# Infinite Multiplets and the Hydrogen Atom*

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Section I of this report deals with attempts to construct infinite-component wave equations with mass spectra that are free of unphysical features. An example is given of a second-order relativistic wave equation whose mass spectrum, both the discrete and the continuous part, is precisely the same as that of the nonrelativistic hydrogen atom. Section II deals with the nonrelativistic hydrogen atom, without any approximations. It is shown that the Schrödinger equation is equivalent to a nonrelativistic analog of Majorana-Nambu wave equations. The Hilbert space of all the states, including the continuum, is profitably and economically used as a representation space for a unitary, irreducible representation of the group $SO(4,2)$. The dipole operator is an $SO(4,2)$ generator. The theory is generalized to arbitrary frames of reference by application of Galilei transformations. The generators of Galilei transformations also belong to the $SO(4,2)$ algebra. What results is a field theory of the hydrogen atom. The exact electromagnetic interaction, including all multipoles, takes the form of a local interaction between the electromagnetic field and the infinite-component hydrogen field. In Sec. III the significance of all this for hadron physics is discussed.

## INTRODUCTION AND SUMMARY

MANY physical ideas that are being applied to elementary-particle physics have been borrowed from the theory of atomic systems, in spite of both conceptual and technical difficulties. Perhaps the most extreme example is the idea[1] that baryons are made up of three quarks, bound together by virtue of a quark-quark "potential," the main difficulty of which is the necessity of postulating a tremendously strong, and completely unknown, potential. In view of the fact that the triton withstood successful theoretical analysis until recently, although the nucleon-nucleon potential was known to a considerable accuracy, one has to admire the power of positive thinking that makes it possible to cull experimental predictions from the quark picture.[2] Conversely, some ideas that have originated in strong interaction physics could profitably be applied to atomic systems, not because of the intrinsic interest of atomic physics, but for the purpose of testing and sharpening the ideas. The Gell-Mann–Okubo mass formula[3,4] has been applied to comparatively small multiplets of elementary particles, and for this reason the precise analytic form of the formula cannot yet be obtained from experiment; thus it is a matter of controversy whether the mass, or the square of the mass, is to be represented by the formula. If it could be shown that the infinite multiplet of levels of the hydrogen atom satisfies a version of the Gell-Mann–Okubo formula, then this would probably be instructive. Another example, about which the same comments can be made, is the notion that weak and electromagnetic currents satisfy certain local commutation relations,[5] or that they are matrix elements of generators of Lie algebras.[6]

The suggestion that elementary-particle theory be tested on atomic systems became particularly attractive when the close analogy between the $SU(6)$ theory of Gürsey and Radicati,[7] and Sakita,[8] on the one hand, and the $SO(4)$ symmetry of the nonrelativistic hydrogen atom[9] on the other, was appreciated.[10,11] However, initial attempts to implement this program failed because of the great contrasts that appeared to distinghish atomic from hadronic physics. First, the "mass formula" for hydrogen does not betray a close relationship to the Gell-Mann–Okubo formula. Second, the idea that electromagnetic currents are somehow associated with generators of a Lie algebra seemed to be contradicted by the facts. According to the original phenomenological approach[12] to the $SO(4,1)$ group of the hydrogen atom, the generators have the property of connecting states with principal quantum numbers $n, n'$ that differ by at most one unit, while the experimental transition amplitudes are appreciable for very large values of $n-n'$.[13]

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[1] M. Gell-Mann, Phys. Letters **8**, 214 (1964); G. Zweig, CERN Report Nos. Th. 401 and 412, 1964 (unpublished).

[2] For a recent summary of many applications of the quark picture, see the talk by R. H. Dalitz, in *Proceedings of the XIIIth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967).

[3] M. Gell-Mann, California Institute of Technology Synchrotron Laboratory Report No. CTSL-20, 1961 (unpublished).

[4] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

[5] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964); Phys. Letters **8**, 214 (1964).

[6] The electromagnetic charge was associated with a generator of isospin [W. Heisenberg, Z. Physik **77**, 1 (1932)]. The strangeness-nonchanging weak currents were associated with the other generators of isospin [R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958)]. Other weak currents have been associated with generators of higher symmetry groups [R. E. Behrends and A. Sirlin, *ibid.* **121**, 324 (1961); and M. Gell-Mann, Ref. 3]. Finally, magnetic moments have been related to $SU(6)$ generators [M. A. B. Bég, B. W. Lee, and A. Pais, Phys. Rev. Letters **13**, 514 (1964)].

[7] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 299 (1964).

[8] B. Sakita, Phys. Rev. **136**, B1756 (1964); Phys. Rev. Letters **13**, 643 (1964).

[9] V. Fock, Z. Physik **98**, 145 (1935).

[10] Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17**, 148 (1965).

[11] C. Fronsdal, in *Proceedings of the Seminar on High-Energy Physics and Elementary Particles, Trieste, 1964* (International Atomic Energy Agency, Vienna, 1964).

[12] A. O. Barut, P. Budini, and C. Fronsdal, Proc. Roy. Soc. (London) **A291**, 106 (1966).

[13] See, e.g., the review article by H. A. Bethe and E. E. Salpeter, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. XXXV, p. 8.

It is our intention to show that the great obstacles that confront attempts to apply theoretical atomic physics to experimental hadron physics, or *vice versa*, can be overcome by recasting both atomic and hadronic physics into the formalism of infinite-component local fields.

