tion of (I 24), is[28]

$$\left\{(\Gamma_\mu p_\mu-\mu e^2)\Gamma_4^{-1}(\Gamma_\mu p_\mu-\mu e^2) -\left(1+\frac{\mu}{M}\right)p^2\Gamma_4+\mu M\Gamma_4\right\}\psi=0. \quad \text{(III 17)}$$

Some properties of this equation are:

A. The mass spectrum is

$$\frac{1}{2M}(p^2-M^2)=-\frac{\mu e^4}{2n^2},$$

where n has the discrete spectrum $n=1, 2, 3, \cdots$, and a continuous spectrum $0>n^2>-\infty$. This is precisely the spectrum of hydrogen, including bound states and scattering states, and there are no unphysical solutions.

B. The spurion is

$$n(k)^A=(2\mu B)^{-1/2}(k_0, -\mathbf{k}, -(k_0{}^2-k^2-2\mu B)^{1/2}).$$

Therefore $z=1-t/4\mu B$, and the anomalous threshold singularity appears at the correct position. We learn from this example that the spatial extension of the particle states is determined by the spurion, and that the latter is not fixed by the mass spectrum.

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Shape of the $N^*(1236)$ Resonance

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Corrections to the Breit-Wigner shape of the $N^*(1236)$ resonance are calculated using analyticity and inelastic unitarity incorporating the $N\pi$ and $N^*\pi$ channels in a propagator formalism. This method, which includes only a right-hand cut and which evaluates the effect of bubble insertions in the propagator, is motivated by the rigorous results which have been proved for the nucleon. It is argued that even though background and left-hand cuts have been neglected, it is the inclusion of inelasticity that enables the P_{33} phase-shift data to be reproduced, even at energies well above resonance. The structureless-vertex decay model with a Breit-Wigner propagator gives an N^* shape too asymmetric, and inclusion of an inelastic channel with an analytic propagator serves to correct this. Assuming $N^*\pi$ as the only inelastic channel, the $N^{*++}N^{*++}\pi^0$ coupling is estimated as 170 ± 50, where the unknown behavior of the vertices far off the mass shell causes the uncertainty. This estimate can be compared with about 75 from relativistic $SU(6)$, and 136 using Adler-Weissberger techniques. The P_{33} partial-wave amplitude constructed on this model has a left-hand pole which simulates the effect of the neglected nucleon-exchange short cut, and which tends to lie too far left and with too large a residue. The application of the method to other resonances and bound states is discussed.

1. INTRODUCTION

The $N^*(1236)$ resonance is interpreted as the $J^P=\frac{3}{2}^+, I=\frac{3}{2}$ contribution to πN scattering, so that the P_{33} phase-shift analyses are the experimental source of data. This is assuming that there is no nonresonant background to the N^* which has the same quantum numbers. The intention is not to calculate the resonance parameters, but to consider the detailed consequences of the resonant behavior of the partial wave from threshold to center-of-mass (c.m.) energies of 1500 MeV or higher. The unstable particle will be treated in a way motivated by the field-theoretic behavior of an off-mass-shell stable particle above threshold. Thus we will try to treat the resonance as a stable particle that has wandered above the threshold.

The general form for the partial-wave amplitude $a_{33}(E)$, deduced from unitarity and the requirement of a phase shift δ of 90° at the real resonance mass m, is

$$a_{33}(E)=\frac{\epsilon(E)}{\cot\delta(E)-i}=\frac{\frac{1}{2}\Gamma_1(E)}{G(E)(m-E)-\frac{1}{2}i\Gamma_T(E)}, \quad (1.1)$$

where E is the c.m. energy, $\epsilon(E)$ is the elasticity, and by definition $G(m)=1$, so that the total width at resonance is defined by $\Gamma_T(m)=2(m-E)/\cot\delta(E)$ as E tends to m. $\Gamma_1(E)$ is the elastic width into the πN channel.

The relativistic generalization of the Breit-Wigner formula corresponds to (1.1) with $G(E)=1$ and the threshold for a P-wave decay to $N\pi$ requires, nonrelativistically, $\Gamma_1(E)\sim q^3$, where q is the π momenta in the N^* rest frame or c.m. system. The energy variation of $\Gamma(E)$ is discussed in detail in Sec. 3 using unitarity and a model for the decay vertex amplitude. This model, which is covariant and has the correct threshold factors in all crossed channels, i.e., P-wave $N^*\to N\pi$, P-wave $\bar{N}\to N^*\pi$, D-wave $\pi\to N^*\bar{N}$, D-wave $0\to N^*\bar{N}\pi$, etc., is a decay amplitude

$$V_\lambda(E,q,\lambda_f)=(g_1/m_0)\bar{U}^\mu(E,\lambda)q_\mu U(q,\lambda_f)\,, \qquad (1.2)$$

where $U^\mu(E,\lambda)$ is a Rarita-Schwinger spinor for a spin-$\frac{3}{2}$ N^* of mass E, spin projection λ, at rest; and $U(q,\lambda_f)$ is a Dirac spinor for a nucleon of mass m_N, helicity λ_f, and momentum q. This expression is unique, except for the energy dependence of the factor g_1/m_0, which will be taken as g_1/E with a constant g_1. This leads to an expression for the decay width to $N^+\pi^+$ of

$$\Gamma_1(E)=\frac{g_1^2}{4\pi}\frac{q^3}{6E^2}\frac{(E+t_1)(E+a_1)}{E^2}\,, \qquad (1.3)$$

where $t_1=m_N+m_\pi$, $a_1=m_N-m_\pi$.

