more accurate. However, it is expected that the results obtained here are qualitatively, and at worst semiquantitatively, correct.

V. SUMMARY

By judiciously comparing the results of theory and experiment, we have been able to extract information from experiment as to position and width for the lowest ^1S and ^3P resonances, even though the widths of those resonances are of the same magnitude as the energy resolution of the electrons used in the experiment. The very strong dependence of resonance structure in the (e-H) system upon θ and $\Delta\theta$ is found when S-, P-, and D-wave scattering is considered.

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Radiation from Nonlinearly Excited Plasmas

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Longitudinal plasma oscillations can be excited nonlinearly by two strong incident electromagnetic waves. The reradiation from such nonlinearly excited plasma is calculated for the cases of a thin plasma film and a small plasma column. Finite-temperature effects are neglected, but the dominant mechanism of coupling between longitudinal and transverse waves by boundaries is properly taken into account. Resonances in such reradiation from both the thin film and the small column are found at frequencies ω_p and $\omega_p/\sqrt{2}$, respectively, where ω_p is the plasma frequency. The intensity of such reradiation is compared with the scattered intensity of another electromagnetic wave from the same nonlinearly excited plasma. It is found quite generally that the reradiation dominates the scattering, thereby explaining the result of a recent experiment.

I. INTRODUCTION

THROUGH nonlinear interaction, two strong electromagnetic waves may induce longitudinal current or density fluctuations in matter. Considerable attention has been focused on the incoherent scattering of an electromagnetic wave by such nonlinearly induced density fluctuations in a plasma.[1] (From now on, we shall refer to such incoherent scattering as the *optical-mixing effect*). In this paper, we shall study instead the reradiation from such nonlinearly excited plasmas. We find that the reradiation from a nonlinearly excited thin film exhibits resonant behavior when the sum or the difference of the frequencies of the two electromagnetic waves is equal to the plasma frequency, although no such resonant behavior exists for a semi-infinite medium. The ratio of the intensity of the reradiation to that of one of the incoming waves is in fact found to be of the same order as the ratio of *optical mixing* to the reradiation. A similar conclusion is obtained for a small plasma column, with the resonant frequency at $\omega_p/\sqrt{2}$, where ω_p is the plasma frequency. As an application of our findings we see why, in a recent experiment,[2] reradiation from a nonlinearly excited plasma column was detected while the *optical-mixing effect* eluded observation.

II. THIN PLASMA FILM

Consider two electromagnetic waves with electric vectors, $\mathbf{E}_1 \exp(-i\omega_1 t+i\mathbf{k}_1\cdot\mathbf{x})$ and $\mathbf{E}_2 \exp(-i\omega_2 t+i\mathbf{k}_2\cdot\mathbf{x})$ respectively, entering into a bounded electron gas. For simplicity we adopt the collisionless-cold-plasma model, thereby neglecting the fine structure due to temperature. However, the dominant mechanism of coupling

[1] N. M. Kroll, A. Ron, and N. Rostoker, Phys. Rev. Letters **13**, 83 (1964); H. Cheng and Y. C. Lee, Phys. Rev. **142**, 104 (1966). For the related phenomena of light-light scattering, see P. M. Platzman, S. J. Buchsbaum, and N. Tzoar, Phys. Rev. Letters **12**, 573 (1964); D. F. Dubois and V. Gilinsky, Phys. Rev. **135**, A995 (1964).

[2] R. A. Stern and N. Tzoar, Phys. Rev. Letters **16**, 785 (1966).

between longitudinal and transverse waves by boundaries is properly taken into account. By solving the Vlasov equation to the second order in the external fields, it is seen that the influence of the two waves on the plasma due to nonlinear mixing is equivalent to that of an external longitudinal current placed inside the plasma (see Appendix for derivation):

$$\mathbf{J}^{\mathrm{ext}}(\mathbf{x},t)=J_0\hat{e}_{\mathrm{k}}\exp(-i\omega t+i\mathbf{k}\cdot\mathbf{x}),\tag{1}$$

where[3], in mks units,

$$J_0=-\epsilon_0\frac{e}{m}\frac{\omega_p{}^2}{\omega_1\omega_2}\frac{k}{\omega}(\mathbf{E}_1\cdot\mathbf{E}_2)\tag{2}$$

and where $\omega=\omega_1+\omega_2$, $\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2$, $k=|\mathbf{k}|$, $\hat{e}_{\mathrm{k}}$ is a unit vector in the direction of $\mathbf{k}$, and ω_p is the plasma frequency. Correspondingly, there is also an equivalent external charge density in the plasma,

$$\rho^{\mathrm{ext}}(\mathbf{x},t)=\frac{k}{\omega}J_0\exp(-i\omega t+i\mathbf{k}\cdot\mathbf{x}),\tag{3a}$$

as required by the equation of continuity. The total charge density in the bulk of the plasma characterized by the usual dielectric constant $\epsilon(\omega)$ is then given by

$$\rho^{\mathrm{tot}}=\rho^{\mathrm{ext}}/\epsilon(\omega),\tag{3b}$$

which has a resonance when $\omega_2=-(\omega_1-\omega_p)$. It is this resonance effect which enhances the scattering of an electromagnetic wave by such a density fluctuation.[1]