Section I of this report deals with attempts to construct infinite-component wave equations with mass spectra that are free of the unphysical features that characterize the first example of such equations: the Majorana theory[14] of 1932. In this theory the mass spectrum has a discrete part, consisting of an infinite number of points that accumulate at $p^2=0$, and a continuous part that is entirely unphysical since for this part $p^2<0$.[15] The discrete part of the spectrum has been improved by Nambu,[16] who obtained hydrogenlike bound-state spectra. An example is given (in Part I, Sec. 4) of a second-order relativistic wave equation whose mass spectrum, both the discrete and the continuous part, is precisely the same as that of the nonrelativistic hydrogen atom.[17]

Section II deals with the nonrelativistic hydrogen atom, without any approximations. It is shown that the Schrödinger equation is equivalent to a nonrelativistic analog of the wave equations studied in Sec. I, and that the methods used there to find the discrete and continuous parts of the mass spectrum yield, respectively, the bound states and the scattering states of the hydrogen atom. The Hilbert space of all the states, including the continuum, is profitably and economically used as a representation space for a unitary, irreducible representation of the group $SO(4,2)$.[18] It turns out that the dipole operator is a generator of the $SO(4,2)$ algebra. The theory, first treated in the hydrogen-atom rest frame, is generalized to arbitrary frames of reference by the application of Galilei transformations. The generators of Galilei transformations also belong to the $SO(4,2)$ algebra. What results is a field theory of the hydrogen atom. It is found that the exact electromagnetic interaction, including all multipoles, takes the form of a local interaction between the electromagnetic field and the infinite-component hydrogen field.

In Sec. III the significance of all this for hadron physics is discussed. It is shown that the quark picture can be transformed into a relativistic theory of infinite-component local fields, and that results derived from the quark picture can be justified even if quarks do not exist. The mass spectrum of the 35-plet of mesons is accounted for by means of a second-order relativistic wave equation. The relevance of the hydrogen atom for the understanding of current algebras is discussed. It is shown that the methods that have been developed for the calculation of hadron from factors in relativistic $SU(6)$ theory[19] can be applied to the hydrogen atom, providing a very short and direct calculation of atomic transition form factors. Finally a relativistic wave equation is presented that, in addition to an entirely physical mass spectrum that coincides with that of the nonrelativistic hydrogen atom, gives hydrogenlike form factors with the correct position of the anomalous threshold singularities, and reduces to the Schrödinger equation in the nonrelativistic limit.

## I. SEARCH FOR EQUATIONS WITH REALISTIC MASS SPECTRA

### 1. The Internal Symmetry Group

Investigations of first- and second-order wave equations for multiplets that transform irreducibly under the spin group $SL(2,C)$ have shown that unphysical mass spectra always result[14,16,20]; hence it becomes necessary to consider reducible representations. If $SL(2,C)$ is embedded in a larger symmetry group $S$, and if the fields are associated with an irreducible representation of $S$, then the reduction with respect to the $SL(2,C)$ subgroup will usually yield an infinite number of irreducible $SL(2,C)$ multiplets, and the fields will transform reducibly with respect to $SL(2,C)$. The reducible representations that are constructed in this way are, of course, of a highly specialized sort. It is worth emphasizing, however, that the level structure of important physical systems are precisely of this nature. This is true of the harmonic oscillator,[21] and it is true of the hydrogen atom.[10,12]

The simplest nontrivial choice of the group $S$ is the 4+1 Lorentz group, $SO(4,1)$. Because the study of representations of this group is greatly facilitated by the use of a five-dimensional notation, it is perhaps necessary to emphasize that the system under discussion is an entirely physical one, confined to a four-dimensional space time. The infinitesimal generators $s_{AB}=-s_{BA}$, $A, B=0, 1, 2, 3, 4$ of $SO(4,1)$ satisfy the following commutation relations:

$$[s_{AB},s_{CD}]=-i(g_{AC}s_{BD}+g_{BD}s_{AC}-g_{BC}s_{AD}-g_{AD}s_{BC}),$$

[14] E. Majorana, Nuovo Cimento **9**, 335 (1932). See also D. M. Fradkin, Am. J. Phys. **34**, 314 (1966). The Majorana theory was rediscovered and generalized by I. M. Gel'fand, A. M. Yaglom, and M. A. Naimark; the work of this group is summarized by M. A. Naimark, in *Linear Representations of the Lorentz Group* (Pergamon Press Ltd., London, 1964).

[15] The existence of the continuous part, with its unphysical spacelike momenta, was pointed out by E. Majorana (Ref. 14) and V. Bargmann [Math. Rev. **10**, 583 (1949)].

[16] Y. Nambu, University of Chicago Report No. EFINS 66-65, July, 1966 (unpublished); and in *Proceedings of the XIIIth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967).

[17] A. O. Barut and H. Kleinert [Phys. Rev. (to be published)] have considered an equation that, besides the hydrogen-like solutions considered by the authors, has additional unphysical solutions.

[18] The role that this group is found to play differs considerably from that assigned to it in Ref. 10.

[19] G. Cocho, C. Fronsdal, H. Ar-Rashid, and R. White, Phys. Rev. Letters **17**, 275 (1966); and Trieste Report No. IC/66/84 (unpublished).

[20] C. Fronsdal, preceding paper, Phys. Rev. **156**, 1653 (1967). For reasons that are explained in that paper, our search for wave equations is always restricted to differential equations.

[21] A. O. Barut, Phys. Rev. **139**, B1433 (1965).

where the metric tensor $g_{AB}$ is a diagonal matrix with $g_{00}=1$, $g_{11}=\cdots=g_{44}=-1$. The subalgebra spanned by the generators $s_{\mu\nu}$, $\mu$, $\nu=0$, 1, 2, 3, is the algebra of $SL(2,C)$; and the generators of Lorentz transformations are[20,22]

$$L_{\mu\nu}=i\left(x_\mu\frac{\partial}{\partial x^\nu}-x_\nu\frac{\partial}{\partial x^\mu}\right)+s_{\mu\nu}.$$

The subalgebra spanned by $s_{ab}$, $a$, $b=1$, 2, 3, 4 generates the maximal compact subgroup $SO(4)$; this group will be loosely associated with the degeneracy group of the nonrelativistic hydrogen atom. Common to both of these subalgebras is the subalgebra of the three-dimensional rotation group; it is spanned by $s_{ij}$, $i$, $j=1$, 2, 3.

