High-Energy Deuteron Cross Sections: Charge-Exchange Effects*

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The effects of charge exchange in collisions of high-energy particles of isotopic spin $\frac{1}{2}$ with deuterons are investigated. The diffraction approximation is used to express the deuteron charge-exchange cross section in terms of the elastic-scattering amplitudes of the neutron and proton and the deuteron ground-state wave function. Examinations are also made of the influence of the charge-exchange mechanism on the total cross section of the deuteron, the elastic differential cross section, and the summed angular distribution of scattering (elastic plus inelastic scattering with initial charges preserved). The important role played by double-scattering processes in shaping the differential cross sections of the deuteron is illustrated in a discussion of proton-deuteron elastic scattering at 2 BeV. Double scattering is shown to be the dominant collision mechanism at scattering angles which are not too close to the forward direction. The effect of the double charge-exchange process on the elastic differential cross section is shown to be small but of some significance in the angular range which is dominated by double scattering. The theory developed for charge-exchange reactions is applied to the K^0 angular distribution which has been observed from K^+-deuteron collisions at 2.27 BeV/c. Estimates are found for the differential and integrated K^+-neutron charge-exchange cross sections, and for some related parameters. The effects of double collisions on these cross sections are relatively small and easily evaluated. The effect of the charge-exchange mechanism on the values of the pn, $\bar{p}n$, and $K^{\pm}n$ total cross sections which are reached indirectly via measurements on the deuteron is shown to be exceedingly small for momenta above $\sim$2 BeV/c.

I. INTRODUCTION

SINCE most varieties of elementary particles are both scarce and short lived, no direct method is available for measuring their cross sections for collisions with neutrons. An alternative procedure which has been widely applied at high energies involves measuring deuteron cross sections and subtracting from them the measured cross sections of the free proton. It has been important in applying this method to recognize that the deuteron cross section may deviate appreciably from the sum of the free neutron and proton cross sections. The principal source of this deviation lies in double-collision processes and shadowing effects to which the incident particle is subject when passing through the two-nucleon system. We have given a detailed discussion of these effects in an earlier paper[1] in which we have assumed that the charges of all particles remain fixed during the collisions.

It is clear that double-collision and shadowing effects may play an analogous role in the determination of charge-exchange cross sections of the neutron from the corresponding measurements made upon the deuteron. We shall therefore extend the analysis of the previous paper to include the effects of charge exchange. Since scattering takes place predominantly near the forward direction at high energies whether charge is exchanged or not, the dynamical considerations presented in I remain essentially unaltered. All of the approximations introduced earlier apply equally well when the charge state of the incident particle is changed in the collision process.

Charge-exchange processes also have a certain bearing on deuteron cross sections for collisions in which there is no net transfer of charge. This effect, which has been noted by Wilkin,[2] is due to a pair of successive collisions in which two cancelling charge exchanges occur. For charge-independent particle interactions, the effect may be seen to depend quadratically on the difference of the elastic-scattering amplitudes of the neutron and proton. It tends therefore to be a small effect relative to the other cross-section corrections, as we shall show explicitly for the cases of incident nucleons, antinucleons and K mesons.

The kinds of collision processes which lead to charge-exchange scattering by the deuteron are illustrated schematically in Fig. 1 for an incident positive particle which is the $I_3=\frac{1}{2}$ member of a charge doublet. Only one type of single-collision charge-exchange process can take place, that shown by Fig. 1(a), but two varieties of double-collision processes may occur as shown by Figs. 1(b) and 1(c).[3] All three processes lead to identical

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[1] V. Franco and R. J. Glauber, Phys. Rev. **142**, 1195 (1966). We shall refer to this paper as I. Earlier papers on the cross-section correction are: R. J. Glauber, Phys. Rev. **100**, 242 (1955); in *Proceedings of the Conference on Nuclear Forces and the Few-Nucleon Problem*, edited by T. C. Griffith and E. A. Power (Pergamon Press, Inc., London, 1960), Vol. I, p. 233; in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Interscience Publishers, Inc., New York, 1959), Vol. I, p. 315; V. Franco, Ph.D. thesis, Harvard University, 1963 (unpublished); D. R. Harrington, Phys. Rev. **135**, B358 (1964); **137** AB3(E) (1965).

[2] C. Wilkin, Phys. Rev. Letters **17**, 561 (1966).

[3] Triple- and higher-order multiple collision processes have negligibly small amplitudes when the scattering takes place predominantly at small angles.

final states and are experimentally indistinguishable; the amplitudes they contribute add together coherently. In the place of the initial deuteron the charge-exchange process leaves either two protons or two neutrons, and the states accessible to these particles are significantly restricted by the exclusion principle. In particular, since the final nucleons cannot remain bound, charge-exchange collisions with the deuteron must be inelastic in character.

The types of collision processes which may take place with no net transfer of charge are illustrated in Fig. 2. The charges of all particles remain fixed in the two single-collision processes shown in Figs. 2(a) and 2(b) and the two double-collision processes indicated in Figs. 2(c) and 2(d). Shown in Fig. 2(e) is the additional process, mentioned earlier, which is introduced by allowing for the possibility of two compensating charge exchanges. The amplitudes for all five processes add coherently and contribute both to elastic and inelastic scattering by the deuteron.

We derive expressions for the various deuteron cross sections in terms of charge-independent scattering amplitudes in Sec. II and express them in terms of the more directly observable neutron and proton scattering amplitudes in Sec. III. The angular distribution of elastic proton-deuteron scattering is discussed in Sec. IV. The theory suggests that the scattering observed at angles which are not too close to the forward direction is contributed predominantly by double-collision processes. We discuss the effect of charge exchange on this angular distribution and note that a recent calculation[4] of the distribution is in approximate agreement with its observed form.[5] In particular the shape of the observed angular distribution seems to give

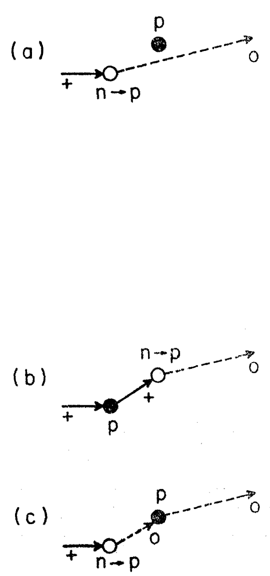

FIG. 1. Schematic representation of collision processes which contribute to charge-exchange scattering by the deuteron. The incident positively charged particle is assumed to be the $I_3=\frac{1}{2}$ member of a charge doublet.

[4] V. Franco and E. Coleman, Phys. Rev. Letters **17**, 827 (1966).
[5] E. Coleman, R. M. Heinz, O. E. Overseth, and D. E. Pellet, Phys. Rev. Letters **16**, 761 (1966).

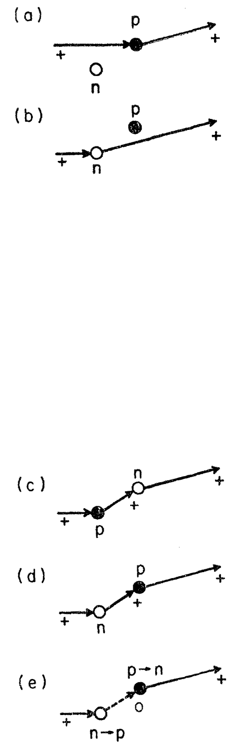

FIG. 2. Schematic representation of collision processes which contribute to scattering by the deuteron with no net transfer of charge. The incident particle is again the $I_3=\frac{1}{2}$ member of a charge doublet.

fairly direct evidence for the detection of double-collision processes.

In Sec. V we illustrate the way in which our results can be used to analyze charge-exchange processes by considering the reaction $K^+ + d \rightarrow K^0 + p + p$. We derive the differential cross section for K^+-neutron charge exchange scattering and several related parameters from the data of Butterworth *et al.*[6] Finally, we indicate in Sec. VI the quantitative effect which charge-exchange scattering has on the total cross section of the deuteron for collisions with incident protons, antiprotons, and K mesons.

II. FORMULATION OF CROSS SECTIONS

The high-energy approximation technique we use permits us to express the amplitudes and cross sections for various kinds of scattering by the deuteron directly in terms of the individual amplitudes for scattering by the neutron and proton. We shall assume that the particles of the incident beam have isotopic spin $\frac{1}{2}$ without specifying their nature further. They may thus be nucleons, antinucleons, K mesons, etc. We shall furthermore assume, for simplicity, that all of the interactions considered are precisely charge-independent and omit the effects of spin dependence (some of which are discussed in detail in I).

[6] I. Butterworth, J. L. Brown, G. Goldhaber, S. Goldhaber, A. A. Hirata, J. A. Kadyk, B. M. Schwarzschild, and G. H. Trilling, Phys. Rev. Letters **15**, 734 (1965); also B. M. Schwarzschild (private communication).