Using this model for $\Gamma_1(E)$, $G(E)$ is calculated in Sec. 2 from the experimental data on $\epsilon(E)$, $\delta(E)$, and m:

$$G(E)=\Gamma_1(E)\,\cot\delta(E)/2\epsilon(E)(m-E)\,. \qquad (1.4)$$

The result is that $G(E)$ is not a constant, so that the Breit-Wigner model with a Born-model decay amplitude is insufficient to explain the shape of the N^*, giving too asymmetric an effect. This has long been known[1] and one explanation was to retain the Breit-Wigner approximation that $G(E)=1$ and to invoke a form factor or structure effect at the decay vertex. The alternative method presented here is to keep the Born-model decay vertex and to seek an explanation in the possibility that an analytic expression for the propagator will determine $G(E)$. The inverse propagator $P^{-1}(E)$, which is the denominator of (1.1), is not analytic as it stands since $\Gamma_1(E)$ has a factor $\theta(E-t_1)$ from the phase-space integral. An analytic expression is deduced in Sec. 3 from a discussion motivated by the Lehmann spectral representation[2] for the nucleon. The contribution of $N\pi$ intermediate states in the nucleon propagator, with Born pseudoscalar-vertex coupling, is evaluated and leads to correction factors which are considerable and in the direction required by experiment. These corrections reduce the s-channel nucleon-pole contribution by 1.24 and 6 at threshold in P_{11} and S_{11} partial waves, respectively.

In extending these considerations to unstable particles, one has no such specific guide although the general method is the same. In particular, one must assume that the type of corrections included in this approach are the most important class and so neglect any other singularities than those in the s channel. The singularities retained are, in the E plane, the N^* resonant pole and the elastic and inelastic thresholds and discontinuities in both P_{33} and the related D_{33} partial wave.

The simplest vertex amplitudes, as (1.2), that one would write for each different N^* decay channel give partial widths which have high-energy behavior worse than in the field-theory-motivated nucleon case. This is overcome by retaining the threshold dependences while excluding high-spin contributions such as the F-wave coupling of $N^*\to N^*\pi$, and choosing scaling masses as in (1.3), so that each partial width increases asymptotically as E at most, as in the nucleon example. The dependence of the final result on the method of ensuring that the widths increase as E, far off mass shell, is tested and the final result is insensitive to the details.

The results are discussed in Sec. 4 and it is found that including $N\pi$ states in the dispersion relation makes very little difference and is even of the wrong sign; however, the contribution of closed channels such as $N^*\pi$ does not cancel and is of the correct sign. Then one can constrain the inelastic spectrum coupled to the resonance to fit the observed shape correction factor $G(E)$ near the resonance peak. Assuming that $N^*\pi$ is the dominant channel, the $N^{*++}N^{*++}\pi^0$ coupling is estimated as about 170, which is compared with other estimates. Furthermore, the inelasticity in the P_{33} amplitude so predicted is approximately correct.

The solutions that fit the experimental data all have a zero of $G(E)$ at about 800 to 900 MeV. This is consistent with the analytic properties assumed for $P^{-1}(E)$ and corresponds to a pole in the partial-wave amplitude which is simulating the t and u channel singularities that have been omitted. The known short cut due to nucleon exchange is situated nearer to the physical region at 938 MeV and has a residue less than half that of the pole predicted. It is instructive that only when analyticity is included does a left-hand pole appear and determine the nucleon exchange parameters.

In Sec. 5, the application of the method to other resonances and bound states is discussed.

2. EXPERIMENTAL DATA

The data near resonance have been compiled by Olsson,[3] who subtracted the small partial waves from the experimental total cross section to get the P_{33} data, a procedure which should be particularly accurate for π^+p where the background is very small near resonance. Then he finds $m=1236.0\pm0.5$ MeV for N^{*++} and using

[1] M. Gell-Mann and K. Watson, Ann. Rev. Nucl. Sci. 4, 219 (1954).

[2] H. Lehmann, Nuovo Cimento 11, 342 (1954).

[3] M. G. Olsson, Phys. Rev. Letters 14, 118 (1965).

the parametrization

$$\Gamma(E)=\gamma q^3/(1+q^2R^2)\,, \tag{2.1}$$

where γ and R are constants, he finds $\Gamma(m)=120\pm 2$ MeV and $R=0.91m_\pi^{-1}$. These values of m and $\Gamma(m)$ are used and fix $g_1^2/4\pi=29$ from (1.3).

Then $G(E)$ may be calculated from the phase-shift data using (1.4); the results are shown in Table I.[3–8]

G is plotted against the c.m. energy E in Fig. 1, and the results from Table I are shown with the error bars. The Breit-Wigner approximation is shown and the expression (2.1) used by Olsson is also plotted with the value $R=0.91$. This parametrization is good near resonance but leads to a scattering length of 0.276 and 310-MeV phase shift of 133°, both of which are well outside the experimental errors. A phenomenological form like (2.1) was used by Gell-Mann and Watson[1] and revived by Layson,[9] and has a theoretical justification if the $N^*N\pi$ interaction is assumed in a nonrelativistic-potential model to have a square-well potential barrier of radius R when the factor $1+q^2R^2$ comes from the barrier penetration of a P wave.