### 2. Some Representations of $SO(p,1)$

The representations of the group $SO(4,1)$ have been studied by many authors,[23] but a short description of a special series of representations must be included here for completeness. In order to facilitate comparison with the more familiar representations of $SO(3,1)$ and $SO(2,1)$, it will be convenient to study $SO(p,1)$ for arbitrary (positive integer) values of $p$, although the value $p=4$ is the only one that has actual interest.

An irreducible representation of $SO(p,1)$, suitable for describing the hydrogen atom, belongs to a family of particularly simple representations that may be labeled $D(N)$, where $N$ is a complex number. These representations are unitary if *either*

$$N=-\tfrac{1}{2}(p-1)+i\rho,\quad \rho=\text{real} \tag{I 1}$$

or

$$-(p-1)<N<0. \tag{I 2}$$

The irreducible representation $D(N)$, with $N$ in the range (I 1), may be described analytically as follows. Let $z_A$ denote a set of $p+1$ real variables that are restricted to the cone

$$z^2=z_A{}^2=z_0{}^2-\sum_{a=1}^{p}z_a{}^2=0. \tag{I 3}$$

These variables have nothing whatever to do with space-time coordinates. The carrier space for $D(N)$ is the Hilbert space $H_N$ of all functions $f(z)$ that satisfy the condition of *homogeneity*

$$f(\lambda z)=\lambda^N f(z), \tag{I 4}$$

and of *normalizability*

$$\|f\|^2\equiv(f,f)<\infty, \tag{I 5}$$

where the inner product in $H_N$ is denfied by

$$(f,g)\equiv\int f^*(z)g(z)\delta(z^2)dz. \tag{I 6}$$

It is easy to verify that (I 4) and (I 5) are compatible if and only if $N$ is in the range (I 1).[24] Let $g$ be an element of $SO(p,1)$; that is, $g$ is a real, unimodular, $(p+1)$-dimensional square matrix with the property that $z'=zg$ implies $z'^2=z^2$ for all $z$. Then a unitary irreducible representation of $SO(p,1)$ is obtained by associating with every $g$ in $SO(p,1)$ the linear operator $T_g$ in $H_N$, where

$$T_g: f(z)\rightarrow f(zg). \tag{I 7}$$

The unitarity of this representation is ensured by the invariance and positive definiteness of the inner product (I 6); irreducibility will become obvious later.

An algebraic description of $D(N)$ may be obtained by introducing a basis in $H_N$. A complete orthogonal set of basis functions in $H_N$ is provided by the functions

$$f_{t,\alpha}=z_0{}^{N-t}Y_{t,\alpha}(z_1,\cdots,z_p), \tag{I 8}$$

where the $p$-dimensional spherical polynomials $Y_{t,\alpha}$ are homogeneous of degree $t$ in the $p$ variables. For each fixed value of $t$, the $Y_{t,\alpha}$ span a finite-dimensional subspace that carries an irreducible representation of the compact subgroup $SO(p)$. In the case $p=4$, $\alpha$ is identified with the pair $l$, $m$ of angular-momentum quantum numbers, with the range $m=-l$, $-l+1$, $\cdots$, $l$; and $l=0$, 1, $\cdots$, $t$. The infinitesimal generators $s_{AB}$ of the transformations (I7) are

$$s_{AB}=i\left(z_A\frac{\partial}{\partial z^B}-z_B\frac{\partial}{\partial z^A}\right),\quad A,B,=0,1,\cdots,p. \tag{I 9}$$

Application of one of these operators to one of the basis functions (I 8) yields a finite linear combination of the basis functions, and hence an algebraic representation of the Lie algebra. Examination of this algebraic representation shows that the representatives of $s_{AB}$ are Hermitian[25] not only when $N$ lies in the range (I 1), but for real $N$ in the range (I 2) as well.

In order to exhibit this algebraic structure in some detail, it is convenient to introduce an alternative description of $D(N)$. The following method has already been applied to other Lie algebras.[26] Consider the symmetric and traceless tensor

$$\psi_{A_1\cdots A_N}=z_{A_1}\cdots z_{A_N},\quad A_1,\cdots=0,1,\cdots p. \tag{I 10}$$

For every positive integer value of $N$ this tensor provides a basis for an irreducible, finite-dimensional represenation of $SO(p,1)$. The reduction of this representa-

[22] P. Budini and C. Fronsdal, Phys. Rev. Letters **14**, 968 (1965).

[23] L. H. Thomas, Ann. Math. **42**, 113 (1940); T. D. Newton, *ibid.* **51**, 730 (1949). J. Dixmier, Bull. Soc. Math. France **89**, 9 (1961).

[24] C. Fronsdal, Lectures given at the NATO Advanced Study Institute held in Istanbul, 1966 [W. H. Freeman and Company, San Francisco, California (to be published)].

[25] It would be more accurate to say that the $s_{AB}$ can be made Hermitian by appropriate normalization of the basis vectors. The simplest way to see that this is possible only when $N$ is in the range (I 1) or (I 2) is to examine the invariant (I 15). The requirement is that all the coefficients must be positive.