In the experiments we are considering, incident particles of either of two charge states ($I_3=\pm\frac{1}{2}$) collide with neutrons and protons. The scattering amplitudes which describe such collisions may be dealt with most compactly by defining an isotopic spin-dependent scattering amplitude operator. With the assumptions noted earlier we may write the amplitude operator for scattering by a nucleon labeled by the index 1 in the general form

$$a_1(\mathbf{q})=f(\mathbf{q})+\boldsymbol{\tau}\cdot\boldsymbol{\tau}_1 g(\mathbf{q}), \tag{2.1}$$

where $\hbar\mathbf{q}$ is the momentum transferred to the target nucleon, and $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_1$ are the isotopic spin operators for the incident particle and target nucleon, respectively. We shall always take the functions f and g to have the values characteristic of the laboratory system, which is the initial rest frame of the nucleon.

If we take the initial state of the incident particle and deuteron to be $|i\rangle$ and the final state to be $|f\rangle$, we may write the corresponding amplitude for scattering by the deuteron as $F_{fi}(\mathbf{q})$. In the diffraction-type approximation which we use, the amplitude $F_{fi}(\mathbf{q})$ may be regarded as the matrix element of an operator F which contains terms which are linear or bilinear in the scattering amplitude operators a_j, i.e., we may write

$$F_{fi}(\mathbf{q})=\langle f|F(\mathbf{q},\mathbf{r},\boldsymbol{\tau},\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)|i\rangle, \tag{2.2}$$

where the indices 1 and 2 label the two nucleons of the deuteron and $\mathbf{r}$ is the internal coordinate of the deuteron, $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$. The operator F which we require is an immediate generalization of the one derived in I, Eq. (3.6), for the case in which charge exchange is neglected. The only change necessary stems from the fact that the scattering amplitudes are now represented by operators rather than c numbers. Since a_1 and a_2 do not commute in general, the order in which they occur is an important feature of the term which describes double scattering. To express the operator F most conveniently, we separate the internal deuteron coordinate $\mathbf{r}$ into two orthogonal components. We let z be the component of $\mathbf{r}$ in the direction of the propagation vector $\mathbf{k}$ for the incident particle and let $\mathbf{s}$ be the component of $\mathbf{r}$ in the plane perpendicular to $\mathbf{k}$. The operator F is then given by

$$F(\mathbf{q},\mathbf{r},\boldsymbol{\tau},\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)=a_1(\mathbf{q})e^{i\frac{1}{2}\mathbf{q}\cdot\mathbf{s}}+a_2(\mathbf{q})e^{-i\frac{1}{2}\mathbf{q}\cdot\mathbf{s}} +\frac{i}{2\pi k}\int e^{i\mathbf{q}'\cdot\mathbf{s}}[a_1(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')a_2(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')\theta_+(z) +a_2(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')a_1(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')\theta_-(z)]d^{(2)}\mathbf{q}', \tag{2.3}$$

in which the integral over $d^{(2)}\mathbf{q}'$ is an integration over the plane of momentum transfers perpendicular to $\mathbf{k}$, and the functions $\theta_\pm(z)$ are defined by

$$\begin{aligned}\theta_+(z)&=1 \quad \text{for} \quad z>0\\ &=0 \quad \text{for} \quad z<0\\ \theta_-(z)&=1-\theta_+(z).\end{aligned} \tag{2.4}$$

The first two terms of F describe single-scattering processes by each of the two nucleons. The third term describes the effect of double interactions. Since our basic approximation assumes that scattering processes take place predominantly through small angles, the order in which the two interactions take place is determined by the instantaneous configuration of the deuteron. For $z>0$, the incident particle must first strike nucleon 2 and then 1. This requirement explains the presence of the function $\theta_+(z)$ in the double-scattering term of Eq. (2.3). The function $\theta_-(z)$ plays a corresponding role for the reverse ordering.

An alternative way of writing the ordered operator product which occurs in Eq. (2.3) is based on the identity

$$\theta_\pm(z)=\tfrac{1}{2}[1\pm\epsilon(z)], \tag{2.5}$$

where

$$\epsilon(z)=z/|z|. \tag{2.6}$$

By substituting the identity in Eq. (2.3) we see that the double-scattering term may also be expressed as

$$\frac{i}{2\pi k}\int e^{i\mathbf{q}'\cdot\mathbf{s}}\tfrac{1}{2}[\{a_1(\tfrac{1}{2}\mathbf{q}+\mathbf{q}'),\, a_2(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')\} +[a_1(\tfrac{1}{2}\mathbf{q}+\mathbf{q}'),\, a_2(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')]\epsilon(z)]d^{(2)}\mathbf{q}', \tag{2.7}$$

in which the brackets [,] designate the commutator and { , } the anticommutator.[7]

The charge dependence of the operator F can be made more evident by writing it in the form

$$F(\mathbf{q},\mathbf{r},\boldsymbol{\tau},\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)=A(\mathbf{q},\mathbf{s}) +B_1(\mathbf{q},\mathbf{s})\boldsymbol{\tau}\cdot\boldsymbol{\tau}_1+B_2(\mathbf{q},\mathbf{s})\boldsymbol{\tau}\cdot\boldsymbol{\tau}_2 +C(\mathbf{q},\mathbf{s})[\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2+i\boldsymbol{\tau}\cdot(\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2)\epsilon(z)], \tag{2.8}$$

where the functions A, B_1, B_2, and C are defined as

$$A(\mathbf{q},\mathbf{s})=2f(\mathbf{q})\cos(\tfrac{1}{2}\mathbf{q}\cdot\mathbf{s})+\frac{i}{2\pi k}\int\exp(i\mathbf{q}'\cdot\mathbf{s}) \times f(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')d^{(2)}\mathbf{q}', \tag{2.9}$$

$$B_1(\mathbf{q},\mathbf{s})=g(\mathbf{q})\exp(i\tfrac{1}{2}\mathbf{q}\cdot\mathbf{s})+\frac{i}{2\pi k}\int\exp(i\mathbf{q}'\cdot\mathbf{s}) \times g(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')d^{(2)}\mathbf{q}', \tag{2.10}$$

$$B_2(\mathbf{q},\mathbf{s})=g(\mathbf{q})\exp(-i\tfrac{1}{2}\mathbf{q}\cdot\mathbf{s})+\frac{i}{2\pi k}\int\exp(i\mathbf{q}'\cdot\mathbf{s}) \times f(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')g(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')d^{(2)}\mathbf{q}' \tag{2.11}$$

$$=B_1(\mathbf{q},-\mathbf{s}),$$

[7] In the analysis of the effects of spin dependence on the deuteron elastic-scattering amplitude and total cross sections presented in I the term analogous to the latter of the two in the expression (2.7) has been dropped. This term, which is proportional to $\epsilon(z)$, has no effect on either the elastic-scattering amplitude or the total cross section. It does contribute, however, to the cross sections for breakup of the deuteron and must be included when these are calculated.

$$C(\mathbf{q},\mathbf{s})=\frac{i}{2\pi k}\int \exp(i\mathbf{q}'\cdot\mathbf{s})\times g(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')g(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')d^{(2)}\mathbf{q}'. \quad (2.12)$$

The amplitude for elastic scattering by the deuteron is obtained by finding the diagonal matrix element of the operator F in the state $|i\rangle$. For this purpose, let us write the deuteron ground-state wave function as $\phi(\mathbf{r})$ and introduce the deuteron form factor $S(\mathbf{q})$ which is defined as

$$S(\mathbf{q})=\int e^{i\mathbf{q}\cdot\mathbf{r}}|\phi(\mathbf{r})|^2 d\mathbf{r} = S(-\mathbf{q}). \quad (2.13)$$

Since the deuteron ground state is an isotopic singlet state, the diagonal matrix element of F evidently reduces to

$$F_{ii}(\mathbf{q})=\langle i|A-3C|i\rangle, \quad (2.14)$$

which may be written more explicitly as

$$F_{ii}(\mathbf{q})=2f(\mathbf{q})S(\tfrac{1}{2}\mathbf{q})+\frac{i}{2\pi k}\int S(\mathbf{q}')h(\mathbf{q},\mathbf{q}')d^{(2)}\mathbf{q}', \quad (2.15)$$

where $h(\mathbf{q},\mathbf{q}')$ is the function

$$h(\mathbf{q},\mathbf{q}')=f(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f(\tfrac{1}{2}\mathbf{q}-\mathbf{q}') -3g(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')g(\tfrac{1}{2}\mathbf{q}-\mathbf{q}'). \quad (2.16)$$

The total cross section of the deuteron, σ_d, is easily found from the elastic forward scattering amplitude by means of the optical theorem,

$$\sigma_d=(4\pi/k)\,\mathrm{Im}F_{ii}(0). \quad (2.17)$$

To evaluate this expression, we note that the average of the neutron and proton total cross sections, σ_n and σ_p, is given similarly by

$$\tfrac{1}{2}(\sigma_n+\sigma_p)=(4\pi/k)\,\mathrm{Im}f(0). \quad (2.18)$$

We may therefore write the deuteron total cross section in the form

$$\sigma_d=\sigma_n+\sigma_p-\delta\sigma, \quad (2.19)$$

where the cross section defect $\delta\sigma$ is given by

$$\delta\sigma=-\frac{2}{k^2}\,\mathrm{Re}\int S(\mathbf{q}')h(0,\mathbf{q}')d^{(2)}\mathbf{q}'. \quad (2.20)$$

We shall look further into the structure of this term in Sec. III.