3. ANALYTICITY AND UNITARITY FOR THE RESONANCE PROPAGATOR

Before embarking on a discussion of propagators for unstable particles it is necessary to recall the rigorous results proved in field theory for stable particles, in particular for the nucleon since the complication of spin is present in that case. The notation of Jin and MacDowell[10] will be followed and other references may be found in their paper. The motivation is to discover

TABLE I. The experimental P_{33} phase shifts δ, elasticities ϵ, and scattering length a from the references quoted are used at each center-of-mass energy E to calculate the shape correction factor G using (1.4).

$T^\pi{}_{\text{lab}}$ (MeV)	$E_{\text{c.m.}}$ (MeV)	δ	ϵ	Reference	G
0	1078	$a=0.215\pm 0.005$		4	0.57 ± 0.013
120	1178	31.7 ± 1	1	5	0.82 ± 0.03
195	1236	90	1	3	1.00 ± 0.02
247.5	1275	119.3 ± 1.3	1	6	1.20 ± 0.12
310	1320	$136\ \pm 1$	1	7	1.39 ± 0.04
370	1362	$145\ \pm 1$	1	8	1.58 ± 0.06
410	1390	$148\ \pm 1$	1	8	1.68 ± 0.06
492	1444	$157\ \pm 1$	1	8	2.28 ± 0.11

[4] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

[5] A. Loria, P. Mittner, R. Santangelo, I. Scotoni, G. Zago, B. Aubert, A. Brenner, Y. Goldschmitt-Clermont, F. Grard, G. Macleod, A. Minguzzi-Rangi, and L. Montanet, Nuovo Cimento **22**, 820 (1961).

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[9] W. Layson, Nuovo Cimento **27**, 724 (1963).

[10] Y. S. Jin and S. W. MacDowell, Phys. Rev. **137**, B688 (1965).

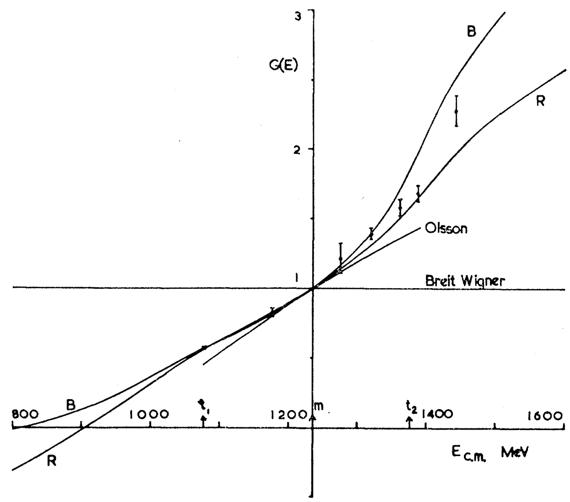

FIG. 1. G defined by (1.4) is plotted against the center-of-mass energy E in MeV. G calculated as in Table I is shown by the error bars, the Breit-Wigner approximation corresponds to G constant at 1; the Olsson curve corresponds to (2.1), and the curves B and R correspond to (4.1) and (4.3), and are the best fit to the experimental points with the two-channel model, having $g_2^2/4\pi=231$ and 112, respectively.

suitable scalar functions for propagator and vertex which can be generalized to the case of unstable particles.

For a stable particle one has a Feynman-diagram expansion and one can divide diagrams into two classes, the first class containing those with a pole at $s=m^2$, and the second class containing those background contributions with no single line corresponding to the intermediate one-particle state in s. It is the sum of diagrams of the first class that will be discussed, and for the nucleon one can include contributions from two-body channels to which it is coupled in a straightforward manner. For the propagator, the corrections due to bubble insertions are recalled and only πN states will be considered explicitly.

The sum of diagrams of the first class for πN elastic scattering with a nucleon s-channel pole is[10]

$$\bar{u}g\bar{\Gamma}\Delta\Gamma u\,, \tag{3.1}$$

where the proper vertex function $\Gamma=[\mathbf{P}\Gamma_1(s)+\Gamma_2(s)]\gamma_5$ for pseudovector and pseudoscalar couplings. The fermion propagator $\Delta=\mathbf{P}\Delta_1(s)+\Delta_2(s)$ may be expressed, using the $E=\sqrt{s}$ plane, in terms of one scalar function

$$\Delta=((\mathbf{P}+E)/2E)P(E)-((\mathbf{P}-E)/2E)P(-E)\,, \tag{3.2}$$

$$P(E)=E\Delta_1(s)+\Delta_2(s)\,, \tag{3.3}$$

where $P(E)$ is a Herglotz function analytic in the complex E plane cut along the segments $(\pm >t, \pm\infty)$ and with a pole at $E=m$. t is the threshold of the lowest mass physical state coupled to the particle, and is $N\pi$ for the nucleon in strong interaction.