[26] C. Fronsdal, Trieste Report No. IC/66/51 (unpublished).

tion with respect to the subgroup $SO(p)$ is given by

$$\psi_{A_1\cdots A_N}$$
$$=S\sum_{t=0,1,\cdots}\tilde{\psi}_{A_1\cdots A_t}\sum_{n=t,t+2,\cdots}\frac{(2N+t-n+p-3)!!}{(N+t+p-2)!}$$
$$\times a(N,t,n)g_{A_{t+1}A_{t+2}}\cdots g_{A_{n-1}A_n}$$
$$\times\delta_{A_{n+1}}{}^0\cdots\delta_{A_N}{}^0, \quad \text{(I 11)}$$

$$=S\sum_{t=0,1,\cdots}\tilde{\psi}_{A_1\cdots A_t}\sum_{n=t,t+2,\cdots}\frac{a(N,t,n)}{(n+t+p-2)!!}$$
$$\times\Theta_{A_{t+1}A_{t+2}}\cdots\Theta_{A_{n-1}A_n}\delta_{A_{n+1}}{}^0\cdots\delta_{A_N}{}^0, \quad \text{(I 12)}$$

where $S$ stands for symmetrization in the indices,

$$\Theta_{AB}=g_{AB}-\delta_A{}^0\delta_B{}^0,$$
$$a(N,t,n)=\frac{(2t+p-2)!!N!}{t!(n-t)!!(N-n)!}(-1)^{(n-t)/2},$$

and $\tilde{\psi}_{A_1\cdots A_t}$ is the traceless projection of

$$\psi_{A_1\cdots A_t}=\Theta_{A_1}{}^{B_1}\cdots\Theta_{A_t}{}^{B_t}\psi_{B_1\cdots B_t0\cdots0}.$$

From (I 9), (I 10), and the definition of $\tilde{\psi}_{A_1\cdots A_t}$, it follows that the generators $s_{AB}$ act on these tensors as follows ($a,b=1,\cdots,p$):

$$-is_{ab}\tilde{\psi}_{A_1\cdots A_t}=\sum_{r=1}^{t}(\delta_{aA_r}\tilde{\psi}_{A_1\cdots b\cdots A_t}-\delta_{bA_r}\tilde{\psi}_{A_1\cdots a\cdots A_t}), \quad \text{(I 13)}$$

$$-is_{a0}\tilde{\psi}_{A_1\cdots A_t}=(N-t)\tilde{\psi}_{aA_1\cdots A_t}+\frac{t(N+t+p-2)}{2t+p-2}$$
$$\times S\left(\delta_{aA_1}\tilde{\psi}_{A_2\cdots A_t}-\frac{t-1}{2t+p-4}\delta_{A_1A_2}\tilde{\psi}_{aA_3\cdots A_t}\right). \quad \text{(I 14)}$$

These formulas provide an irreducible representation of $SO(p,1)$ for almost every complex value of $N$, although the derivation is strictly valid for positive integer values only. For positive integer values of $N$, the range of $t$ is from 0 to $N$, but for other values of $N$ there is no upper limit on $t$. The meaning of the number $t$ is exactly the same here as in (I 8); each tensor $\tilde{\psi}_{A_1\cdots A_t}$ carries an irreducible representation of the $SO(p)$ subalgebra, and the components of this tensor are the spherical functions $Y_{t,\alpha}$.

The invariant inner product may be determined in the manner described previously[20] for the case $p=3$, with the result[27]

$$(\psi,\psi')=\sum_{t=0}^{\infty}\frac{(t-N^*-1)!}{(t+N+p-2)!}\frac{(2t+p-2)!!}{t!}$$
$$\times\tilde{\psi}^{*A_1\cdots A_t}\tilde{\psi}'_{A_1\cdots A_t}, \quad \text{(I 15)}$$
$$\tilde{\psi}^{*A_1\cdots A_t}=(\tilde{\psi}_{A_1\cdots A_t})^*.$$

Hermiticity of the operators (I 13) and (I 14) is seen to be satisfied if and only if $N$ is in the range (I 1) or in the range (I 2). Irreducibility of these representations is now obvious. Finally it may be noted that $D(N)$ is equivalent to $D(N')$ if $N+N'=-(p-1)$.

Of special interest in connection with wave equations is the question of the existence of a set of $p+1$ matrices $\Gamma_A$ that transform among themselves in the same way as the $z_A$ (generalized Dirac matrices, or more properly generalized Majorana matrices). At first sight it may appear that the $z_A$ themselves can be interpreted as operators in $H_N$, and hence represented by matrices. This is not the case, however, because $z_Af(z)$ has a different degree of homogeneity, namely $N+1$; thus $z_Af(z)$ is an element of $H_{N+1}$. A special situation arises, however, when $N=-p/2$, for $D(-p/2)$ is equivalent to $D(-p/2+1)$. In this case, matrices $\Gamma_A$ exist and may be determined as follows. From (I 9) and (I 14) the result of applying $s_{a0}$ to $z_0\tilde{\psi}_{A_1\cdots A_t}$ is calculated. If $N=-p/2$, then there exists, because of the equivalence just referred to, a numerical function $\Gamma_0(t)$ such that the tensors $\Gamma_0(t)\tilde{\psi}_{A_1\cdots A_t}$ transform among themselves like the tensors $z_0\tilde{\psi}_{A_1\cdots A_t}$, and a short calculation shows that this function is determined up to a constant factor to be

$$\Gamma_0=t+\tfrac{1}{2}p-1. \quad \text{(I 16)}$$

The other $\Gamma$ matrices may be computed in a similar way, or they may be determined by means of their transformation properties:

$$[s_{a0},\Gamma_0]=i\Gamma_a, \quad a=1,\cdots,p. \quad \text{(I 17)}$$

The result is

$$\Gamma_a\tilde{\psi}_{A_1\cdots A_t}=(t+\tfrac{1}{2}p)\tilde{\psi}_{aA_1\cdots A_t}$$
$$+\tfrac{1}{2}tS\left(\frac{2t+p-4}{2t+p-2}\delta_{aA_1}\tilde{\psi}_{A_2\cdots A_t}-\frac{t-1}{2t+p-2}\delta_{A_1A_2}\tilde{\psi}_{aA_3\cdots A_t}\right). \quad \text{(I 18)}$$

It may be noted that

$$[\Gamma_A,\Gamma_B]=-is_{AB}, \quad \text{(I 19)}$$

that is, the operators $s_{AB}$ and $\Gamma_A$ realize a unitary, irreducible representation of $SO(p,2)$. (Thus, it would be quite appropriate to write $s_{A5}$ instead of $\Gamma_A$.)