Our problem now is reduced to the study of radiation from a medium when a longitudinal current of the form (1) is present in the medium. To avoid confusing such radiation with the incident radiations $\mathbf{E}_1$ and $\mathbf{E}_2$ which induce the current (1), we shall call such radiation the *reradiation* from now on. For a semi-infinite medium, such reradiation in the form of second harmonic generation has in fact been detected and observed.[4]

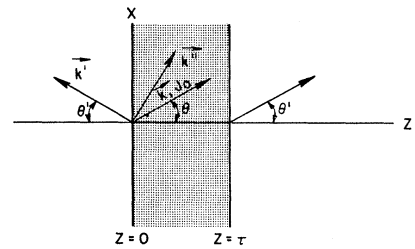

FIG. 1. Radiation from an infinite plasma slab in the presence of a longitudinal current of magnitude J_0 and directed along $\mathbf{k}$.

We first consider the case when the medium is an infinite slab of thickness τ. (See Fig. 1.) Solving the Maxwell equations with the current and charge density in the medium given by (1) and (3) and by imposing the usual boundary conditions of continuity of the magnetic field and the tangential component of the electric field across the boundaries, we may find in a straightforward manner that electromagnetic (EM) waves are radiated from the two sides of the slab. The wave vector of the radiation makes an angle θ' with the normal of the slab:

$$\sin\theta'=\frac{ck\sin\theta}{\omega},\tag{4}$$

where θ is the angle between the longitudinal current vector and the z axis, which is parallel to the normal to the surface of the slab (see Fig. 1).

In the mks units, the electric field in the vacuum is given by

$$\mathbf{E}=E_0\hat{e}_y\exp\left[-i\left(\omega t+\frac{\omega}{c}x\sin\theta'-\frac{\omega}{c}z\cos\theta'\right)\right],\quad z<0\tag{5}$$

where

$$E_0=\frac{iJ_0\sin\theta}{\epsilon_0\omega}\,\frac{[i\epsilon(\omega)\omega\cos\theta'/ck_z'']\sin k_z''\tau-\cos k_z''\tau+\exp(ik\tau\cos\theta)}{2\epsilon(\omega)\cos\theta'\cos k_z''\tau-i(c/\omega)k_z''\sin k_z''\tau[1+(\epsilon(\omega)\omega\cos\theta'/ck_z'')^2]}\tag{6}$$

where $k_z''=[(\epsilon\omega^2/c^2)-k^2\sin^2\theta]^{1/2}$, is the z component of the wave vector $\mathbf{k}''$ of the transverse wave (of frequency ω) in the plasma film, and $\epsilon(\omega)$ is the dielectric function. The amplitude of the electric field in the region $z>\tau$ can be obtained from (6) by replacing θ' with $\pi-\theta'$.

When $k_z''\tau\ll1$, (6) is approximated by

$$E_0=\frac{-iJ_0\sin\theta}{\epsilon_0\omega}\times\frac{\exp(ik\tau\cos\theta)-1+(i\omega\tau/c)\epsilon(\omega)\cos\theta'}{2\epsilon(\omega)\cos\theta'-i(c\tau/\omega)\{k_z''^2+[\omega\epsilon(\omega)\cos\theta'/c]^2\}}.\tag{7}$$

When

$$\epsilon(\omega)\sim\frac{c^2k^2}{\omega^2}\sin^2\theta\ll1,$$

we further have

$$E_0=\frac{-iJ_0\theta}{2\epsilon_0\omega\epsilon(\omega)}\,\frac{\exp(ik\tau)-1}{1-(i\omega/2c)\tau(1-(c^2k^2\theta^2/\omega^2\epsilon))}.\tag{8}$$

At the plasma frequency ω_p, where $|\epsilon(\omega)|$ is small, E_0

[3] We remark here that if $\mathbf{k}_1=\mathbf{k}_2$, we should multiply the right side of (2) by an extra factor $\frac{1}{2}$ to avoid double counting.