Introducing Dirac spinors with spin component λ, normalized to 1, for the virtual intermediate nucleon or antinucleon of momentum p and mass E which satisfy

$$\sum_{\lambda} u(p,\lambda)\bar{u}(p,\lambda)=(\mathbf{P}+E)/2E\,, \tag{3.4}$$

$$\sum_{\lambda} v(p,\lambda)\bar{v}(p,\lambda)=-(\mathbf{P}-E)/2E\,, \tag{3.5}$$

Eq. (3.1) may be rewritten with the help of (3.2) as

$$\sum_{\lambda} \bar{u}g\bar{\Gamma}u(p,\lambda)P(E)\bar{u}(p,\lambda)g\Gamma u +\sum_{\lambda} \bar{u}g\bar{\Gamma}v(p,\lambda)P(-E)\bar{v}(p,\lambda)g\Gamma u\,. \tag{3.6}$$

This expression enables one to define scalar vertex functions which have a physical interpretation:

$$\bar{V}_{\lambda}^{1/2}(E,q_i,\lambda_i)=\bar{u}(E,\lambda)g\Gamma u(q_i,\lambda_i)\,. \tag{3.7}$$

In the rest frame of the intermediate state, which is the over-all c.m. system, V is the amplitude from a nucleon of spin $\frac{1}{2}$, magnetic quantum number λ, and mass E, to a πN state with momentum along the direction q_i which can be specified by two spherical polar angles θ, ϕ relative to a fixed coordinate system such as q_f. This expression may be readily generalized to other channels when (3.6) will factorize. The importance of (3.7), which allows a natural extension to unstable particles, is that the virtual particle is represented as if it were a stable particle of mass E. The angular distribution of the decay of a spin-J state into a two-body final state with one spin-zero particle is given by

$$V_{\lambda}^{J}(E,q_i,\lambda_i)=\left(\frac{2J+1}{4\pi}\right)^{1/2} D_{\lambda\lambda_i}{}^{J*}(\phi,\theta,0)M(E,J,\lambda_i), \tag{3.8}$$

where the D functions are the usual Wigner rotation matrices.

Since the total angular momentum J and component λ are conserved, the unitarity equation for this contribution to πN elastic scattering can be simplified considerably. The first term in (3.6) corresponds[10] to the intermediate nucleon with $J^P=\frac{1}{2}^+$ and the second corresponds to an additional nucleon-antinucleon pair which has negative intrinsic parity and so contributes to $J^P=\frac{1}{2}^-$ partial waves. These contributions separate because of parity conservation, so that the unitarity equation for the nucleon intermediate state with spin component λ is

$$\mathrm{Im}V_{\lambda}(E,q_f,\lambda_f)P(E)\bar{V}_{\lambda}(E,q_i,\lambda_i) =\tfrac{1}{2}\sum_{\lambda n}\int d\Omega^n V_{\lambda}(E,q_f,\lambda_f)P(E)\bar{V}_{\lambda}(E,q_n,\lambda_n) \times\bar{V}_{\lambda}{}^{*}(E,q_n,\lambda_n)P^{*}(E)V_{\lambda}{}^{*}(E,q_i,\lambda_i)\,, \tag{3.9}$$

where $d\Omega^n$ is the Lorentz-invariant phase-space element for the intermediate πN state. Now in the case of no background, the M amplitudes are real and cancel in (3.9), giving

$$\mathrm{Im}P(E)=\tfrac{1}{2}\sum_{\lambda n}\int d\Omega^n |V_{\lambda}(E,q_n,\lambda_n)P(E)|^2 \tag{3.10}$$

or

$$\mathrm{Im}P^{-1}(E)=-\tfrac{1}{2}\sum_{\lambda n}\int d\Omega^n |V_{\lambda}(E,q_n,\lambda_n)|^2\,, \tag{3.11}$$

and for a two-body channel with one spin-zero particle this simplifies to

$$\mathrm{Im}P^{-1}(E)=-\frac{q}{8\pi E}\frac{2m}{4\pi}\sum_{\lambda n}|M(E,J,\lambda_n)|^2. \tag{3.12}$$

Equation (3.11) can also be derived when the background is nonzero provided that the background itself is unitary as well as the total amplitude.[10,11]

For no background, the reality of the M amplitudes allows the P_{11} partial wave amplitude to be constructed:

$$e^{i\delta}\sin\delta=-\mathrm{Im}P^{-1}(E)/[\mathrm{Re}P^{-1}(E)+i\,\mathrm{Im}P^{-1}(E)]\,, \tag{3.13}$$

and for the S_{11} partial wave one could repeat the above derivation and find, in agreement with MacDowell symmetry,[12] that E must be replaced by $-E$ in (3.13).

Our intention is to calculate corrections to the Born term $(m-E)$ for $\mathrm{Re}P^{-1}(E)$. The expression for $\mathrm{Im}P^{-1}(E)$ in (3.11) together with the analytic properties of $P(E)$ allow this. $P^{-1}(E)$ will have no poles assuming that $P(E)$ had no zeros—this is the simplest case and the alternatives have been discussed.[10,13] The number of subtractions needed in the dispersion relation for $P^{-1}(E)$ can be determined since the existence of unsubtracted dispersion relations[2] for $\Delta_1(s)$ and $\Delta_2(s)$ implies an unsubtracted relation for $P(E)$ in the cut E plane. Thus $P^{-1}(E)$ increases less rapidly than E, and two subtractions are sufficient and will be taken at m.

$$P^{-1}(E)=(m-E)+\frac{(m-E)^2}{\pi}\left[\int_t^{\infty}-\int_{-t}^{-\infty}\right]\frac{\mathrm{Im}P^{-1}(E')dE'}{(m-E')^2(E'-E)}. \tag{3.14}$$

However, $P^{-1}(E)$ will only increase as E or less if the coefficient of E in (3.14) is finite,[4] and so a once-subtracted dispersion relation exists:

$$P^{-1}(E)=Z_2(m-E)+\frac{(m-E)}{\pi}\left[\int_t^{\infty}-\int_{-t}^{-\infty}\right]\frac{\mathrm{Im}P^{-1}(E')dE'}{(m-E')(E'-E)}, \tag{3.15}$$

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[12] S. W. MacDowell, Phys. Rev. **116**, 774 (1960).
[13] I. S. Gerstein and N. G. Deshpande, Phys. Rev. **140**, B1643 (1965).

where Z_2 is the nucleon wave-function renormalization coefficient.