[27] The tensor $\psi^{*A_1\cdots A_N}$ transforms like $z^{A_1}\cdots z^{A_N}$, and $\tilde{\psi}^{*A_1\cdots A_t}$ corresponds to $z_0{}^{N-t}z^{A_1}\cdots z^{A_t}$. Consequently, the relationship between the function $f(z)$ that transforms according to (I 7), and the tensors $\tilde{\psi}_{A_1\cdots A_t}$, is given by

$$f(z)=\sum_{t=0}^{\infty}\frac{(t-N^*-1)!}{(t+N+p-2)!}\frac{(2t+p-2)!!}{t!}z_0{}^{N-t}z^{A_1}\cdots z^{A_t}\tilde{\psi}_{A_1\cdots A_t}.$$

### 3. First-Order Wave Equations

The first example of a soluble model, with a mass spectrum that is more reasonable than that of the Majorana theory,[14] was given by Nambu.[16] The equation is[28]

$$(\Gamma_A p_A - \kappa_0)\psi = 0. \tag{I 20}$$

Here $p_A = (p_0, p_1, p_2, p_3, p_4)$ is a set of five real numbers, the first four of which is the energy-momentum four-vector; the number $p_4$ is a fixed constant. The number $\kappa_0$ is another real constant.

Let $n_A$ be any timelike five-vector of unit length ($n_0^2 - n_1^2 - n_2^2 - n_3^2 - n_4^2 = 1$ and $n_0 > 0$), and let

$$\begin{aligned} p_A &= (p^2 - p_4^2)^{1/2} n_A, \\ p^2 &= p_\mu^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2. \end{aligned} \tag{I 21}$$

Let $n_A'$ be another timelike vector of unit length, then there exists a unitary transformation, namely the representative $T_g$ of a rotation $g$ in the $(n_A, n_A')$ plane, such that

$$T_g n_A \Gamma_A T_g^{-1} = n_A' \Gamma_A.$$

Consequently, the eigenvalue spectrum of $n_A \Gamma_A$ is the same for any two timelike $n_A$, and is the same as that of $\Gamma_0$. According to (I 16), with $p = 4$, the eigenvalues of $\Gamma_0$ are the natural numbers, and one concludes that the positive eigenvalues of $p^2 - p_4^2$ are given by[28]

$$p^2 = p_4^2 + \kappa_0^2/n^2, \quad n = 1, 2, \cdots \text{("bound states")}. \tag{I 22}$$

A similar analysis may be applied to the case of spacelike $n_A$. ["Spacelike" in this context means only that $n_0^2 - n_1^2 - n_2^2 - n_3^2 - n_4^2 < 0$, which does not imply that the four-vector $n_\mu = (n_0, n_1, n_2, n_3)$ is spacelike.] In this case $n_A \Gamma_A$ has the same spectrum as $\Gamma_1$, namely, the entire real line. The result is formally the same expression (I 22), except that now $n^2$ is negative,

$$0 > n^2 > -\infty \quad \text{("scattering states")}. \tag{I 23}$$

For integer $n$, the degeneracy of each mass level is $n^2$, which reflects the fact that $p_A \Gamma_A$ is invariant under the subgroup of $SO(4,1)$ of rotations in the four-dimensional plane that is orthogonal to $p_A$. This group is isomorphic to the compact rotation group $SO(4)$, and each set of degenerate states forms an irreducible representation of this group. In the case of "spacelike" $n_A$, the stability group of $p_A \Gamma_A$ is isomorphic to the noncompact group $SO(3,1)$, and each level is infinitely degenerate.

The spectrum (I 22) is "hydrogenlike"; it is the same as the spectrum of the nonrelativistic hydrogen atom, except for the sign of the last term. However, a complete set of solutions of (I 20) includes the "scattering states" (I 23), and for some of these the momentum $p_\mu$ is spacelike. It is not possible to exclude these unphysical states by fiat, for any interesting local interaction, e.g., the minimal electromagnetic coupling, will cause transitions from states of timelike momentum to states of spacelike momentum. In this respect, (I 20) has the same shortcomings as the Majorana theory.[14,15] In addition, if the equation is considered as an attempt to describe the hydrogen atom, then it has another defect. Nambu[16] calculated the electric form factor in this theory and found that it has an anomalous threshold singularity at $t = 8MB$, where $B$ is the binding energy and $M$ is the total mass (i.e., the mass of the hydrogen atom). This is incorrect; the anomalous threshold of hydrogen is at $t = 8\mu B$, where $\mu$ is the reduced mass. This question is discussed in Sec. III.4.

The method we used to solve (I 20) is due to Nambu.[16] It will be applied to a number of similar wave equations whose mass spectra are characterized by an $SO(4)$ degeneracy. It is possible to write down equations that do not possess this symmetry, but not so easy to solve them.

### 4. Second-Order Wave Equations

Nambu,[16] noticing that Eq. (I 20) gave a hydrogenlike spectrum for the bound states, except for the sign of the last term in (I 22), has constructed another example in which the sign is reversed. In order to accomplish this with a first-order wave equation, he introduced a considerable expansion of the multiplet structure. Although this can be justified, at least in part, by reference to the spinning electron, it is unnatural to expect that the spin of the electron is an essential ingredient of a theory of the hydrogen atom. In fact it is possible to eliminate all reference to the spin by "squaring" the first-order wave operator, and obtain a second-order wave equation with the same mass spectrum as the first-order wave equation.