The differential cross section for elastic scattering is obtained by squaring the modulus of $F_{ii}(\mathbf{q})$. In this way we find

$$(d\sigma/d\Omega)_{\mathrm{el}}=4|f(\mathbf{q})|^2S^2(\tfrac{1}{2}\mathbf{q}) -\frac{2}{\pi k}S(\tfrac{1}{2}\mathbf{q})\,\mathrm{Im}\left[f^*(\mathbf{q})\int S(\mathbf{q}')h(\mathbf{q},\mathbf{q}')d^{(2)}\mathbf{q}'\right] +\frac{1}{(2\pi k)^2}\left|\int S(\mathbf{q}')h(\mathbf{q},\mathbf{q}')d^{(2)}\mathbf{q}'\right|^2. \quad (2.21)$$

The total angular distribution of scattering, which includes inelastically scattered particles as well as elastically scattered ones, can be found by summing $|F_{fi}(\mathbf{q})|^2$ over a complete set of possible final states f. Since the final states f include all possible charge states of the particles considered, the angular distribution we find in this way is the sum of the differential cross section for charge-exchange scattering, $(d\sigma/d\Omega)_{\mathrm{ex}}$, and the differential cross section for scattering with charges preserved, $(d\sigma/d\Omega)_{\mathrm{sc}}$. We thus have

$$(d\sigma/d\Omega)_{\mathrm{ex}}+(d\sigma/d\Omega)_{\mathrm{sc}}=\textstyle\sum_f |F_{fi}(\mathbf{q})|^2 =\langle i|F^\dagger(\mathbf{q})F(\mathbf{q})|i\rangle. \quad (2.22)$$

If we substitute the form (2.8) for $F(\mathbf{q})$ in this expression and evaluate the expectation values of the isotopic spin operators which occur in it, we find

$$(d\sigma/d\Omega)_{\mathrm{ex}}+(d\sigma/d\Omega)_{\mathrm{sc}} =\langle i||A-3C|^2+3|B_1-B_2|^2+12|C|^2|i\rangle. \quad (2.23)$$

To separate the contributions of the charge-exchange and charge-retention scattering we must consider the corresponding varieties of final states separately. The deuteron is initially in an isotopic singlet state and therefore the third component of its isotopic spin $I_3=\frac{1}{2}(\tau_{13}+\tau_{23})$, has initially the eigenvalue zero. If the incident particle exchanges one unit of charge with the two nucleons, their final value of I_3 must be ± 1. A projection operator corresponding to those states with $I_3=\pm 1$ is clearly

$$\tfrac{1}{4}(\tau_{13}+\tau_{23})^2=\tfrac{1}{2}(1+\tau_{13}\tau_{23}). \quad (2.24)$$

Hence the differential cross section for charge exchange scattering alone is

$$(d\sigma/d\Omega)_{\mathrm{ex}}=\tfrac{1}{2}\langle i|F^\dagger(\mathbf{q})(1+\tau_{13}\tau_{23})F(\mathbf{q})|i\rangle \quad (2.25)$$

$$=2\langle i||B_1-B_2|^2+4|C|^2|i\rangle. \quad (2.26)$$

By subtracting these equations from Eqs. (2.22) and (2.23), we find the differential cross section for charge retention scattering

$$(d\sigma/d\Omega)_{\mathrm{sc}}=\tfrac{1}{2}\langle i|F^\dagger(\mathbf{q})(1-\tau_{13}\tau_{23})F(\mathbf{q})|i\rangle \quad (2.27)$$

$$=\langle i||A-3C|^2+|B_1-B_2|^2+4|C|^2|i\rangle. \quad (2.28)$$

When we express the charge exchange angular distribution in terms of the scattering amplitudes f

and g by means of Eqs. (2.10)–(2.12) and (2.13), we find

$$(d\sigma/d\Omega)_{\rm ex}=4|g(\mathbf{q})|^2[1-S(\mathbf{q})] -\frac{4}{\pi k}\,\mathrm{Im}\left[g^*(\mathbf{q})\int S(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')j(\mathbf{q},\mathbf{q}')d^{(2)}\mathbf{q}'\right] +\frac{2}{(2\pi k)^2}\int S(\mathbf{q}'-\mathbf{q}'')[j(\mathbf{q},\mathbf{q}')j^*(\mathbf{q},\mathbf{q}'') +m(\mathbf{q},\mathbf{q}')m^*(\mathbf{q},\mathbf{q}'')]d^{(2)}\mathbf{q}'d^{(2)}\mathbf{q}'', \quad (2.29)$$

where we have written

$$j(\mathbf{q},\mathbf{q}')=g(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f(\tfrac{1}{2}\mathbf{q}-\mathbf{q}') -f(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')g(\tfrac{1}{2}\mathbf{q}-\mathbf{q}') \quad (2.30)$$

and

$$m(\mathbf{q},\mathbf{q}')=2g(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')g(\tfrac{1}{2}\mathbf{q}-\mathbf{q}'). \quad (2.31)$$

The exclusion principle has an important bearing on the final states available for the charge-exchange reaction. When charge exchange has taken place, the two nucleons of the deuteron are left in identical charge states and can not remain in their original even-parity orbital state. Charge exchange thus can not be an elastic-scattering process, and, in particular, it can not take place in collisions with zero momentum transfer. This conclusion is not altered by the introduction of spin variables since the only bound state of the two-nucleon system is the deuteron ground state.

The first term of the charge-exchange angular distribution (2.29) clearly vanishes as $\mathbf{q}\rightarrow 0$ since $S(0)=1$. Furthermore, two of the remaining terms also vanish in the forward direction since $j(0,\mathbf{q}')=0$. The double-scattering term which contains the function $m(\mathbf{q},\mathbf{q}')m^*(\mathbf{q},\mathbf{q}'')$, however, does not vanish for $\mathbf{q}=0$, and the cross section $(d\sigma/d\Omega)_{\rm ex}$ therefore does take on a nonvanishing value in the forward direction. The apparent contradiction between this statement and the requirement of the exclusion principle is easily resolved.

The term containing the function m in Eq. (2.29) has its origin in the term proportional to $\epsilon(z)$ in the expression for the scattering amplitude operator F given by Eq. (2.8). Since the expectation value of $\epsilon(z)=z/|z|$ vanishes in the ground state of the deuteron, it is clear that this term contributes only to inelastic scattering; in particular, the momentum transferred to the deuteron must have a longitudinal component if the matrix element for scattering is not to vanish. The approximations we have made in summing over final states, on the other hand, have assumed the energy transfer to be negligible and have thus omitted the longitudinal component of the momentum transfer. The charge-exchange scattering which is described as having $\mathbf{q}=0$ is, therefore, scattering in the forward direction which is slightly inelastic; the transverse component of its momentum transfer vanishes but a small longitudinal component remains.

The differential cross section $(d\sigma/d\Omega)_{\rm sc}$ is the sum of the intensities of both elastically and inelastically scattered particles which retain their original charge state. We may use Eqs. (2.9)–(2.12) and (2.13) to write Eq. (2.28) in the form

$$(d\sigma/d\Omega)_{\rm sc}=2|f(\mathbf{q})|^2[1+S(\mathbf{q})]+2|g(\mathbf{q})|^2[1-S(\mathbf{q})] -\frac{2}{\pi k}\,\mathrm{Im}\left[f^*(\mathbf{q})\int S(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')h(\mathbf{q},\mathbf{q}')d^{(2)}\mathbf{q}'\right] -\frac{2}{\pi k}\,\mathrm{Im}\left[g^*(\mathbf{q})\int S(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')j(\mathbf{q},\mathbf{q}')d^{(2)}\mathbf{q}'\right] +\frac{1}{(2\pi k)^2}\int S(\mathbf{q}'-\mathbf{q}'')[h(\mathbf{q},\mathbf{q}')h^*(\mathbf{q},\mathbf{q}'') +j(\mathbf{q},\mathbf{q}')j^*(\mathbf{q},\mathbf{q}'')+m(\mathbf{q},\mathbf{q}')m^*(\mathbf{q},\mathbf{q}'')]d^{(2)}\mathbf{q}'d^{(2)}\mathbf{q}''. \quad (2.32)$$

The angular distribution which corresponds to inelastic scattering with charge retention, i.e., simple dissociation of the deuteron, is given by

$$(d\sigma/d\Omega)_{\rm dis}=(d\sigma/d\Omega)_{\rm sc}-(d\sigma/d\Omega)_{\rm el}, \quad (2.33)$$

which is found by subtracting Eq. (2.21) from Eq. (2.32).