A similar relation has also been obtained for a stable spin-zero particle[14] assuming no zeros of $P(s)$;

$$P^{-1}(s)=Z_3(m^2-s)+\frac{(m^2-s)}{\pi}\int_{t^2}^{\infty}\frac{\mathrm{Im}P^{-1}(s')ds'}{(m^2-s')(s'-s)}. \quad (3.16)$$

Now from a model for the vertex function V, one can evaluate $\mathrm{Im}P^{-1}$ from (3.11) and then $\mathrm{Re}P^{-1}$ from (3.15) or (3.16) and hence the partial-wave amplitude using (3.13). For instance, treating the deuteron as a spin-zero particle with an S-wave coupling to neutron-proton leads to a result using (3.16) very similar to that obtained by Weinberg[15] in a nonrelativistic manner.

The existence of an unsubtracted dispersion relation for $P(E)$ and thus (3.15) means that $\mathrm{Im}P^{-1}(E)$ and hence $|M|^2$ must increase less rapidly than E. However, for the $NN\pi$ Born pseudoscalar coupling with $\Gamma_1(s)=0$ and $\Gamma_2(s)=1$, one finds from (3.7) and (3.8)

$$M(E,\tfrac{1}{2},\lambda_f)/(2\pi)^{1/2}=g((E_f-m)/2m)^{1/2}2\lambda_f\delta_{\lambda\lambda_f}, \quad (3.17)$$

where E_f is the final nucleon total energy. Therefore, $|M|^2$ increases as E and the Born vertex model has incorrect high-energy behavior. Moreover, $\Gamma(E)=E\Gamma_1(s)+\Gamma_2(s)$ has the same cuts as $P(E)$ and it obeys an unsubtracted dispersion relation. Its discontinuity across the cuts is proportional to the background, so that if one neglects background, $\Gamma(E)$ is identically zero unless $\Gamma(E)$ has a pole.[10] However, within the spirit of keeping only $NN\pi$ vertices, the background will include nucleon exchange contributions from the u channel.

For an unstable particle such as the $N^*(1236)$, the division into two classes of Feynman diagrams discussed for the nucleon cannot be proved, although the assumption made is that the resonance contributes entirely by an analog of the first class of diagram. In the approximation that the $N^*N\pi$ vertex is negligible compared to $N^*N^*\pi$, one would treat the N^* as a stable particle, and so the propagator formulation for an unstable particle is more likely to be a useful approach if closed channels on aggregate have stronger couplings to the resonant state than do open channels. The factorization (3.6) is easily generalized to the unstable-particle case since the vertex amplitudes are readily defined. That the factorization is valid away from the resonance peak is an assumption suggested by the virtual-stable-particle result.

The singularities of $P^{-1}(E)$ for an unstable fermion will include the cuts $E>t$, $E<-t$ suggested by the stable case and also $\mathrm{Re}P^{-1}(E)$ will have a zero at $E=m$, where m is the real resonance mass defined in this manner. Assuming again no zeros in $P(E)$, one will have the dispersion relation for $P^{-1}(E)$ to evaluate the real part from the imaginary part given by (3.12). The model with no background implies that P or V must have all the known left-hand partial-wave singularities, whereas in the stable-particle case they have only right-hand cuts. In keeping with the approach of treating the unstable particle as a stable one which has wandered above a threshold, the dispersion relation for $P^{-1}(E)$ is approximated by retaining in case B both cuts $E>t$, $E<t$ or in case R keeping only the right-hand particle cut $E>t$. In this latter case, diagrams with an $N^*\bar{N}^*$ pair produced are also neglected, and this is reasonable since these contributions have vertex functions which are off mass shell by at least twice the nucleon mass. In both cases the content is the N^* s channel resonance with all inelastic channels to which it is coupled in an analytic and unitary model.[16]

The model for the decay vertices must now be constructed generalizing (3.7). The N^* is described by a Rarita-Schwinger spinor for spin $\frac{3}{2}$ and mass E normalized to 1, in analogy with the nucleon spinor of mass E used above. Thus the N^* polarization sum operator, off mass shell, has been constructed in such a way that no spin-$\frac{1}{2}$ contributions due to failure of the Rarita-Schwinger subsidiary conditions off mass shell will appear. Thus an irreducible spin-$\frac{3}{2}$ theory would give the same results. The unstable particle spinor of virtual energy E is represented as a stable-particle spinor with the same quantum numbers and mass E, just as the stable-particle spinor was represented when the stable particle had a virtual energy above a threshold and so could decay. This approach gives decay amplitudes with the correct angular and threshold properties.

These considerations give the motivation for writing the resonance contribution to scattering in a vertex-propagator-vertex formalism for which rules similar to Feynman rules can be established. One must be careful not to count twice since the N^* direct-channel contribution includes the effect of nucleon exchange, so that the latter contribution must not be thought of as background. Now one has a formalism which is a relativistic generalization of the Breit-Wigner formalism and which applies even if the resonance width is comparable to the energy of the resonance above the relevant threshold.