[4] N. Bloembergen and P. S. Pershan, Phys. Rev. **126**, 606 (1962); R. Kronig and J. I. Bonkema, Koninkl. Ned. Akad. Wetenschap., Proc. Ser. B **66**, 8 (1963); H. Cheng and P. B. Miller, Phys. Rev. **134**, A683 (1964); S. S. Jha, *ibid.* **140**, A2020 (1965); N. Bloembergen and Y. R. Shen, *ibid.* **141**, 298 (1966).

becomes resonantly large. At the angle $\theta \approx |\epsilon(\omega_p)|^{1/2}$, E_0 is of the order of $J_0/[\omega\epsilon_0\sqrt{\epsilon(\omega)}]$. Also, from (8), we see that to obtain a large value for E_0, we should choose τ so that $k\tau$ is not very small, while satisfying $k_z''\tau \ll 1$. This is possible because we have $k_z'' \ll k$ at resonance. If $\theta \sim \sqrt{\epsilon(\omega_p)}$, ω/c and k are of comparable magnitudes, then τ must satisfy $k\tau\sqrt{\epsilon(\omega_p)} \ll 1 \lesssim k\tau$.

The physical mechanism is as follows: In an infinite medium the induced current (1) would produce a longitudinal wave

$$\mathbf{E} = \frac{-i}{\omega\epsilon_0\epsilon(\omega)} \mathbf{J}^{\text{ext}}(\mathbf{x},t)$$

in the medium. When this longitudinal wave arrives at a boundary of the medium, two transverse waves, one propagating into the medium, the other propagating into vacuum, are produced. Although at the plasma frequency ω_p, the amplitude of the longitudinal wave is enhanced by the resonant factor ϵ^{-1}, the amplitude of the transverse wave in the vacuum exhibits no resonant behavior, and the wave is almost totally reflected back into the medium. We may understand this by noting that the normal component of the electric vector in the vacuum is equal to that in the medium multiplied by ϵ. This explains the absence of resonant behavior for a semi-infinite medium. For a slab, the transverse waves in the medium are then multiply reflected and refracted by the front and back plane surfaces. If the slab is sufficiently thin so that no appreciable attenuation is caused as the wave travels from one surface to another, the refracted waves in the vacuum may add up constructively to produce a large amplitude. The conditions for no appreciable phase difference between two refracted waves are given precisely by $k_z''\tau \ll 1$ and $\epsilon \sim k^2c^2\sin^2\theta/\omega^2 \ll 1$. Under these conditions, one sees from (8) that the energy flux of the reradiation, being proportional to $|E_0|^2$, is enhanced by a factor of $|\epsilon(\omega_p)|^{-2}$ at resonance.[5] When $|k_z''\tau - N\pi| \ll 1$, $N=1, 2, \cdots$, the consecutively refracted waves may also add up constructively. Owing to attenuation the total amplitude in this case is somewhat smaller.

When Eq. (2) is substituted into Eq. (8), we find that the peak value of the ratio of the reradiated power to the power of one of the incident waves (say E_1) is of the order of

$$\left(\frac{kc}{\omega_1}\right)^2 |\epsilon(\omega_p)|^{-1} \left(\frac{v_2}{c}\right)^2,$$

where $v_2 = (eE_2/m\omega_2)$ is the velocity of an electron linearly driven by the electric field $\mathbf{E}_2$.

For second harmonic generation, we let $\omega_1 = \omega_2 = \omega/2 \approx \omega_p/2$, $\mathbf{k}_1 = \mathbf{k}/2$. In this case, the ratio above is by a factor $|\epsilon(\omega_p)|^{-1}$ larger than that for a semi-infinite medium ($\tau = \infty$). Since such reradiation from a semi-infinite medium has been observed,[3] the resonance effect discussed here should well be observable. Next we consider the case when $\omega_1 = |\omega_1| \gg \omega_p$, $-\omega_2 = |\omega_2| \gg \omega_p$, but $\omega_2 = -(\omega_1 - \omega_p)$ and $\mathbf{k} = \mathbf{k}_1 + (-\mathbf{k}_2)$ which are the conditions under which *optical mixing* is observed.[1] With a plasma slab of $\omega_p \sim 10^{12}$ sec^{-1} and $|\epsilon(\omega_p)| \sim 0.001$, irradiated by two laser beams with frequencies ω_1, ω_2 of the order of 10^{15} sec^{-1} and field intensities of the order of 10^9 volt/cm, this ratio is of the order of 10^{-6} which should also be observable. In some experimental situations, one may employ, instead of laser beams, two microwaves which have less power but also lower frequencies.