The diffraction approximation, as we have noted earlier, assumes that all scattered intensities are heavily concentrated near the forward direction.[8] To find the integrated cross section, it is therefore sufficiently accurate to replace the differential element of solid angle $d\Omega$ by the nearly equivalent element $d^{(2)}\mathbf{q}/k^2$ and to integrate over a plane in momentum space perpendicular to the incident momentum $\hbar\mathbf{k}$. We may then write

$$\sigma_{\rm ex}=\int\left(\frac{d\sigma}{d\Omega}\right)_{\rm ex}\frac{d^{(2)}\mathbf{q}}{k^2}, \quad (2.34)$$

and similar expressions hold for the integrated cross sections for scattering and dissociation. The absorption cross section, which is the cross section for all incoherent processes including particle production, is

$$\sigma_{\rm abs}=\sigma_{\rm d}-\sigma_{\rm ex}-\sigma_{\rm sc}. \quad (2.35)$$

III. CROSS SECTIONS IN TERMS OF NEUTRON AND PROTON SCATTERING AMPLITUDES

In order to compare the cross sections we have derived with experimental results, it is convenient to

[8] Neutron-proton charge-exchange scattering is frequently described as backward scattering in the center-of-mass system. The definition of the scattering angle implicit in this terminology is not the convenient one, however, for use with the isotopic spin description of nucleon states. In the isotopic spin formalism we regard all nucleon-nucleon elastic scattering amplitudes as defined for scattering angles $\pi/2\geq\theta\geq 0$. Both charge-exchange and charge-retention scattering then tend to be concentrated near the forward direction at high energies.

express them in terms of the experimentally measured amplitudes for scattering by neutrons and protons. Let us assume for definiteness that the incident particle has the charge state $I_3=\frac{1}{2}$. Then no charge exchange can take place when the particle is scattered by a proton. The amplitude for elastic scattering by the proton, according to Eq. (2.1), is

$$f_p(\mathbf{q})=f(\mathbf{q})+g(\mathbf{q}). \tag{3.1}$$

The wave scattered by the neutron, on the other hand, contains a charge exchanged $I_3=-\frac{1}{2}$ component as well as one for $I_3=\frac{1}{2}$. To separate these components we may introduce the charge-exchange operator

$$P_1=\tfrac{1}{2}(1+\boldsymbol{\tau}\cdot\boldsymbol{\tau}_1) \tag{3.2}$$

and use it to write the scattering amplitude operator $a_1(\mathbf{q})$ of Eq. (2.1) in the form

$$a_1(\mathbf{q})=f(\mathbf{q})-g(\mathbf{q})+2g(\mathbf{q})P_1. \tag{3.3}$$

It is evident from this expression that the amplitude for scattering by the neutron without alteration of charge states is

$$f_n(\mathbf{q})=f(\mathbf{q})-g(\mathbf{q}), \tag{3.4}$$

while the charge exchange amplitude is

$$f_c(\mathbf{q})=2g(\mathbf{q}). \tag{3.5}$$

If the incident particle has $I_3=-\frac{1}{2}$, the same charge-exchange amplitude is associated with charge-exchange scattering by the proton while the signs of g are interchanged in the elastic-scattering amplitudes (3.1) and (3.4).

The amplitudes f and g which we have used to construct the cross sections are therefore related to the directly observable amplitudes f_p, f_n, and f_c via the three equations

$$f(\mathbf{q})=\tfrac{1}{2}[f_p(\mathbf{q})+f_n(\mathbf{q})], \tag{3.6}$$

$$g(\mathbf{q})=\tfrac{1}{2}(-1)^{\frac{1}{2}-I_3}[f_p(\mathbf{q})-f_n(\mathbf{q})], \tag{3.7}$$

$$g(\mathbf{q})=\tfrac{1}{2}f_c(\mathbf{q}). \tag{3.8}$$

The possibility of finding the amplitude g in two ways, either through direct measurement of the charge-exchange amplitude or by taking the difference of the proton and neutron elastic amplitudes, leads to a variety of useful ways of expressing our results.

We may, for example, write the cross-section defect of the deuteron, given by Eqs. (2.19) and (2.20), in terms of the observable amplitudes in the form[2]

$$\delta\sigma=-\frac{2}{k^2}\,\mathrm{Re}\int S(\mathbf{q})\tfrac{1}{2}[f_p(\mathbf{q})f_n(-\mathbf{q}) +f_n(\mathbf{q})f_p(-\mathbf{q})-f_c(\mathbf{q})f_c(-\mathbf{q})]d^{(2)}\mathbf{q}. \tag{3.9}$$

The three terms of this expression correspond to the three possible varieties of elastic double-collision processes. The first two terms correspond to charge-preserving collisions with both the neutron and proton in the two possible orders. The third term corresponds to double charge exchange. The two steps of the latter process only take place in a predetermined order, i.e., the incident particle can only exchange charge initially with one of the two nucleons. The sign of the double charge-exchange term, which is the term disregarded in the analysis of I, is negative since it corresponds to the exchange of two nucleons in an antisymmetric state.

We may alternatively write the cross-section defect completely in terms of the scattering amplitude for charge-preserving processes. If we note that these amplitudes can only depend on the magnitude, and not on the direction of $\mathbf{q}$, we may write

$$\delta\sigma=-\frac{2}{k^2}\,\mathrm{Re}\int S(q)[2f_n(q)f_p(q) -\tfrac{1}{2}f_p^2(q)-\tfrac{1}{2}f_n^2(q)]d^{(2)}\mathbf{q}. \tag{3.10}$$

If the range of the forces between the incident particle and the nucleons is small in comparison to the size of the deuteron, the scattering amplitudes $f_p(q)$ and $f_n(q)$ will tend to decrease in magnitude more slowly with increasing q than the deuteron form factor $S(q)$. The integral in Eq. (3.10) may then be approximated by evaluating the scattering amplitudes at $q=0$. The integral of the form factor which remains is seen to be

$$\int S(q)d^{(2)}\mathbf{q}=2\pi\langle r^{-2}\rangle_d, \tag{3.11}$$

where $\langle r^{-2}\rangle_d$ is the mean inverse squared neutron-proton separation in the deuteron ground state. Thus when the force range is small the cross section defect may be approximated as

$$\delta\sigma=-(4\pi/k^2)\,\mathrm{Re}\{f_n(0)f_p(0) -\tfrac{1}{2}[f_n(0)-f_p(0)]^2\}\langle r^{-2}\rangle_d. \tag{3.12}$$

If we make use of the optical theorem in the form

$$\sigma_i=(4\pi/k)\,\mathrm{Im}f_i(0),\quad i=n,\,p \tag{3.13}$$

where σ_n and σ_p are total cross sections, and define the ratios

$$\rho_i=\mathrm{Re}f_i(0)/\mathrm{Im}f_i(0),\quad i=n,\,p \tag{3.14}$$

of the real and imaginary parts of the scattering amplitudes, we may write the expression (3.12) in the form

$$\delta\sigma=(1/4\pi)\{(1-\rho_n\rho_p)\sigma_n\sigma_p -\tfrac{1}{2}[(\sigma_n-\sigma_p)^2-(\rho_n\sigma_n-\rho_p\sigma_p)^2]\}\langle r^{-2}\rangle_d. \tag{3.15}$$

If the scattering amplitudes are purely imaginary ($\rho_n=\rho_p=0$), the effect of the double charge-exchange correction is always to decrease the cross-section defect, i.e., to increase the total cross section of the deuteron. While the expression (3.15) makes the sign and magnitude of the cross-section defect clear, its use for detailed comparisons of cross sections requires some caution. In particular when the incident particles are nucleons or antinucleons the force range is not particularly small compared with the radius of the deuteron or the dimension $\{\langle r^{-2}\rangle_d\}^{-1/2}$. For such cases, as we have noted in I, Sec. IV, the cross-section defect can only be estimated accurately by returning to more general expressions for it, such as that in Eq. (3.10), and approximating the required integrals more closely.

The angular distribution of elastic scattering may be written in terms of the neutron and proton scattering amplitudes by substituting Eqs. (3.6)–(3.8) into Eq. (2.21). In this way we find the differential cross section

$$(d\sigma/d\Omega)_{\rm el}=\{|f_n(q)|^2+|f_p(q)|^2+2\,{\rm Re}[f_n^*(q)f_p(q)]\}S^2(\tfrac{1}{2}q)-\frac{1}{\pi k}S(\tfrac{1}{2}q)\,{\rm Im}[f_n^*(q)+f_p^*(q)]\int S(\mathbf{q}')$$
$$\times\tfrac{1}{2}[f_p(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_n(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')+f_n(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')-f_c(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_c(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')]d^{(2)}\mathbf{q}'$$
$$+\frac{1}{(2\pi k)^2}\left|\int S(\mathbf{q}')\tfrac{1}{2}[f_p(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_n(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')+f_n(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')-f_c(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_c(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')]d^{(2)}\mathbf{q}'\right|^2. \quad (3.16)$$

The first term of this expression represents the effects of single scattering processes. Within it, we see the squared amplitudes of the waves scattered individually by the neutron and proton and the interference term for these amplitudes. The last term is the squared amplitude for double-collision processes. The two charge-preserving processes of the type illustrated in Figs. 2(c) and 2(d), and the double-charge-exchange process illustrated in Fig. 2(e) all contribute coherently to the double-collision amplitude. The middle term of the expression (3.16) represents the interference of the amplitudes for single- and double-collision processes.

