partial x)_P=nV_0$ neglecting an exchange contribution in the nondegenerate region. Combining this with (118) and (115) we have

$$\begin{aligned}\frac{n_4}{|V_0|}\frac{\partial|V_0|}{\partial n_4}=-1-\frac{m_4s^2}{n_4|V_0|}\Bigg[\frac{1}{v_{04}}\left(\frac{\partial^2 v}{\partial x^2}\right)_P\\+KT\left(\frac{\partial\alpha}{\partial P}+\frac{3}{2}\frac{\partial^2\ln m^*}{\partial x\partial P}\right)\Bigg],\end{aligned} \tag{119}$$

where v is the molar volume. The term in square brackets may be estimated by letting $T\rightarrow 0$ and extrapolating to $T=0$ the value of $\partial^2v/\partial x^2$ inferred from Kerr's data[24] in the nondegenerate region. Unfortunately the loss of accuracy in this extrapolation allows us to deduce only that $(\partial^2v/\partial x^2)_P/v_{04}$ is at least greater than 0.05 as $T\rightarrow 0$. Thus the right side of (118) is negative and greater in magnitude than 1.6. This is consistent with the result (111), and is a further check that V_0 should decrease in magnitude with applied pressure. [Turning the argument around, Eq. (119) with (111) and (112) may be regarded as a very preliminary microscopic calculation of $(\partial^2v/\partial x^2)_P$.]

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[32] K. R. Atkins and R. A. Stasior, Can. J. Phys. **31**, 1156 (1953).