To calculate the scattering cross section of a third EM wave by the same thin film via the optical mixing effect, we first solve the Vlasov equation to the third order to obtain a current proportional to E_1, E_2, E_3. The calculation is more lengthy but similar to the one carried out in the Appendix and will not be repeated here. The *optical mixing effect* can then be viewed as the radiation from this current. We find

$$\frac{\text{optical mixing}}{\text{reradiation}} \sim \frac{v_3^2}{c^2} \epsilon^{-1}(\omega) \left(\frac{\omega}{\omega_3}\right)^2$$

where $v_3 = eE_3/m\omega_3$. This ratio is of the same order of magnitude as the ratio of the reradiation intensity to the intensity of one of the EM waves. This is because *optical mixing* is a higher order effect as compared to the reradiation, in the same way that the reradiation is a higher order effect as compared to the ordinary reflection of a wave from a medium.

III. PLASMA COLUMN

Next we consider the case when the medium is an infinitely long plasma column of radius a. For simplicity the two electromagnetic waves which induce the longitudinal current (1) in the plasma column are assumed to have their polarization vectors $\mathbf{E}_1$, $\mathbf{E}_2$ and their wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$ perpendicular to the axis of the column (see Fig. 2). As before, the longitudinal current (1) induced by the two electromagnetic waves will in turn cause the plasma column to reradiate in radial directions

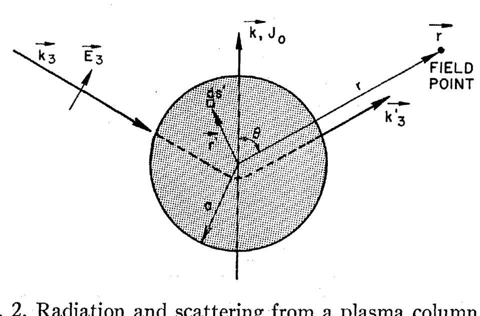

FIG. 2. Radiation and scattering from a plasma column in the presence of a longitudinal current of magnitude J_0 and directed along $\mathbf{k}$.

[5] For a linearly excited medium, the existence of such resonance for a thin film has been pointed out previously. See, for example, R. A. Ferrell and E. A. Stern, Am. J. Phys. 30, 810 (1962), and references contained therein.

perpendicular to the axis of the column (the z axis). We assume here that $\omega_1 \gg \omega_p$, $\omega_2 \gg \omega_p$. If we split the total field induced by the equivalent external current (1) and charge (3a) into two parts,

$$\mathbf{E}^i = \mathbf{E}' + \mathbf{E}, \quad \mathbf{B}^i = \mathbf{B} = \hat{z} B_z \tag{9a}$$

where

$$\mathbf{E}' = 0 \quad \text{for} \quad r > a$$
$$= \frac{-i}{\omega \epsilon_0 \epsilon(\omega)} \mathbf{J}^{\text{ext}}(\mathbf{x}, t) \quad \text{for} \quad r < a,$$

it can be easily shown that Maxwell's equations in cylindrical coordinates can be reduced to

$$\mathbf{E} = \frac{ic^2}{\omega \epsilon(r)} \nabla \times (\hat{z} B_z) \tag{9b}$$

and

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \epsilon(r) k_0^2 \right) B_z = 0 \tag{9c}$$

where

$$\epsilon(r) = 1 \quad \text{for} \quad r > a$$
$$= \epsilon(\omega) \quad \text{for} \quad r > a,$$

and

$$k_0 \equiv \frac{\omega}{c}.$$

Together with the standard boundary conditions of continuity of $\mathbf{B}^i$ and $E^i_{\text{tangential}}$ across the boundary, Eq. (9) can be solved in a straightforward manner,[6] the solution being given in the form of an infinite series involving the Bessel and Hankel functions. If we make the approximations that $k_0 a \ll 1$, $|[\sqrt{\epsilon(\omega)}] k_0 a| \ll|$ and $k_0^2 a^2 < |\text{Im}\epsilon(\omega)|$, the reradiation field takes the simple form

$$E_\theta(r,\theta) = (2\pi)^{1/2} e^{i3\pi/4} \frac{1-\epsilon(\omega)}{1+\epsilon(\omega)} J_1(ka)$$
$$\times (k_0 a)(k_0 \Phi_0) \frac{e^{ik_0 r}}{(k_0 r)^{1/2}} \sin\theta, \tag{10}$$

where the potential Φ_0 is related to the current J_0 of (2) by $\Phi_0 = \omega J_0 / (\epsilon_0 \omega_p^2 k)$ and $J_1(x)$ is the Bessel function of order 1. We see from Eq. (10) that a resonance occurs when $\epsilon(\omega) + 1 \approx 0$ or $\omega \approx \omega_p / \sqrt{2}$ for an electron gas. This is the familiar surface mode resonance. However, within our present cold plasma approximation, the reradiation does not exhibit an $\epsilon(\omega) \approx 0$ resonance. The factor $\sin\theta$ in Eq. (10) suggests that the reradiation pattern is similar to that of an electric dipole directed along $\mathbf{k}$, which is expected in view of our $k_0 a \ll 1$ approximation.