Consider an irreducible representation of $SO(4,1)$ as before, and the second-order equation

$$\{[p_\mu \Gamma_\mu (\Gamma_4 + \alpha)^{-1}]^2 - x^2 p^2 + \kappa_0^2\}\psi = 0, \tag{I 24}$$

where $\alpha$, $x^2$, and $\kappa_0^2$ are real constants. To find the spectrum of the mass is the same as finding the spectrum of the operator

$$Y \equiv p_\mu \Gamma_\mu (\Gamma_4 + \alpha)^{-1}.$$

Thus, let $y$ be a pure number and ask under what conditions there exist solutions of the equation

$$p_\mu \Gamma_\mu - y\Gamma_4 = \alpha y.$$

But this is the same as Eq. (I 20), with $p_4$ replaced by $y$ and $\kappa_0$ replaced by $\alpha y$; therefore,

$$\begin{aligned} &(p^2 - y^2) n^2 = y^2 \alpha^2, \\ \text{and} \quad &y^2 = p^2 n^2 (n^2 + \alpha^2)^{-1}, \end{aligned} \tag{I 25}$$

[28] Notation: From now on, $p = 4$. Indices $A, B, \cdots = 0, 1, 2, 3, 4$; $\mu, \nu \cdots = 0, 1, 2, 3$; $a, b, \cdots = 1, 2, 3, 4$; $i, j, \cdots = 1, 2, 3$. The mass $p_0^2 - p_1^2 - p_2^2 - p_3^2$ is written $p_\mu^2$ or $p^2$.

with

$$n=\begin{cases}1, 2, \cdots, & \text{"bound states"}\\ \text{pure imaginary}, & \text{"scattering states".}\end{cases} \quad \text{(I 26)}$$

Substitution of (I 25) into (I 24) gives the mass spectrum

$$p^2=\kappa_0^2(1+\alpha^2/n^2)(x^2-1+\alpha^2x^2/n^2)^{-1},$$

which agrees with Nambu's result,[16] and which closely approximates the hydrogen bound-states spectrum if $x^2>1$ and $\alpha^2\ll 1$. Unfortunately, this improved spectrum still contains states with spacelike momentum.

A second-order wave equation that yields a positive definite spectrum for $p^2$ is

$$\{2p^2-[\Gamma_4^{-1}(p_\mu\Gamma_\mu-\alpha)]^2-\kappa_0^2\}\psi=0. \quad \text{(I 27)}$$

This may be solved in the same way as the preceding example, with the following exact result:

$$p^2=k_0^2-\alpha^2/n^2. \quad \text{(I 28)}$$

The range of the quantum number $n$ is the same as before, as given by (I 26). The anomalies of the Majorana theory have here been purged completely, but closer examination reveals a fresh one. To diagonalize the mass operator defined by (I 27) is to diagonalize the operator

$$Y\equiv\Gamma_4^{-1}(p_\mu\Gamma_\mu-\alpha).$$

Let $y$ be a pure number and ask under what conditions there exist solutions to the equation

$$p_\mu\Gamma_\mu-y\Gamma_4=\alpha. \quad \text{(I 29)}$$

Certainly it is necessary that $(p^2-y^2)n^2=\alpha^2$, which, when inserted into (I 27) gives the result (I 28). The solutions of (I 29), for positive $p_0$, account for both the bound states and the scattering states of the hydrogen atom in a most precise and complete fashion. However, provided $p^2-y^2<0$, solutions exist for negative $p_0$ as well. This means that, to every scattering state with positive energy, there is associated a similar state with negative energy. But if $p^2-y^2>0$, then (I 29) implies that $p_0$ has the same sign as $\alpha$. Thus scattering states have antistates, but bound states do not—a very droll state of affairs indeed.[29]

Up to now the search for a relativistic wave equation that can be applied to the hydrogen atom has not been completely successful. Nevertheless, it has been demonstrated that the occurrence of solutions with spacelike momentum can be avoided. The continuum in the mass spectrum remains, however, and it is therefore necessary to show that these "scattering states" do in fact have that interpretation. We now proceed to that task. The defect of the wrong position of the anomalous threshold, noted above for Eq. (I 20), is present in the instance of (I 24) as well. The cure for this, and hence the closest approach to a relativistic theory of hydrogen that we have discovered so far, is given in Sec. III.4.

[29] Y. Nambu has suggested that the extra states may be interpreted as proton-positron hole-states (private communication).

## II. REFORMULATION OF THE SCHRÖDINGER THEORY OF THE H ATOM

### 1. The $SO(4)$ Symmetry

The Schrödinger equation for the hydrogen atom in momentum space is[30]

$$\left(\frac{q^2}{2\mu}-E\right)\varphi(q)=\frac{e^2}{2\pi^2}\int\frac{d^3q'\varphi(q')}{(\mathbf{q}-\mathbf{q}')^2}, \quad \text{(II 1)}$$

where $\mu$ is the reduced mass. Fock[9] brought out the hidden $SO(4)$ symmetry of this equation by making the following change of variables:

$$\mathbf{u}=\frac{2q_0\mathbf{q}}{q^2+q_0^2}, \quad u_4=\frac{q^2-q_0^2}{q^2+q_0^2}, \quad \text{(II 2)}$$

where $q_0^2=-2\mu E$. The operator $q_0$ is here treated as a pure number, so that the following is valid when applied to a set of wave functions associated with a fixed, negative value of $E$. The four-vector $u=(\mathbf{u},u_4)$ is of unit length[28]:

$$u^2=u_a{}^2=1.$$

The inverse transformation is

$$\mathbf{q}=q_0\frac{\mathbf{u}}{1-u_4}, \quad q^2=q_0^2\frac{1+u_4}{1-u_4}, \quad \text{(II 3)}$$

and the Jacobian of the transformation is given by

$$d^3q=\tfrac{1}{4}q_0^{-3}(q^2+q_0^2)^3\delta(u^2-1)d^4u. \quad \text{(II 4)}$$

Instead of the Schrödinger wave function $\varphi(q)$, Fock[9] studied the function

$$\hat{\varphi}(u)=\tfrac{1}{4}q_0^{-5/2}(q_0^2+q^2)^2\varphi(q) \quad \text{(II 5)}$$

in terms of which the Schrödinger equation takes a form that clearly exhibits its four-dimensional symmetry

$$\hat{\varphi}(u)=\frac{\mu e^2}{q_0\pi^2}\int\frac{\hat{\varphi}(v)}{(v-u)^2}\delta(v^2-1)d^4v. \quad \text{(II 6)}$$