We may express the differential cross section for charge-exchange collisions in terms of the neutron and proton scattering amplitudes by substituting Eqs. (3.6)–(3.8) into Eqs. (2.29)–(2.31). We then find

$$(d\sigma/d\Omega)_{\rm ex}=|f_c(\mathbf{q})|^2[1-S(\mathbf{q})]$$
$$-\frac{2}{\pi k}{\rm Im}\left[f_c^*(\mathbf{q})\int S(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')j(\mathbf{q},\mathbf{q}')d^{(2)}\mathbf{q}'\right]$$
$$+\frac{2}{(2\pi k)^2}\int S(\mathbf{q}'-\mathbf{q}'')[j(\mathbf{q},\mathbf{q}')j^*(\mathbf{q},\mathbf{q}'')$$
$$+m(\mathbf{q},\mathbf{q}')m^*(\mathbf{q},\mathbf{q}'')]d^{(2)}\mathbf{q}'d^{(2)}\mathbf{q}'', \quad (3.17)$$

where we have written

$$j(\mathbf{q},\mathbf{q}')=\tfrac{1}{4}\{f_c(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')[f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')+f_n(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')]$$
$$-f_c(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')[f_p(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')+f_n(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')]\} \quad (3.18)$$

and

$$m(\mathbf{q},\mathbf{q}')=\tfrac{1}{2}f_c(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_c(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')$$
$$=\tfrac{1}{4}(-1)^{\frac{1}{2}-I_3}\{f_c(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')$$
$$\times[f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')-f_n(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')]+f_c(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')$$
$$\times[f_p(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')-f_n(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')]\}. \quad (3.19)$$

The first term of the expression (3.17) represents the effect of charge exchange in single collisions such as the one illustrated in Fig. 1(a). The last term of the cross section represents the squared amplitude for double-collision processes in which a single charge exchange takes place. Two different pairs of collision processes contribute coherently to this term. If the incident particle has $I_3=\frac{1}{2}$, the process illustrated in Fig. 1(b), in which the proton is struck before the neutron, contributes terms containing the product f_cf_p. On the other hand, the process in which the neutron is struck first, as in Fig. 1(c), contributes a different product of scattering amplitudes. The charge-exchange collision with the neutron is again represented by the amplitude f_c, but when the $I_3=-\frac{1}{2}$ particle which results collides subsequently with the proton, its amplitude for elastic scattering by the proton may be seen through charge reflection to be equal to f_n. The process shown in Fig. 1(c) thus contributes terms containing the product f_cf_n. The middle term of Eq. (3.17) represents the interference of the amplitudes for the single-collision and double-collision charge-exchange processes.

The scattering cross section $(d\sigma/d\Omega)_{\rm sc}$ has been defined as the summed angular distribution of particles scattered both elastically and inelastically in their original charge state. The form given by Eq. (2.32) for this cross section may be written in terms of the neutron and proton scattering amplitudes as

$$(d\sigma/d\Omega)_{\rm sc}=|f_n(q)|^2+|f_p(q)|^2+2S(q)\,{\rm Re}[f_n^*(q)f_p(q)]$$
$$-\frac{1}{\pi k}{\rm Im}\left\{[f_n^*(q)+f_p^*(q)]\int S(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')h(\mathbf{q},\mathbf{q}')d^{(2)}\mathbf{q}'\right\}-\frac{1}{\pi k}{\rm Im}\left\{f_c^*(\mathbf{q})\int S(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')j(\mathbf{q},\mathbf{q}')d^{(2)}\mathbf{q}'\right\}$$
$$+\frac{1}{(2\pi k)^2}\int S(\mathbf{q}'-\mathbf{q}'')[h(\mathbf{q},\mathbf{q}')h^*(\mathbf{q},\mathbf{q}'')+j(\mathbf{q},\mathbf{q}')j^*(\mathbf{q},\mathbf{q}'')+m(\mathbf{q},\mathbf{q}')m^*(\mathbf{q},\mathbf{q}'')]d^{(2)}\mathbf{q}'d^{(2)}\mathbf{q}'', \quad (3.20)$$

where we have made use of the definitions (3.18) and (3.19) for $j(\mathbf{q},\mathbf{q}')$ and $m(\mathbf{q},\mathbf{q}')$ and have also written

$$h(\mathbf{q},\mathbf{q}')=\tfrac{1}{2}[f_n(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')+f_p(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_n(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')-f_c(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_c(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')]. \quad (3.21)$$

The various terms of the cross section (3.20) have their origins in the same scattering processes as the corresponding terms of the elastic cross section given by Eq. (3.16).

IV. DOUBLE SCATTERING IN PROTON-DEUTERON COLLISIONS

We shall illustrate the results of some of the foregoing calculations by considering high-energy elastic scattering of protons by deuterons. Before discussing the specific effect of the double-charge-exchange process, however, it will be interesting to discuss in somewhat more general terms the over-all effect of double-collision processes.

The differential cross section for elastic scattering, as we have indicated in connection with Eqs. (2.21) and (3.16), contains three types of contributions. There are the terms connected with pure single-scattering processes, those connected with pure double-scattering processes, and the ones arising from the interference of single and double scatterings. For the case of proton-deuteron scattering, the angular distribution contributed by the single-scattering terms alone tends to decrease quite rapidly from its value near the forward direction as the angle of scattering increases. The pure double-scattering contribution is, of course, much smaller in magnitude near the forward direction, but it falls away from this value considerably more slowly as the scattering angle increases. While double scattering is relatively unimportant at small angles, therefore, its contribution becomes the dominant one for larger scattering angles. The angle which separates what we may call the single scattering and double scattering regions is roughly 12° for p-d scattering at 2 BeV.

The magnitude of the term contributed by the interference of single and double scattering tends naturally to be intermediate between the magnitudes of the pure single- and double-scattering terms. The importance of this term relative to the entire scattered intensity therefore tends to be greatest in the region of transition which separates the single- and double-scattering regions. The sign of the interference term significantly affects the shape of the differential cross section in this intermediate region. Let us recall that the amplitudes for high-energy elastic nucleon-proton scattering have been shown[9] to be predominantly imaginary near the forward direction, and that the charge exchange amplitudes are relatively small in magnitude. Then we may see from Eq. (3.16) that the sign of the interference term tends to be negative at small angles.[10] The destructive character of the interference term means that the single- and double-scattering regions tend to be separated by a dip in the elastic-scattering cross section. The different behavior of the three contributions to elastic scattering and the dominance of double scattering at the larger angles are illustrated for antiproton-deuteron scattering in Figs. 3 and 5 of I, in which these

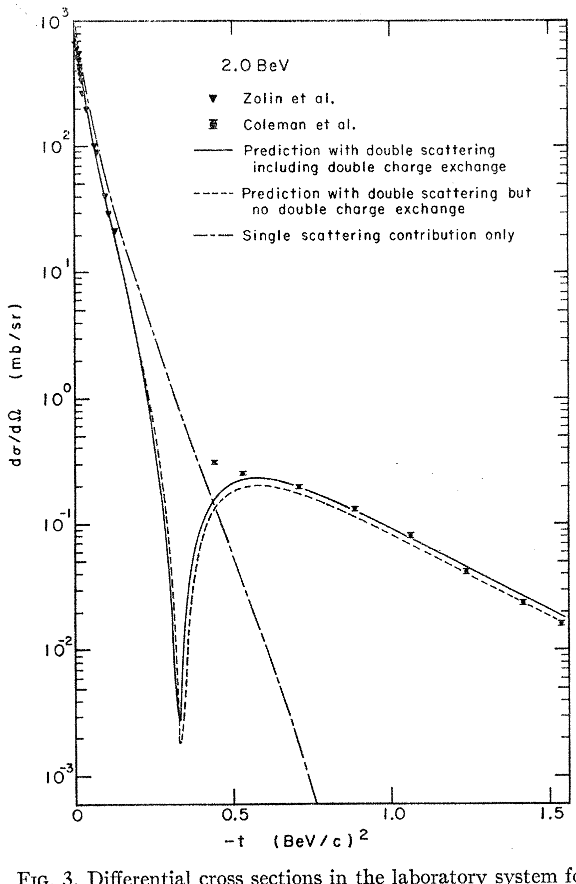

FIG. 3. Differential cross sections in the laboratory system for proton-deuteron elastic scattering at 2.0 BeV as a function of the negative squared four-momentum transfer $-t$. The solid curve is the theoretical prediction, using nucleon-nucleon data, with all double-collision processes, including double charge exchange, taken into account. The dashed curve is the theoretical prediction when double charge exchange is omitted. The dash-dot curve represents the contribution of single scattering processes alone. The experimental points are from Refs. 5 and 14. The shape of the interference minimum which occurs near $t=-0.33$ (BeV/c)^2 is quite sensitive to the phases of the nucleon-nucleon scattering amplitudes which are not yet known accurately.

[9] See, for example, G. Bellettini, G. Cocconi, A. N. Diddens, E. Lillethun, G. Matthiae, J. P. Scanlon, and A. M. Wetherell, Phys. Letters **19**, 341 (1965); G. Bellettini, G. Cocconi, A. N. Diddens, E. Lillethun, J. Pahl, J. P. Scanlon, J. Walters, A. M. Wetherell, and P. Zanella, *ibid.* **14**, 164 (1965). See also Refs. 12–14.