If we consider the scattering (see Fig. 2) of a third electromagnetic wave $\mathbf{E}_3$, $\mathbf{B}_3$ of $\mathbf{k}_3$ and $\omega_3 (\omega_3 \gg \omega_p)$ from the induced current and density fluctuations of frequency ω of the infinitely long plasma column (the *optical mixing effect*), we can apply the usual scattering formula,[7] modified slightly for the present two-dimensional problem:

$$\langle |E_{\text{scatt}}(r,\theta)|^2 \rangle_{\text{av}} = \frac{8\pi r_0^2}{r} (\mathbf{E}_3 \cdot \hat{\theta})^2$$
$$\times \left\{ \frac{1}{k_0 + k_3} \left| \int ds' N(\mathbf{r}') \exp i [\mathbf{k}_3 - (k_0 + k_3) \hat{r}] \cdot \mathbf{r}' \right|^2 \right.$$
$$\left. + \frac{1}{|k_3 - k_0|} \left| \int ds' N(\mathbf{r}') \exp(-i) [\mathbf{k}_3 - (k_3 - k_0) \hat{r}] \cdot \mathbf{r}' \right|^2 \right\} \tag{11}$$

where

$$r_0 = \frac{e^2}{4\pi \epsilon_0 m c^2}, \quad k_0 \equiv \omega / c, \quad k_3 = \omega_3 / c.$$

Also, in Eq. (11), ds' is an area element and $N(\mathbf{r}')$ is the number of electrons per unit volume. In $N(\mathbf{r}')$ there are two terms, namely, the volume density fluctuation given by Eq. (3b) and the surface density fluctuation. These two terms give rise to two resonances of comparable magnitudes in the scattering cross section, one at $\epsilon(\omega) \approx 0$, the other at $\epsilon(\omega) + 1 \approx 0$. For an order of magnitude estimate and also for the sake of comparing the scattered intensity with the reradiation intensity from Eq. (10), we can consider just the surface density fluctuation, which is the sole contributor to the $\epsilon(\omega) + 1 \approx 0$ resonance. This surface density, being related to the radial components of the total induced field of Eq. (9a) by

$$\sigma = -\frac{\epsilon_0}{e} [(E_r^i)_{r=a+\delta} - (E_r^i)_{r=a-\delta}]$$

could already be known in the process of deriving the result of Eq. (10). Thus, to the same approximations as Eq. (10), the surface charge density is found to be

$$\sigma(a,\theta') = \frac{2\epsilon_0 (\epsilon - \frac{1}{2})}{e \epsilon (1+\epsilon)} \frac{\Phi_0}{a} \sum_{n=1,2,\cdots}^{\infty} i^n \cos n\theta'$$
$$\times [(1-\epsilon) n J_n(ka) - (1+\epsilon) k_0 a J_n'(ka)]. \tag{12}$$

Comparing the contribution of this surface charge density to the scattered intensity with the reradiation intensity from Eq. (10), we find, with the additional assumptions of $k_3 a \lesssim 1$, $ka \lesssim 1$, that

$$\frac{\text{scattering (optical mixing effect)}}{\text{reradiation}} \sim \frac{v_3^2}{c^2} \left(\frac{\omega_3}{\omega} \right)^3, \tag{13}$$

where

$$v_3 = e E_3 / m \omega_3.$$

Recently,[2] reradiation from a nonlinearly excited plasma column has indeed been detected while the scattering by the same excited plasma (the *optical mixing effect*)

[6] See, for example, W. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Cambridge, Massachusetts 1955), Chap. 12.

[7] See, for example, M. N. Rosenbluth and N. Rostoker, Phys. Fluids **5**, 776 (1962).

has eluded observation. In this case, Eq. (13) shows why reradiation greatly dominates over scattering.

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APPENDIX

When a bounded, uniform plasma is under the influence of incident transverse electromagnetic waves

$$\begin{aligned}\mathbf{E}^{\text{ext}}(\mathbf{x},t)&=\mathbf{E}_1{}^{\text{ext}}(\mathbf{x},t)+\mathbf{E}_2{}^{\text{ext}}(\mathbf{x},t)\,,\\ \mathbf{B}^{\text{ext}}(\mathbf{x},t)&=\mathbf{B}_1{}^{\text{ext}}(\mathbf{x},t)+\mathbf{B}_2{}^{\text{ext}}(\mathbf{x},t)\,,\end{aligned}\tag{A1}$$

where

$$\begin{aligned}\mathbf{E}_i{}^{\text{ext}}(\mathbf{x},t)&=2\mathbf{E}_i{}^{\text{ext}}\cos(\mathbf{k}_i\cdot\mathbf{x}-\omega_i t)\,,\\ \mathbf{B}_i{}^{\text{ext}}(\mathbf{x},t)&=2\mathbf{B}_i{}^{\text{ext}}\cos(\mathbf{k}_i\cdot\mathbf{x}-\omega_i t),\quad i=1,\,2\end{aligned}$$