The correct normalization is given invariantly by

$$1=\int|\varphi(q)|^2d^3q=2\int|\hat{\varphi}(u)|^2\delta(u^2-1)d^4u. \quad \text{(II 7)}$$

Equation (II 6) is soluble if and only if

$$n\equiv\mu e^2/q_0 \quad \text{(II 8)}$$

is a positive integer. For each value of $n$, a complete set of solutions is given by the four-dimensional spherical

[30] In this section we follow closely the notation of M. Bander and C. Itzykson, Rev. Mod. Phys. 38, 330 (1966); 38, 346 (1966).

harmonics $Y_{t,l,m}(u)$, where $t=n-1=0, 1, \cdots$. The eigenvalues of $E$ are given by (II 8):

$$E=-q_0^2/2\mu=-\mu e^2/2n^2. \tag{II 9}$$

The set $Y_{t,l,m}(u)$ of wave functions associated with a single value of $E$ (that is, a fixed, integral value of $t$) form the basis for an irreducible $(t+1)^2$ dimensional representation of $SO(4)$. Naturally, this symmetry is a property of the original form of the Schrödinger equation; it was studied in that form by Bargmann.[31]

### 2. The Role of the Group $SO(4,1)$

The wave functions $Y_{t,l,m}$, for fixed $t$, carry a representation of $SO(4)$. The totality of all the bound-state wave functions carry a representation of $SO(4,1)$.[12] This is an entirely trivial observation that depends solely on the fact that $t$ has the range 0, 1, 2, $\cdots$. For it was shown in Sec. I.2 that the $Y_{t,l,m}$, for fixed $t$, are the components of a tensor $\tilde{\psi}_{A_1\cdots A_t}$, and that a family of irreducible, unitary representations of $SO(4,1)$ could be defined algebraically on these tensors. In fact it is only necessary to know that $SO(4,1)$ has a representation whose reduction according to the compact subgroup contains precisely those representations of $SO(4)$ that are realized by the bound states of the hydrogen atom. However, this phenomenological, "spectrum-generating" interpretation of the role of the $SO(4,1)$ group is not the most interesting one.

It is convenient to return to the discussion of Sec. I.2, specializing to the case $p=4$. For every function $f(z)$, $z=(z_0,z_1,\cdots,z_4)$, introduce a new funcion $h(\xi)$ by writing

$$f(z)=z_0{}^N h(\xi), \tag{II 10}$$

$$\xi=(\xi_1,\cdots,\xi_4), \quad \xi_a=z_a/z_0. \tag{II 11}$$

The new functions $h(\xi)$ form a Hilbert space $H_N'$, with an inner product that may be calculated by direct substitution of (II 10) into (I 6)[32]:

$$(h,k)=2\int h^*(\xi)k(\xi)\delta(\xi^2-1)d^4\xi. \tag{II 12}$$

To find the action of the operators

$$T_g: f(z)\rightarrow f(zg)$$

on the functions $h(\xi)$, notice that the transformation[28]

$$z_A\rightarrow z_B g_{BA}$$

is the same as[28]

$$\xi_a\rightarrow\frac{\xi_b g_{ba}+g_{a0}}{\xi_b g_{b0}+g_{00}}\equiv(\xi*g)_a, \tag{II 13}$$

$$z_0\rightarrow z_0(\xi_b g_{b0}+g_{00}),$$

so that $h(\xi)$ transforms according to the "multiplier representation":

$$T_g: \quad h(\xi)\rightarrow(\xi_b g_{b0}+g_{00})^N h(\xi*g). \tag{II 14}$$

For a pure $SO(4)$ transformation $g_{0a}=g_{a0}=0$, $g_{00}=1$, and the expression on the right simplifies to $h(\xi g)$.

A purely phenomenological approach would suggest that the four variables $\xi_a$ be identified with the Fock variables $u_a$. In that case the spherical harmonics $Y_{t,l,m}(\xi)$, which for each value of $t$ carry an irreducible representation of the compact subgroup $SO(4)$, would be directly associated with the eigenstates of the Hamiltonian. The most important item of new insight that has been gained recently is that this phenomenological approach is not the most profitable one. The correct approach is indicated by a number of independent considerations that will be discussed in Sec. III. In the present context the most relevant point is this. If the $\xi_a$ are identified with the $u_a$, then the relationship between $\mathbf{q}$ and $\xi$ is not just a change of variables, but rather an operator relationship, because it depends on the operator $E$. This has the practical disadvantage that the observables of the Schrödinger theory, and the rules for evaluating their matrix elements, become extremely involved. In addition, it would be necessary to give a completely separate treatment of the scattering states, since the Hilbert space $H_N$ would be exhausted by the bound states. This is one reason for exploring the possibility of a direct geometrical relationship between $\mathbf{q}$ and $\xi_a$.

In (II 2), which defines the Fock variables in terms of $\mathbf{q}$, replace $q_0$ by a constant, and relate $\xi_a$ to $\mathbf{q}$ as follows:

$$\begin{aligned}&\xi_i=2aq_i/(q^2+a^2), \quad i=1,2,3\\ &\xi_4=(q^2-a^2)/(q^2+a^2),\end{aligned} \tag{II 15}$$

or

$$q_i=a\xi_i/(1-\xi_4), \quad q^2=a^2(1+\xi_4)/(1-\xi_4). \tag{II 16}$$

At the same time, in analogy with (II 5), we introduce the function

$$\hat{\varphi}(\xi)=\tfrac{1}{4}a^{-5/2}(a^2+q^2)^2\varphi(q). \tag{II 17}$$

The simplest way to find the functions $f(\xi)$ that corresponds to the physical eigenfunction $Y_{t,l,m}(u)$ is to notice that the relationship between $u$ and $\xi$ can be written [compare Eq. (II 13)]

$$u=\xi*g(q_0), \tag{II 18}$$

where the operation indicated by the asterisk was defined by Eq. (II 13), and $g(q_0)$ is a rotation in the (0,4) plane by the hyperbolic angle

$$\vartheta(q_0)=\ln(q_0/a). \tag{II 19}$$

Therefore, the physical states are related to the basis states $Y_{t,l,m}(\xi)$ by this rotation.