[10] The influence of the interference term on $(d\sigma/d\Omega)_{el}$ and $(d\sigma/d\Omega)_{sc}$ at small angles was calculated by V. Franco [Phys. Rev. Letters **16**, 944 (1966)] for proton-deuteron scattering between 2.8 and 19.3 BeV/c and comparisons were made with the measurements. Since the theory was applied to scattering at quite small angles, where the influence of the Coulomb interaction was appreciable, both the single- and double-scattering effects of the Coulomb field were treated.

properties of the angular distribution were first pointed out.

The differential cross section for p-d elastic scattering is given by Eq. (3.16). The charge-exchange amplitude $f_c(q)$ which occurs in the expression may be written in terms of the pp and pn elastic-scattering amplitudes, according to Eqs. (3.7) and (3.8), as

$$f_c(\mathbf{q})=f_p(\mathbf{q})-f_n(\mathbf{q}). \tag{4.1}$$

Following notation used elsewhere, we shall write the square of the four-momentum transferred in a collision as t. In the region $0\leq -t\lesssim 1.5$ (BeV/c)^2, which will concern us most, the nucleon-nucleon scattering amplitudes which have been measured are fairly well represented by expressions of the form

$$f_j=(i+\rho_j)(k\sigma_j/4\pi)e^{\frac{1}{2}at+\frac{1}{2}bt^2},\quad j=n,\,p \tag{4.2}$$

where σ_j is the proton-nucleon cross section and ρ_j is the ratio of the real part of the proton-nucleon forward-scattering amplitude to its imaginary part. Within the range of momentum transfers mentioned earlier, the magnitude of the amplitudes f_j decreases quite rapidly as the negative squared four-momentum transfer $-t$ increases. Since the parameter b is found to be positive, however, the magnitudes of the f_j will begin to increase without bound at some larger value of $-t$. Such unphysical behavior of the functions (4.2) must be excluded from the integrations over all momentum transfers $\mathbf{q}$ which occur in Eq. (3.16). Since collisions in which the arguments $(\frac{1}{2}\mathbf{q}\pm\mathbf{q}')^2$ exceed 2 (BeV/$\hbar c$)^2 may be shown to contribute negligibly to the integrals, we have applied a cutoff to the functions $f_j(\mathbf{q})$ at $q^2=2$ (BeV/$\hbar c$)$^2\equiv -t_0\hbar^{-2}$.

By making use of the truncated forms of the amplitudes f_j and introducing units in which $\hbar=1$, we may write the elastic differential cross section given by Eq. (3.16) in the form

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm el}=\left|\left(\frac{k}{4\pi}\right)S(\tfrac{1}{2}\sqrt{-t})e^{\frac{1}{2}at+\frac{1}{2}bt^2}[(i+\rho_n)\sigma_n+(i+\rho_p)\sigma_p]\right.$$
$$+\frac{iDk}{16\pi^3}e^{\frac{1}{4}at+\frac{1}{16}bt^2}\int_0^\infty S(\sqrt{t'})e^{-at'+bt'^2-btt'}dt'$$
$$\left.\times\int_0^{\frac{1}{2}\pi}e^{-\frac{1}{2}btt'\cos 2\phi}\,\theta_+[t_0-t'+\tfrac{1}{4}t-(-tt')^{1/2}\cos\phi]d\phi\right|^2, \tag{4.3}$$

where $\theta_+(t)$ is the unit step function defined in Eq. (2.4) and the complex constant D is given by

$$D=\sigma_n\sigma_p(i+\rho_n)(i+\rho_p)$$
$$-\tfrac{1}{2}[\sigma_n(i+\rho_n)-\sigma_p(i+\rho_p)]^2. \tag{4.4}$$

The effect of the double charge-exchange process is felt entirely through the second term of this expression. If charge exchange were neglected, the constant D would reduce to $\sigma_n\sigma_p(i+\rho_n)(i+\rho_p)$.

The parameters σ_p, ρ_p, a, and b which occur in the amplitudes f_j have been found from proton-proton scattering measurements[11–13] to have the approximate values 45.1 mb, -0.12, 7.62 (BeV/c)$^{-2}$, and 1.88 (BeV/c)$^{-4}$, respectively, at the laboratory energy of 2 BeV. The parameters σ_n and ρ_n have been obtained indirectly from pp and pd measurements[11,14] which have yielded the values 43.0 and 0.20 mb, respectively. The deuteron form factor S has been obtained analytically from the ground-state wave function referred to as ϕ_4 in I, Eq. (4.23).[15]

The elastic differential cross section for p-d scattering to which these data lead has been calculated by Franco and Coleman[4] and is plotted as the solid curve in Fig. 3. We note that the curve has all of the qualitative features mentioned earlier, the rapid dropoff in the single-scattering region, i.e., at small angles, the interference dip at intermediate angles, and the much slower rate of decrease at larger angles. The values of the cross section measured by Zolin *et al.*[14] are confined to small momentum transfers but may be seen to fit the curve well. The values of the cross section found by Coleman *et al.*[5] lie at much larger momentum transfers, but seven of the eight measured points again lie quite close to the calculated curve.

Also shown in Fig. 3 is the contribution made to the cross section by the single-scattering terms alone. While this contribution might be adjusted through parameter changes to fit the measurements at small momentum transfers, it clearly could not be adjusted in addition to fit the measurements at large momentum transfers. The single-scattering contribution is some two or three orders of magnitude smaller than the cross sections measured at the larger momentum transfers. The uncertainty of this ratio, which is due to our uncertainty of the deuteron form factor at large momentum transfers, does not seem great enough to influence the conclusion that the scattering observed at the larger

[11] D. V. Bugg, D. C. Salter, G. H. Stafford, R. F. George, K. F. Riley, and R. J. Tapper, Phys. Rev. **146**, 980 (1966).

[12] L. F. Kirillova, V. A. Nikitin, V. A. Sviridov, L. N. Strunov, M. G. Shafranova, Z. Korbel, L. Rob, A. Zlateva, P. K. Markov, T. Todorov, L. Khristov, Kh. Chernev, N. Dalkhazhav, and D. Tuvdendorzh, Zh. Eksperim. i Teor. Fiz. **50**, 76 (1966) [English transl.: Soviet Phys.—JETP **23**, 52 (1966)].

[13] L. F. Kirillova, V. A. Nikitin, V. S. Pantuev, V. A. Sviridov, L. N. Strunov, M. N. Khachaturyan, L. G. Khristov, M. G. Shafranova, Z. Korbel, L. Rob, S. Damyanov, A. Zlateva, Z. Zlatanov, V. Iordanov, Kh. Kanazirski, P. Markov, T. Todorov, Kh. Chernev, N. Dalkhazhav, and T. Tuvdendorzh, Yadern. Fiz. **1**, 533 (1965) [English transl.: Soviet J. Nucl. Phys. **1**, 379 (1965)]; L. F. Kirillova (private communication); A. R. Clyde, University of California Radiation Laboratory Report No. UCRL-16275, 1966 (unpublished). To obtain the value for b we analyzed the data of Clyde, which was taken at 2.2 BeV/c, in the laboratory system.

[14] L. S. Zolin, L. F. Kirillova, Lu Ch'ing-ch'iang, V. A. Nikitin, V. S. Pantuev, V. A. Sviridov, L. N. Strunov, M. N. Khachaturyan, M. G. Shafranova, Z. Korbel, L. Rob, P. Devinski, Z. Zlatanov, P. Markov, L. Khristov, Kh. Chernev, N. Dalkhazhav, and D. Tuvdendorzh, JETP Pis'ma Redaktsiyn **3**, 15 (1966) [English transl.: JETP Letters **3**, 8 (1966)].

[15] A factor r^{-1} has been omitted from ϕ_4 as written there.

momentum transfers is almost entirely double scattering. Calculations done for other plausible representations of the deuteron ground state yield, as shown in Ref. 4, differential cross sections which are quantitatively fairly similar to the solid curve of Fig. 3. As an additional example, we have investigated the influence on the cross section of a hard core in the neutron-proton interaction. We find that while the presence of a hard core may increase the deuteron form factor and raise the contribution of single scattering in the angular region we have considered, this contribution still drops off much too rapidly to approximate the scattering observed at the larger angles. At $t=-0.88$ (BeV/c)^2 and at $t=-1.54$ (BeV/c)^2, for example, the single-scattering contribution for a hard-core potential yields cross sections which are smaller than those observed by factors of ~ 50 and ~ 500, respectively. The double scattering calculation for a hard-core potential, on the other hand, agrees approximately as well as the other calculation we have reported.

Although the representation of charge-exchange scattering which is implicit in the amplitudes f_j is rather crude, it is adequate to establish the magnitude of the double-charge-exchange contribution to elastic scattering. We have shown as the dashed curve in Fig. 3, the differential cross section obtained by dropping the charge-exchange term in the constant D given by Eq. (4.4). The double-charge-exchange contribution evidently does not alter the shape of the differential cross section materially and, relative to the single-scattering contribution, is quite insignificant in magnitude for small momentum transfers. In the double-scattering region, however, it does raise the cross section perceptibly. It increases the double-scattering cross section by about 12% near the secondary maximum at $-t\sim 0.6$ (BeV/c)^2.