the dynamics of the plasma is described by the Vlasov equation

$$\frac{\partial f}{\partial t}(\mathbf{x},\mathbf{v},t)+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}}-\frac{e}{m}[\mathbf{E}^{\text{ext}}(\mathbf{x},t)+\mathbf{E}^{i}(\mathbf{x},t)+\mathbf{v}\times\mathbf{B}^{\text{ext}}(\mathbf{x},t)+\mathbf{v}\times\mathbf{B}^{i}(\mathbf{x},t)]\cdot\frac{\partial(f_0+f)}{\partial\mathbf{v}}=0.\tag{A2}$$

In (A2), $\mathbf{E}^i$ and $\mathbf{B}^i$ are the induced fields, $f_0=f_0(\mathbf{v})$ is the Maxwell distribution function, $f(\mathbf{x},\mathbf{v},t)$ is the perturbed part of the electron distribution function.

We now assume that, in the region occupied by the plasma,

$$\begin{aligned}f(\mathbf{x},\mathbf{v},t)=\sum_{i=1,2}&f(\mathbf{k}_i,\mathbf{v},\omega_i)e^{i(\mathbf{k}_i\cdot\mathbf{x}-\omega_i t)}+f(-\mathbf{k}_i,\,\mathbf{v},\,-\omega_i)e^{-i(\mathbf{k}_i\cdot\mathbf{x}-\omega_i t)}+f(\Delta\mathbf{k},\mathbf{v},\Delta\omega)e^{i(\Delta\mathbf{k}\cdot\mathbf{x}-\Delta\omega t)}\\ &+f(-\Delta\mathbf{k},\,\mathbf{v},\,-\Delta\omega)e^{-i(\Delta\mathbf{k}\cdot\mathbf{x}-\Delta\omega t)}+f(\mathbf{K},\mathbf{v},\Omega)e^{i(\mathbf{K}\cdot\mathbf{x}-\Omega t)}+f(-\mathbf{K},\,\mathbf{v},\,-\Omega)e^{-i(\mathbf{K}\cdot\mathbf{x}-\Omega t)}+\cdots(\text{higher-order terms})\,,\end{aligned}\tag{A3}$$

and corresponding equations for the induced fields $\mathbf{E}^i(\mathbf{x},t)$ and $\mathbf{B}^i(\mathbf{x},t)$, where

$$\begin{aligned}\Delta\mathbf{k}&\equiv\mathbf{k}_1-\mathbf{k}_2,\quad \Delta\omega\equiv\omega_1-\omega_2\,,\\ \mathbf{K}&\equiv\mathbf{k}_1+\mathbf{k}_2,\quad \Omega\equiv\omega_1+\omega_2\,.\end{aligned}$$

Substituting (A3) into (A2), we can easily find that the first-order perturbation of the distribution function is given by

$$f(\mathbf{k}_j,\mathbf{v},\omega_j)=-i\frac{e}{m}\frac{[\mathbf{E}_j{}^{\text{ext}}+\mathbf{E}^i(\mathbf{k}_j,\omega_j)]\cdot(\partial f_0/\partial\mathbf{v})}{\mathbf{k}_j\cdot\mathbf{v}-\omega_j}\,,\quad j=1,\,2\,,\tag{A4}$$

where

$$\mathbf{E}_j{}^{\text{ext}}+\mathbf{E}^i(\mathbf{k}_j,\omega_j)=\mathbf{E}^{(1)}(\mathbf{k}_j,\omega_j)$$

is the total electric field to the first order, representing the linearly refracted field in the medium or the linearly transmitted and the reflected field in the vacuum. The second-order perturbation of the distribution function is given by

$$\begin{aligned}f(\Delta\mathbf{k},\mathbf{v},\Delta\omega)=-\frac{ie}{m}\frac{1}{\Delta\mathbf{k}\cdot\mathbf{v}-\Delta\omega}\bigg\{&[\mathbf{E}^{(1)}(\mathbf{k}_1,\omega_1)+\mathbf{k}\times\mathbf{B}^{(1)}(\mathbf{k}_1,\omega_1)]\cdot\frac{\partial f}{\partial\mathbf{v}}(-k_2,\,\mathbf{v},\,-\omega_2)\\ &+[\mathbf{E}^{(1)}(-\mathbf{k}_2,\,-\omega_2)+\mathbf{v}\times\mathbf{B}^{(1)}(-\mathbf{k}_2,\,-\omega_2)]\cdot\frac{\partial f}{\partial\mathbf{v}}(\mathbf{k}_1,\mathbf{v},\omega_1)+\mathbf{E}^i(\Delta\mathbf{k},\Delta\omega)\cdot\frac{\partial f_0}{\partial\mathbf{v}}\bigg\}\,,\end{aligned}\tag{A5}$$