The result (1.3) for the partial width to $N\pi$ is obtained by using (1.2) and (3.8) to calculate the reduced matrix element $M(E,\frac{3}{2},\lambda)$, and then using (3.12) to obtain $\Gamma_1(E)=-2\,\mathrm{Im}P^{-1}(E)$. In order to have an asymptotic behavior of E as in the nucleon case, the scaling mass m_0 is put equal to E. This is at our disposal since the vertex amplitude (1.1) was introduced only

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[16] R. H. Dalitz and D. G. Sutherland, Phys. Rev. **146**, 1180 (1966).

to give the correct thresholds and to be relativistically invariant. The high-energy behavior is not to be trusted since it corresponds to a derivative coupling; instead one requires the high-energy behavior to be no worse than that in the pseudoscalar-coupling case. The dispersion relation (3.14) that will be used will then converge and the high-energy behavior will not be dominant. This is particularly so in case B, since both cuts have the same discontinuity asymptotically and the cancellation improves the convergence by an extra power of E.

The propagator model has the advantage that any number of inelastic channels can be included in (3.14) which is still a single integral. In an N/D approach, inelasticity may be incorporated,[17] but explicit many-channel calculations are difficult. The result (3.14) in an N/D channel matrix notation is obtained by approximating $N_{ij}=V_iV_j/(m-E)$ which has no other singularities than a simple pole at the resonant or bound-state mass. Then if $D_{ij}=\delta_{ij}D(E)$, one identifies $P^{-1}(E)=(m-E)D(E)$ and therefore $G(E)=\text{Re}\,D(E)$.

Among the closed channels which are coupled to the resonance, one might expect the $N^*\pi$ channel to be dominant. Another possible channel is nucleon with S-wave $\pi\pi$ interaction, which has a low threshold but a D-wave coupling and is less tractable in any case. The $N^*N^*\pi$ vertex has a P- and F-wave coupling, and the former is expected to be the more important and is retained. Then for the P-wave $N^{*++}N^{*++}\pi^0$ vertex,

$$V_\lambda(E,q,\lambda_t)=g_2\bar{U}^\mu(E,\lambda)\gamma_5U_\mu(q,\lambda_f). \qquad (3.18)$$

This corresponds[18] to a partial decay width of $N^{*++}\to N^*\pi$ of

$$\Gamma_2(E)=(\tfrac{2}{3}+1)\frac{g_2^2}{4\pi}\frac{q_2^3}{E^2}\frac{E^2}{(E+t_2)(E+a_2)}\times\left[1+\left(\frac{2E_f-m_{N^*}}{3m_{N^*}}\right)^2\right], \qquad (3.19)$$

where t_2, a_2 are $m_{N^*}\pm m_\pi$, and E_f is the total energy, in the decay rest frame, of the decay product N^* which has been assumed to have a δ-function mass distribution. This expression has too large a high-energy limit because of the last factor, which corresponds to the crossed threshold for $\pi\to N^*\bar{N}^*$ having S- and D-wave parts. This D-wave component is neglected, so that the factor is replaced by its value at threshold, $E=t_2$:

$$\Gamma_2(E)=\frac{5}{3}\frac{g_2^2}{4\pi}\frac{q_2^3}{E^2}\times\frac{10}{9}\frac{E^2}{(E+t_2)(E+a_2)}. \qquad (3.20)$$

In order to test the sensitivity to the high-energy behavior, the last factor in (3.20), which now corresponds to the S-wave threshold for $N^*\bar{N}^*\pi$ production from vacuum, may also be put equal to its threshold value.

The alternatives to employing a twice-subtracted dispersion relation for $P^{-1}(E)$, and truncating the unstable-particle Born terms to converge, are to modify the Born terms either by a straight cutoff, or by a structure factor such as that introduced in Sec. 2. Then one could use the once-subtracted relation (3.15) and discuss the value of Z needed to fit the data. The structure factors for the inelastic channels would introduce too many parameters, however.

4. COMPARISON OF RESULTS

From (3.14) the correction factor $G(E)$ due to bubble insertions in the propagator is, in case B,

$$G(E)=1+\frac{(m-E)}{\pi}\left(\int_t^\infty-\int_{-t}^{-\infty}\right)\frac{-\frac{1}{2}\Gamma_T(E')dE'}{(m-E')^2(E'-E)}, \qquad (4.1)$$

where there is a principal value at $E'=E$ if $|E|>t$, where t is the lowest threshold, and for unstable particles there is a double principal value at $E'=m$, which is interpreted by

$$\int\frac{f(x)dx}{(m-x)^2}=P\int\frac{f'(x)dx}{x-m}-\left[\frac{f(x)}{x-m}\right]. \qquad (4.2)$$

In case R, mentioned above, one has only the particle cut, and

$$G(E)=1+\frac{(E-m)}{\pi}\int_t^\infty\frac{\frac{1}{2}\Gamma_T(E')dE'}{(m-E')^2(E'-E)}. \qquad (4.3)$$