No measurements of the differential cross section appear to have been made to date in the interference region 0.2 (BeV/c)$^2\lesssim -t\lesssim 0.4$ (BeV/c)^2. Observations within this range would be highly desirable as a check of the theory and of the parameters used in it. The precise behavior of the cross section within the interference dip is quite sensitive, for example, to the phases of the nucleon scattering amplitudes. Variations of ρ_p and ρ_n within the quoted experimental errors[12,14] are found to change the value of the cross section at the minimum by an order of magnitude or more without changing it appreciably in the other ranges of t. It is evident that the curve shown in Fig. 3 gives at best only a qualitative indication of the behavior of the cross section within the interference region. Observation of the shape of the cross section in the interference region may be a useful means of securing information about the phases of the nucleon-nucleon scattering amplitudes.

V. K^+-DEUTERON CHARGE-EXCHANGE COLLISIONS

We shall apply our description of deuteron charge-exchange processes to the reaction

$$K^+ + d \rightarrow K^0 + p + p \tag{5.1}$$

at 2.27 BeV/c, and illustrate the way in which the cross section for the K^+n charge-exchange reaction

$$K^+ + n \rightarrow K^0 + p \tag{5.2}$$

may be estimated from observations of the reaction (5.1) and of K^+p collisions. In so doing we shall obtain estimates for the ratio of the real part of the forward K^+n elastic-scattering amplitude to its imaginary part and for the slope of the forward diffraction peak in K^+n elastic scattering.

The differential cross section for the reaction (5.1) may be expressed in terms of the K^+p and K^+n elastic-scattering amplitudes, f_p and f_n, by means of Eq. (3.17). The charge-exchange amplitude f_c in this expression is related to f_p and f_n by Eqs. (3.7) and (3.8) which may be written as

$$f_c = f_p - f_n. \tag{5.3}$$

We shall simplify the evaluation of the integrals in Eq. (3.17) by taking advantage of the fact that the K^+p scattering amplitude varies much more slowly with increasing momentum transfer than does the deuteron form factor. To state this relationship in different terms, the rms radius of the K^+p interaction near 2 BeV/c, as estimated from the K^+p elastic-scattering angular distribution, is approximately 0.5 F,[16] which is appreciably smaller than the average radius of the deuteron. We are evidently justified, therefore, in making use of the approximation already mentioned in connection with the derivation of Eq. (3.12). It corresponds to replacing $S(\mathbf{q})$ in the integrals in Eq. (3.17) by

$$S(\mathbf{q}) \approx \delta^{(2)}(\mathbf{q}) 2\pi \langle r^{-2}\rangle_d, \tag{5.4}$$

where $\delta^{(2)}(\mathbf{q})$ is a two-dimensional delta function in the momentum-transfer variable. In this way we reduce the expression for the charge-exchange cross section to the form

$$(d\sigma/d\Omega)_{\mathrm{ex}} = [1-S(\mathbf{q})]|f_c(\mathbf{q})|^2 - 2k^{-1}\langle r^{-2}\rangle_d \,\mathrm{Im}\{f_c^*(\mathbf{q})[f_p(\mathbf{q})f_n(0)-f_n(\mathbf{q})f_p(0)]\}$$
$$+\frac{\langle r^{-2}\rangle_d}{4\pi k^2}\int [|f_p(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_n(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')-f_n(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')|^2+|f_c(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_c(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')|^2]d^{(2)}\mathbf{q}'. \tag{5.5}$$

16 W. Chinowsky, G. Goldhaber, S. Goldhaber, T. O'Halloran, and B. Schwarzschild, Phys. Rev. 139, B1411 (1965).

The error of the approximation (5.4) may be estimated, by using a Gaussian form factor, to be less than 0.3% for the integrated cross section for the reaction (5.1) at 2.27 BeV/c.

We shall assume that the K^+-nucleon elastic-scattering amplitudes have the form

$$f_j(q) = (k\sigma_j/4\pi)(i+\rho_j)e^{-\frac{1}{2}\alpha_j^2 q^2}, \quad j=n, p \qquad (5.6)$$

at small angles, where σ_j is the K^+-j total cross section and ρ_j is the ratio of the real part of the forward K^+-j elastic-scattering amplitude to its imaginary part. This form is consistent with the available high-energy K^+-nucleon scattering data. The integral in Eq. (5.5) may then be evaluated analytically and the differential cross section may be expressed in terms of $S(\mathbf{q})$ and Gaussian functions of q.

We shall consider the K^+d charge-exchange reaction $K^+d \to K^0pp$ at 2.27 BeV/c where measurements have been made by Butterworth *et al.*[6] In order to calculate the differential cross section for this reaction, we need to know the values of σ_p, σ_n, α_p^2, α_n^2, ρ_p, and ρ_n which are used to parametrize the K^+-nucleon scattering amplitudes (5.6). The K^+p total cross section has been measured[17] to be 17.3 mb, and the K^+n total cross section has been deduced from K^+p and K^+d measurements[17] to be 18.5 mb. A measurement of α_p^2 has been made by Chinowsky *et al.*[16] at 1.96 BeV/c and yields the value 3.1 (BeV/c)$^{-2}$. The magnitude of ρ_p has been measured[16] to be 0.34 at 1.96 BeV/c, and theoretical calculations[18] for this energy region indicate that ρ_p is negative. We shall therefore assume the value -0.34 for ρ_p. In the absence of K^+n elastic-scattering data, we shall allow α_n^2 and ρ_n to be adjustable parameters and fit Eq. (5.5) to the $K^+d \to K^0pp$ data. In so doing, we shall obtain estimates for α_n^2 and ρ_n. Having obtained these, we will be in a position to predict the differential and integrated cross sections for the charge-exchange reaction $K^+n \to K^0p$. In our calculations we will use the deuteron wave function referred to as ϕ_4 in I.[15] With this wave function, we find $\langle r^{-2}\rangle_d$ to be 0.299 F^{-2}.

The observed angular distribution of the K^0 meson in the laboratory system is shown in Fig. 4 for $\cos\theta_{\text{lab}} \gtrsim 0.85$. A least-squares fit to the logarithm of the measured angular distribution, using Eqs. (5.5) and (5.6), is shown by the solid curve in Fig. 4. The value of α_n^2 obtained from this fit is 2.66 (BeV/c)$^{-2}$. This is somewhat smaller than the value 3.1 (BeV/c)$^{-2}$ measured[16] for α_p^2 at the somewhat lower momentum of 1.96 BeV/c. The value of ρ_n obtained from this fit is -0.76.

To illustrate the magnitude of the double-scattering correction in reaction (5.1), we have calculated

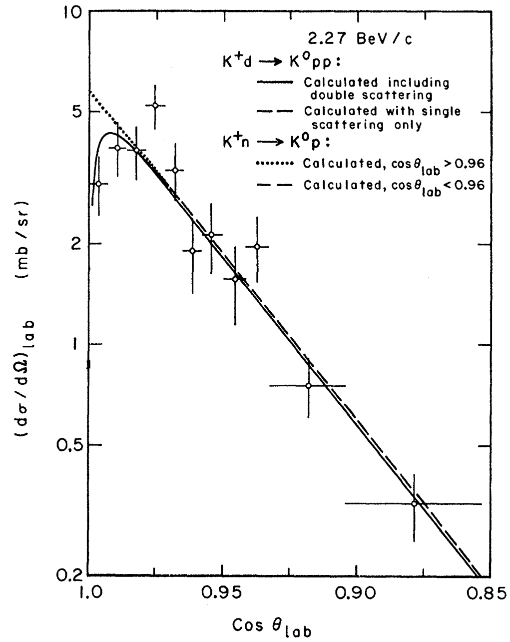

FIG. 4. Differential cross sections in the laboratory system for K^+d and K^+n charge-exchange scattering at 2.27 BeV/c as a function of $\cos\theta_{\text{lab}}$ the cosine of the K^0 scattering angle. The data for $K^+d \to K^0pp$ are from Ref. 6. The solid curve is calculated for $K^+d \to K^0pp$ with double scattering taken into account. The broken curve is calculated for $K^+d \to K^0pp$ with double scattering neglected and coincides with the solid curve for $\cos\theta_{\text{lab}} \gtrsim 0.98$. The dotted curve, which coincides with the broken curve for $\cos\theta_{\text{lab}} \lesssim 0.96$, is the calculated angular distribution for $K^+n \to K^0p$.

$(d\sigma/d\Omega)_{\text{ex}}$ with double-scattering effects neglected. The results, shown by the broken curve in Fig. 4, indicate that double-scattering effects are rather small within the angular region $\theta_{\text{lab}} \lesssim 30°$. For example, the effect of including double scattering is to decrease the differential cross section by approximately 0.2% near $\theta_{\text{lab}} = 7°$, where the intensity is near its maximum, and by approximately 4% near $\theta_{\text{lab}} = 30°$. The effect of including double scattering is to decrease the integrated charge-exchange cross section by approximately 2%.