with similar equations for the other second-order distribution functions. Corresponding to $f(\Delta\mathbf{k},\mathbf{p},\Delta\omega)$, the second-order current density is given by

$$\mathbf{J}(\Delta\mathbf{k},\Delta\omega)=-n_0e\int f(\Delta\mathbf{k},\mathbf{v},\Delta\omega)\mathbf{v}\,d\mathbf{v}=\mathbf{J}_A(\Delta\mathbf{k},\Delta\omega)+\mathbf{J}_B(\Delta\mathbf{k},\Delta\omega)\tag{A6}$$

where

$$\begin{aligned}\mathbf{J}_A\equiv i\epsilon_0\omega_p{}^2\int d\mathbf{v}\,\mathbf{v}&\frac{\mathbf{E}^{(1)}(\mathbf{k}_1,\omega_1)+\mathbf{v}\times\mathbf{B}^{(1)}(\mathbf{k}_1,\omega_1)}{\Delta\mathbf{k}\cdot\mathbf{v}-\Delta\omega}\cdot\frac{\partial f}{\partial\mathbf{v}}(-\mathbf{k}_2,\,\mathbf{v},\,-\omega_2)\\ &+\int d\mathbf{v}\,\mathbf{v}\frac{\mathbf{E}^{(1)}(-\mathbf{k}_2,\,-\omega_2)+\mathbf{v}\times\mathbf{B}^{(1)}(-\mathbf{k}_2,\,-\omega_2)}{\Delta\mathbf{k}\cdot\mathbf{v}-\Delta\omega}\cdot\frac{\partial f}{\partial\mathbf{v}}(\mathbf{k}_1,\mathbf{v},\omega_1)\,,\end{aligned}\tag{A7}$$

$$\mathbf{J}_B\equiv i\epsilon_0\omega_p{}^2\int d\mathbf{v}\,\mathbf{v}\frac{\mathbf{E}^i(\Delta\mathbf{k},\Delta\omega)\cdot\partial f_0/\partial\mathbf{v}}{\Delta\mathbf{k}\cdot\mathbf{v}-\Delta\omega}=\frac{i\epsilon_0\omega_p{}^2}{\Delta\omega}\mathbf{E}^i(\Delta\mathbf{k},\Delta\omega)\quad\text{for}\quad \Delta k v_{\text{thermal}}\ll\Delta\omega\,.\tag{A8}$$

For a cold plasma, $\epsilon(\omega)=1-\omega_p^2/\omega^2$, Eq. (A8) reduces to

$$\mathbf{J}_B(\mathbf{x},t)=\mathbf{J}_B(\Delta\mathbf{k},\Delta\omega)e^{i(\Delta\mathbf{k}\cdot\mathbf{x}-\Delta\omega t)}$$

$$=\epsilon_0[\epsilon(\Delta\omega)-1]\frac{\partial}{\partial t}[\mathbf{E}^i(\Delta\mathbf{k},\Delta\omega)e^{i(\Delta\mathbf{k}\cdot\mathbf{x}-\Delta\omega t)}] \quad \text{(A9)}$$

To calculate $\mathbf{J}_A(\Delta\mathbf{k},\Delta\omega)$, we shall assume $\Delta k v_{\text{th}}\ll\Delta\omega$, $v_{\text{th}}\ll c$, v_{th} being the thermal velocity of the electrons. It is then straightforward to show that

$$\int d\mathbf{v}\ \mathbf{v}\frac{\mathbf{E}^{(1)}(\mathbf{k}_1,\omega_1)\cdot\partial f/\partial\mathbf{v}(-\mathbf{k}_2,\ \mathbf{v},\ -\omega_2)}{\Delta\mathbf{k}\cdot\mathbf{v}-\Delta\omega}\simeq\frac{ie}{m\omega_2(\Delta\omega)^2}$$
$$\times\{[\mathbf{E}^{(1)}(\mathbf{k}_1,\omega_1)\cdot\Delta\mathbf{k}]\mathbf{E}^{(1)}(-\mathbf{k}_2,\ -\omega_2)$$
$$+[\mathbf{E}^{(1)}(-\mathbf{k}_2,\ -\omega_2)\cdot\Delta\mathbf{k}]\mathbf{E}^{(1)}(\mathbf{k}_1,\omega_1)]\}, \quad \text{(A10)}$$

and

$$\int d\mathbf{v}\ \mathbf{v}\frac{[\mathbf{v}\times\mathbf{B}^{(1)}(\mathbf{k}_1,\omega_1)]\cdot\partial f/\partial\mathbf{v}(-\mathbf{k}_2,\ \mathbf{v},\ -\omega_2)}{\Delta\mathbf{k}\cdot\mathbf{v}-\Delta\omega}$$
$$\simeq\frac{ie}{m\omega_2\Delta\omega}\mathbf{E}^{(1)}(-\mathbf{k}_2,\ -\omega_2)\times\mathbf{B}^{(1)}(\mathbf{k}_1,\omega_1). \quad \text{(A11)}$$