The effect of including the $N\pi$ contributions given by (1.3) is small and of the wrong sign; however, including $N^*\pi$ contributions given by (3.20) one has a further effect which depends on g_2. In each of the cases B and R, m and g_1 were fixed as in Sec. 2 and g_2 was adjusted to give a value of $G(E)$ that fitted the experimental scattering length at threshold. The resulting expressions for $G(E)$ are shown in Fig. 1 and compare favorably with the data over a wide region. The $N^{*++}N^{*++}\pi^0$ couplings so obtained were 231 and 112, respectively for B and R. As a check, the last factor in (3.20) was set equal to its threshold value and values of 225 and 192 were obtained. Furthermore, a dispersion relation in the s plane with a cut $s>t^2$, which weighted the high-energy contributions differently again, gave results similar to case R. The results in case B are less dependent on the high-energy behavior, since there is an extra power of convergence due to the cancellation between the cuts. However, the effect of the antiparticle cut at twice threshold energy below the physical region for P_{33} is not significant, since the intervening cuts due to u- and t-channel exchange have not been included. The conclusion is that, assuming the vertex

[17] P. W. Coulter and G. L. Shaw, Phys. Rev. **141**, 1419 (1966).
[18] J. G. Ruthbrooke, Phys. Rev. **143**, 1345 (1966).

model which has Born-term threshold factors and diverges as E, and assuming $N^*\pi$ as the dominant closed channel, the $N^{*++}N^{*++}\pi^0$ coupling $g_2^2/4\pi$ will be about 170.

Including the $SU(3)$ analog channels, which are Σ^+K^+, $N^{*++}\eta^0$, $Y_1^{+*}(1385)K^+$ with $SU(3)$ coupling constants but physical thresholds, would necessitate a reduction of $g_2^2/4\pi$ from 170 to 130. Sakita and Wali's[19] relativistic $SU(6)$ scheme predicts $g_2^2/4\pi=75$, while Sutherland[20] using Adler Weisberger techniques finds a result $g_2^2/4\pi=136_{-30}^{+41}$.

The inelasticity η above the $N^*\pi$ threshold in the P_{33} partial-wave amplitude is also predicted on this resonant-dominated model:

$$a_{33}=\frac{\eta e^{2i\delta'}-1}{2i}=\frac{\frac{1}{2}\Gamma_1}{(m-E)G(E)-i(\frac{1}{2}\Gamma_1+\frac{1}{2}\Gamma_2)}, \quad (4.4)$$

$$\eta^2=\frac{(\Gamma_1-\Gamma_2)^2+4G^2(m-E)^2}{(\Gamma_1+\Gamma_2)^2+4G^2(m-E)^2}. \quad (4.5)$$

Using the values for g_2 obtained by fitting G to the shape of the real part of the phase shift near the resonant region, one can calculate η by (4.5) and it is compared in Fig. 2 with the experimental data from the phase-shift analysis of Bareyre *et al.*[8] The agreement is approximately correct and the phase-shift data could have had even more generous errors. The removal of some of the inelasticity to higher threshold channels would improve the prediction, but the data do not warrant this precision. In principle, the nature of the inelasticity in the P_{33} partial wave could be investigated experimentally to check if it were $N^*\pi$.

The phase shift δ' increases anticlockwise by 180° on this model, as experiment suggests; however this does not imply a Castillejo-Dalitz-Dyson (CDD) pole since the phase shift δ returns to zero. It is ϵ and δ that are significant in this resonance model. For the single-channel cases, $N\to N\pi$ or equivalently $N^*\to N^*\pi$, $G(E)$ decreases and changes sign at an energy of several GeV and this corresponds to a phase shift decreasing by π. In the $N^*\to N\pi$ case, $G(E)$ changes sign similarly and corresponds to a phase shift which returns to zero asymptotically. However, in the two-channel model one has a superposition of the two types of behavior and δ' increases by 180° while δ does not. This model circumvents the usual difficulty with a no CDD pole solution, namely that δ turns back too soon, since in our case the second channel pulls it round. The CDD ambiguity arises in the propagator formalism by the possibility of zeros of the propagator which require poles of the inverse. The simplest assumption of no such poles has been assumed in writing (3.14).

As Fig. 1 shows, $G(E)$ has a zero below threshold which will correspond to a pole of a_{33}. This is consistent with the analytic properties assumed for $P^{-1}(E)$ and is simulating the neglected t- and u-channel singularities. The dominant left-hand singularity is thought to be the nucleon-exchange short cut, which in the Chew-Low model contributes a pole[21]

$$a_{33}=\frac{q^3}{E^2}\frac{g^2/4\pi}{3(E-m_N)}, \quad (4.6)$$

where $g^2/4\pi$ is the $NN\pi^0$ coupling. Equating position and residue of this pole with that predicted corresponds to a $NN\pi^0$ coupling of 60 or 29 and nucleon mass of 812 or 905, respectively, for cases B and R. These poles have too large a residue and are too far left, but the effect of the other known singularities would be in this direction. It is particularly interesting that without including inelastic effects no such left-hand poles would have been predicted.

Part of the original reason for considering resonant propagators in this manner was to calculate the effect of the correction $G(E)$ on the exchange diagrams. Form factors at the vertices, or absorption corrections, are needed but if $G(E)$ were between 3 and 10 for the u-channel region this would reduce baryon exchange contributions to the magnitude required. However, as can be seen from Fig. 1, the value of $G(E)$ is not tied down at all accurately except near the s-channel physical region.