The differential cross section $(d\sigma/d\Omega)_{\text{ex}}$ for K^+d charge exchange may be analyzed in terms of single scattering, interference between single and double scattering, and pure double scattering. In Fig. 5 we show the angular distributions of these three components of the differential cross section. The contribution arising from the interference term is negative; the curve shown corresponds to its absolute value. The magnitude of the interference term is always considerably smaller than the single-scattering term and for $\cos\theta_{\text{lab}} \lesssim 0.9$ has an angular distribution rather similar in form to the single-scattering term. As we have noted in Sec. III, the pure double-scattering term gives the only nonzero contribution in the forward direction. Near $\cos\theta_{\text{lab}} = 0.995$, however, the other two terms each have become greater in magnitude than the double-scattering term. Nevertheless, at larger angles the double-scatter-

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[18] M. Lusignoli, M. Restignoli, G. Violini, and G. A. Snow, Nuovo Cimento 45A, 792 (1966); N. M. Queen, University of Birmingham Report, 1966 (unpublished).

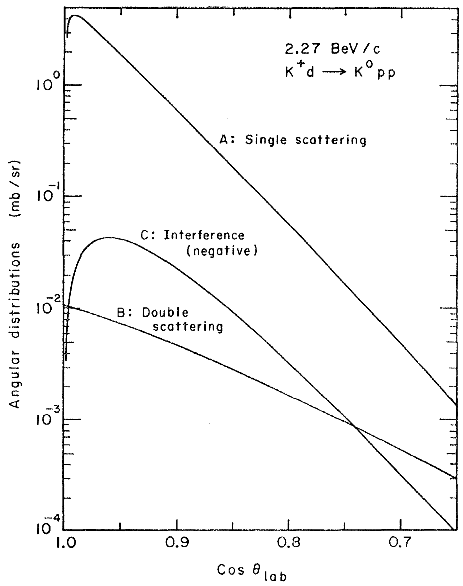

FIG. 5. Angular distributions of the components of the K^+d charge-exchange intensity as a function of $\cos\theta_{lab}$. The incident K^+-meson laboratory momentum is 2.27 BeV/c. The curve labeled A represents the single scattering contribution and curve B the double-scattering contribution. The curve labeled C represents the contribution from the interference between the single- and double-scattering processes, and is negative in sign.

ing term decreases less rapidly than the magnitudes of each of the other two contributions, becoming equal to the absolute value of the interference term near $\cos\theta_{lab}=0.74$. At still larger angles, i.e., $\cos\theta_{lab}\lesssim 0.5$, it seems possible that double scattering may be the dominant mechanism of charge exchange.

We may now use Eq. (5.3) to study the charge exchange reaction $K^+n \rightarrow K^0p$. Using the value $\rho_n=-0.76$, we find that the ratio of the real part of the forward K^+n charge-exchange amplitude to its imaginary part, which we may write as

$$\rho_{ex}=\mathrm{Re}f_c(0)/\mathrm{Im}f_c(0)\,, \tag{5.7}$$

is -6.8 at 2.27 BeV/c. This result is in contrast with the ratio for high-energy K^-p charge exchange which has a predominantly imaginary amplitude.[19]

The differential cross section for the reaction $K^+n \rightarrow K^0p$ is simply $|f_c|^2$. Over the angular range $\theta_{lab}\gtrsim 10°$, where $|S(q)|\ll 1$, the form of $|f_c|^2$ is very close to the angular distribution for $K^+d \rightarrow K^0pp$ given *without* treating double-scattering effects and is therefore represented by the broken curve in Fig. 4. At smaller angles the effects of the exclusion principle must be accounted for and the K^+n charge-exchange angular distribution extrapolates to 0° along the dotted curve. We find a value of 5.7 mb/sr for the $K^+n \rightarrow K^0p$ differential cross section in the forward direction. The integrated cross section for K^+n charge exchange corresponding to the data over the interval $1\geq\cos\theta_{lab}\geq 0.85$ is calculated to be 1.54 mb.

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VI. EFFECT OF CHARGE EXCHANGE ON DEUTERON TOTAL CROSS SECTIONS

Recent measurements have lent support to the conjecture of Pomeranchuk[20] that charge-exchange cross sections vanish in the high-energy limit. The neutron-proton charge-exchange cross section, for example, has been measured to be 0.65 mb at 2.83 BeV/c[21] and 0.06 mb at 8 BeV/c,[22] whereas the corresponding neutron-proton total cross sections are approximately 43 and 41 mb.[11] Additional evidence occurs, for example, in measurements of the antiproton-proton charge-exchange cross sections which yield values of 5 mb at 1.7 BeV/c[23] and 0.28 mb at 9 BeV/c,[24] whereas the corresponding antiproton-proton total cross sections are approximately 96 and 55 mb.[25,26] It is reasonable to expect, therefore, that the effect of charge-exchange processes on the total cross section of the deuteron becomes negligibly small at very high energies. We shall estimate the effect quantitatively in this section in order to determine its bearing on a number of indirectly measured neutron cross sections.

Let us first consider the measurements made by Galbraith *et al.*[26] using incident beams of protons, antiprotons, and $K^\pm$ mesons in the range from 6 to 22 BeV/c. Their measurements of deuteron cross sections were analyzed by assuming the ranges of the strong interactions to be small in comparison to the deuteron radius, and by neglecting the real parts of the forward-scattering amplitudes. The first of these assumptions leads to Eq. (3.15) for the cross-section defect and the second reduces it to the form

$$\delta\sigma=(1/4\pi)\{\sigma_n\sigma_p-\tfrac{1}{2}(\sigma_n-\sigma_p)^2\}\langle r^{-2}\rangle_d\,. \tag{6.1}$$

The term proportional to $(\sigma_n-\sigma_p)^2$ is the charge-exchange correction to the formula used in the analysis of the experiment.

Typical values of the neutron total cross sections,

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derived by Galbraith *et al.*[26] from their data without the charge-exchange correction, are

$$\sigma(pn)=42.6\pm1.7 \text{ mb},\quad \sigma(\bar{p}n)=59.5\pm4.0 \text{ mb},$$
$$\sigma(K^+n)=17.5\pm0.4 \text{ mb},\quad \sigma(K^-n)=21.9\pm0.4 \text{ mb},$$

at 6 BeV/c, and

$$\sigma(pn)=39.2\pm1.7 \text{ mb},\quad \sigma(\bar{p}n)=44.4\pm9.0 \text{ mb},$$
$$\sigma(K^+n)=17.6\pm0.4 \text{ mb},\quad \sigma(K^-n)=20.3\pm1.1 \text{ mb},$$

at 18 BeV/c.

When the experimental data for the cross sections are reanalyzed by taking the charge-exchange correction in Eq. (6.1) into account, we find that six of the eight cross sections quoted remain unaltered to three significant figures. The two cross sections which are changed are $\sigma(K^+n)$ at 6 BeV/c which is decreased to 17.4 mb and $\sigma(\bar{p}n)$ at 18 BeV/c which is decreased to 44.3 mb. The corrections to the neutron cross sections due to charge-exchange effects are generally smaller than 0.1 mb in this momentum range, and thus a good deal smaller than the quoted errors of the measurements.

We consider next the proton-deuteron and proton-proton total cross-section measurements of Bugg *et al.*[11] between 1.1 and 8 BeV/c. Since these measurements were performed at a considerably lower range of incident momenta than those noted earlier, their analysis was carried out differently in two respects. The real parts of the neutron and proton forward-scattering amplitudes were retained in the formula for the cross-section defect, and an attempt was made to correct for the effects of the internal motion of the nucleons in the deuteron. The details of the latter correction are given in Ref. 11. For our present purposes we need only use their values for the averages of the proton-proton and proton-neutron total cross sections taken over the internal motion, which they have referred to as "$\sigma(p-p)$" and "$\sigma(p-n)$," respectively. Typical values of "$\sigma(p-n)$" which have been reached without taking account of the charge-exchange correction are[11]

$$\begin{aligned}``\sigma(p-n)" &= 35.72\pm0.26 \text{ mb at } 1.111 \text{ BeV}/c\\ &= 42.255\pm0.069 \text{ mb at } 4.552 \text{ BeV}/c\\ &= 41.328\pm0.080 \text{ mb at } 7.835 \text{ BeV}/c.\end{aligned}$$

By following the same procedure as is used in Ref. 11, but taking charge-exchange corrections into account by using Eq. (3.15) to represent the cross-section defect, we find the altered cross sections

$$\begin{aligned}``\sigma(p-n)" &= 36.50 \text{ mb at } 1.111 \text{ BeV}/c\\ &= 42.305 \text{ mb at } 4.552 \text{ BeV}/c\\ &= 41.349 \text{ mb at } 7.835 \text{ BeV}/c.\end{aligned}$$

Only at the lowest momentum does the change exceed the quoted experimental error.

The fact that the charge-exchange corrections tend to increase the neutron cross sections in these measurements rather than decreasing them very slightly, as in the earlier discussion, is due to the fact that the real parts of the scattering amplitudes have not been neglected in the analysis. The charge-exchange correction which is proportional to $(\rho_n\sigma_n-\rho_p\sigma_p)^2$ in Eq. (3.15) dominates the one proportional to $(\sigma_n-\sigma_p)^2$ at low incident momenta.

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