On substituting Eqs. (A10), (A11) into (A7) and recalling that $B^{(1)}(\mathbf{k}_i,\omega_i)=\mathbf{k}_i\times\mathbf{E}^{(1)}(\mathbf{k}_i,\omega_i)/\omega_i$, we obtain after some algebra

$$\mathbf{J}_A(\Delta\mathbf{k},\Delta\omega)=-\epsilon_0\frac{e}{m}\frac{\omega_p^2}{\omega_1\omega_2}\mathbf{E}^{(1)}(\mathbf{k}_1,\omega_1)$$
$$\cdot\mathbf{E}^{(1)}(-k_2,\ -\omega_2)\frac{\Delta\mathbf{k}}{\Delta\omega}, \quad \text{(A12)}$$

which is equivalent to Eqs. (1) and (2) of Section II. Corresponding to the currents in (A7), (A8), there is the charge density

$$\rho(\Delta\mathbf{k},\Delta\omega)=\rho_A(\Delta\mathbf{k},\Delta\omega)+\rho_B(\Delta\mathbf{k},\Delta\omega),$$

where

$$\rho_A(\Delta\mathbf{k},\Delta\omega)=\frac{\Delta\mathbf{k}\cdot\mathbf{J}_A(\Delta\mathbf{k},\Delta\omega)}{\Delta\omega},$$

$$\rho_B(\Delta\mathbf{k},\Delta\omega)=\frac{\Delta\mathbf{k}\cdot\mathbf{J}_B(\Delta\mathbf{k},\Delta\omega)}{\Delta\omega}$$
$$=\epsilon_0[1-\epsilon(\Delta\omega)]i\Delta\mathbf{k}\cdot\mathbf{E}^i(\Delta\mathbf{k},\Delta\omega) \quad \text{(A13)}$$

or

$$\rho_B(\mathbf{x},t)=\epsilon_0[1-\epsilon(\Delta\omega)]\nabla\cdot\mathbf{E}^i(\mathbf{x},t). \quad \text{(A14)}$$

The Maxwell equations for the second-order perturbed fields (we write for the $\Delta\mathbf{k}$, $\Delta\omega$ component as an example; equations for the other components of the second order fields can be written down in exactly the same way) are

$$\nabla\cdot\mathbf{E}^i(\mathbf{x},t)=\frac{\rho_A(\mathbf{x},t)+\rho_B(\mathbf{x},t)}{\epsilon_0}$$

or

$$\nabla\cdot\mathbf{E}^i(\mathbf{x},t)=\frac{\rho_A(\mathbf{x},t)}{\epsilon_0\epsilon(\Delta\omega)} \quad \text{(A15)}$$

by using (A14); and

$$\epsilon_0c^2\nabla\times\mathbf{B}^i(\mathbf{x},t)=\epsilon_0\frac{\partial\mathbf{E}^i}{\partial t}(\mathbf{x},t)+[\mathbf{J}_A(\mathbf{x},t)+\mathbf{J}_B(\mathbf{x},t)]$$

or

$$c^2\nabla\times\mathbf{B}^i(\mathbf{x},t)=\epsilon(\Delta\omega)\frac{\partial\mathbf{E}^i}{\partial t}(\mathbf{x},t)+\frac{1}{\epsilon_0}\mathbf{J}_A(\mathbf{x},t) \quad \text{(A16)}$$

by using (A9). Thus, the influence of the incident waves (A1) on the plasma as far as second order fields are concerned is equivalent to that of the current $\mathbf{J}_A(\mathbf{x},t)$ placed inside the plasma with $\epsilon(\Delta\omega)$ as its dielectric constant. Alternatively, one can also say that the waves (A1) is equivalent to a longitudinal electric field

$$\mathbf{E}^{\text{long}}(\mathbf{x},t)=-\nabla\Phi(\mathbf{x},t),$$

where

$$\Phi(\mathbf{x},t)=-\frac{e\mathbf{E}^{(1)}(\mathbf{k}_1,\omega_1)\cdot\mathbf{E}^{(1)}(-\mathbf{k}_2,\ -\omega_2)}{m\omega_1\omega_2}e^{i(\Delta\mathbf{k}\cdot\mathbf{x}-\Delta\omega t)}.$$