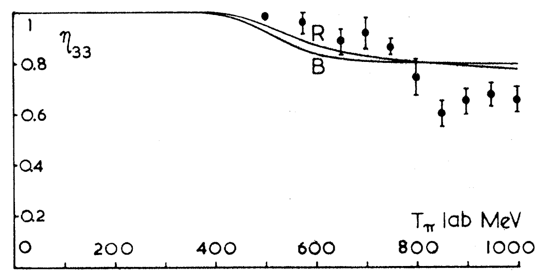

FIG. 2. The inelasticity from Bareyre *et al.* (Ref. 8) is shown by the error bars against pion lab kinetic energy in MeV, and the smooth curves B and R correspond to (4.5) with $g_2^2/4\pi=231$ or 112 and $G(E)$ calculated according to (4.1) or (4.3), respectively.

5. DISCUSSION

The propagator formalism discussed for the $N^*(1236)$ and nucleon in Sec. 3 can be generalized in a natural way to boson and fermion resonances of any spin. The resulting expression factorizes into the incoming and outgoing vertex amplitude and the part common to all channels, the propagator. The spin and threshold dependence of the vertex is most easily accounted for by using the pseudo-Lagrangian method with the spinor or polarization tensor for the resonance constructed as if it were a stable state of mass E. Then the width of the resonance as a function of energy is related

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[20] D. G. Sutherland, Nuovo Cimento **68A**, 188 (1967).
[21] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1494 (1960).

by unitarity to the coupling constant and vertex amplitude. The uncorrected real part of the inverse propagator will be $m-E$ for fermions, and m^2-s for bosons, since the particle and antiparticle contributions to the partial wave have the same parity and can be added in the latter case.

There will be corrections due to background, structure at the decay vertices, and the requirement of an analytic propagator. It is this last correction that will be discussed for different resonances and bound states.

The most straightforward generalization is to the other members of the baryon decuplet. Using $SU(3)$ coupling-constant relations, the effect of channels of the type BP and B^*P on the asymmetries in $Y_1^*(1385)$ and $\Xi^*(1530)$ and the correction to the Ω^- bound state pole in $\Xi\bar{K}$ scattering were obtained. These cannot be checked because elastic-scattering data do not exist; however, one point of interest is that this model is analytic and predicts the position and residue of the complex pole in the lower half-plane. One can then check whether the Gell-Mann–Okubo mass formula is as well satisfied by these predicted pole parameters as by the mass and width defined in Sec. 1.[22] Furthermore, left-hand poles are obtained for each state and may be compared with expected contributions. Thus the pole at 1340 MeV in the Ω^- propagator can be identified with Σ exchange and leads to a $\Xi^0\Sigma^0K^0$ coupling of 8, to be compared with 15 since $SU(3)$ equates this to the $NN\pi^0$ coupling irrespective of the D/F ratio.

For the nucleon, the method of Sec. 3 gives the corrections due to $N\pi$ intermediate states in the propagator and these amount to a reduction by a factor of 1.24 at threshold in P_{11} and by a factor of 6 in S_{11}. The first-order perturbation theory gives scattering lengths respectively 2.1 and -16 times the observed values, so that the disagreement is reduced substantially by the inclusion of $N\pi$ states in the unitarity relation. Exact agreement is not anticipated, since there will be large background effects due to diagrams of the second class. It would be interesting, however, to calculate the nucleon s-channel pole contribution with the effect of all inelastic channels included, since then by subtraction from the observed P_{11} data the resonant structure of this wave could be investigated.

Sufficient data do not exist on other resonant states to check the asymmetries calculated. When e^+e^- colliding-beam experiments are performed, the ρ-meson shape should be well determined and one might usefully relate this to the spectrum of closed channels coupled to this resonance. This would be particularly interesting since the ρ phase shift off mass shell has many applications. In general, the q^{2l+1} factor for the l-wave decay of a resonance does not seem to be supported experimentally, so that we must introduce a structure factor or the propagator correction factor, or even both.

The propagator formalism also provides an alternative method to effective-range theory for discussing S-wave bound states and resonances in an analytic framework. One uses a once-subtracted dispersion relation for P^{-1} with $Z=0$, and as input only the mass and branching ratios are needed since the over-all coupling constant is determined from these. For the deuteron with an S-wave coupling to np, the predicted scattering length is 4.45 F and effective range 0.27 F. The large (1.73 F) experimental effective range presumably comes from the π-exchange contribution.

For the $Y_0^*(1405)$ with S-wave couplings to $\Sigma\pi$ and $\bar{K}N$, the results are again an effective-range prediction for $\bar{K}N$ S-wave scattering. A width of 35 MeV for the Y_0^* gives a scattering length of $1.51-i\,0.51$ F and effective range of 0.39 F. Thus, compared to effective-range theory, one has the bonus of the predicted small positive effective range which is indicated experimentally.[23]

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[22] The author thanks R. H. Dalitz for drawing this point to his attention. For a symmetric N^* resonance, the complex pole would be at $1236-60i$ MeV on the unphysical sheet of the energy plane; however, the analytic propagator model accounts for the observed asymmetry and positions the pole at $1214-52i$ MeV. This decrease of 22 MeV in the real part of the energy is much larger than the corresponding predicted decreases of 5, 1, and 0 MeV for the Y^*, Ξ^*, and stable Ω^-. The equal-spacing rule for the decuplet, which agrees satisfactorily for the masses defined as in Sec. 1, is thus some 15 MeV out of agreement using the complex pole positions. This highlights the difficulty in defining unambiguously the parameters of a broad resonance.

[23] W. Kittel and G. Otter, Phys. Letters **22**, 115 (1966).