[31] V. Bargmann, Z. Physik **99**, 576 (1936).

[32] This form of the inner product is valid when $N$ is in the range (I 1) only.

The $Y_{t,l,m}(\xi)$ are eigenstates of

$$n_A\Gamma_A, \quad n_A=(1,0,0,0,0).$$

Similarly, the physical states are eigenstates of

$$n_A(q_0)\Gamma_A, \quad n_A(q_0)\equiv(nq(q_0))_A, \qquad \text{(II 20)}$$

where $g(q_0)$ is the rotation in the (0,4) plane that relates the physical states to $Y_{t,l,m}(\xi)$. Using (II 19) we find that the rotated vector is

$$n_A(q_0)=(2aq_0)^{-1}(q_0{}^2+a^2,\ 0,\ 0,\ 0,\ q_0{}^2-a^2). \qquad \text{(II 21)}$$

Consider now the $n^2$ states with principal quantum number $n$; they all have the same energy, so the hyperbolic angle $\vartheta(q_0)$ is the same for all of them. These states form the basis for an irreducible representation of the subgroup of $SO(4,1)$ that leaves the five-vector (II 21) invariant. This group, the stability group of $n_A(q_0)$, is isomorphic to $SO(4)$, and may therefore be called $SO(4)_n$. The subscript is a reminder of the fact that it is a different subgroup of $SO(4,1)$ for a different value of $n$, although all the $SO(4)_n$ are isomorphic to each other.

This description of the bound states of the hydrogen atom, in terms of a family of equivalent compact subgroups, has already been noted.[30,33] However, the physical relevance of this point of view, as an alternative to the purely phenomenological one, first became clear to us when we noticed exactly the same features in the Nambu theory (Sec. I.3). In fact, it may be noted that the five-vector $n_A(q_0)$ plays exactly the same role here as the five-vector $n_A$ defined by Eq. (I 21).

The operator (II 20) is the principal quantum number $n=1, 2, \cdots$. Thus, using (II 21) and (II 8), we obtain

$$(2aq_0)^{-1}[(q_0{}^2+a^2)\Gamma_0-(q_0{}^2-a^2)\Gamma_4]=\mu e^2/q_0$$

or[34]

$$\{[E-(a^2/2\mu)]\Gamma_0-[E+(a^2/2\mu)]\Gamma_4+ae^2\}\psi=0. \qquad \text{(II 22)}$$

This is a nonrelativistic analog of the relativistic wave equations studied in Sec. I. It is completely equivalent to the Schrödinger equation, not only for the bound states, for which the above derivation is valid, but for the scattering states as well. In the next section we give a direct transmutation of the Schrödinger equation, from the usual form as a differential equation in configuration space, to the algebraic form (II 22).

[33] H. Bacry, CERN Report No. TH579, 1965 (unpublished); E. C. G. Sudarshan, N. Mukunda, and L. O'Raifeartaigh, Phys. Letters **19**, 322 (1965). In the use of a five-dimensional representation space we have been anticipated by P. Budini [Nuovo Cimento **44**, 363 (1966)].

[34] Notation: The letter $\varphi$ is used to designate the Schrödinger wave function, whether expressed in terms of the space coordinates $\mathbf{r}$, the internal momentum coordinates $\mathbf{q}$, or Fock coordinates. The letter $\psi$, or $\psi(x)$ or $\psi_\sigma(x)$ may mean the same thing physically, but is used to denote an infinite column vector. The argument of $\varphi$ is always an internal coordinate. The argument of $\psi$ is always an external coordinate, center-of-mass position, or total momentum, the internal degree of freedom being relegated to the index.

### 3. Transformation of the Schrödinger Equation to Algebraic Form

In configuration space, the Schrödinger equation for the hydrogen atom is[35]

$$[E+(1/2\mu)\nabla^2+(e^2/r)]\varphi(\mathbf{r})=0. \qquad \text{(II 23)}$$

The kinetic term is transformed as follows[36]:

$$-\nabla^2=q^2=a^2(1+\xi_4)/(1-\xi_4)$$
$$=a^2(\Gamma_0-\Gamma_4)^{-1}(\Gamma_0+\Gamma_4). \qquad \text{(II 24)}$$

The operator $r_i$, when applied to the function (II 17), is

$$r_i=i[(\partial/\partial q_i)-(2/a)\xi_i],$$
$$=\frac{1}{ia}\left[\xi_i\xi_k\frac{\partial}{\partial\xi_k}+(\xi_4-1)\left(\frac{\partial}{\partial\xi_i}+\xi_i\frac{\partial}{\partial\xi_4}\right)+2\xi_i\right],$$
$$=\frac{1}{ia}\left[-is_{i4}+\xi_i\left(\xi_a\frac{\partial}{\partial\xi_a}+2\right)-\frac{\partial}{\partial\xi_i}\right].$$

When $N=-\frac{1}{2}p=-2$, this reduces to

$$ar_i=s_{i0}-s_{i4}. \qquad \text{(II 25)}$$

Next, one may verify that

$$[s_{AB},s_{AC}]_+ + [\Gamma_B,\Gamma_C]_+=(2-p)g_{BC}, \qquad \text{(II 26)}$$

and thus obtain

$$a\mathbf{r}=\Gamma_0-\Gamma_4. \qquad \text{(II 27)}$$

Inserting (II 24) and (II 27) into (II 23), and multiplying by $\Gamma_0-\Gamma_4$, one gets (II 22). Note that in this new derivation it was not necessary to give a separate treatment of the scattering states.

It may be instructive to solve (II 22) by the method that was used in Sec. II. In fact, if $E<0$, then a rotation in the (0,4) plane brings (II 22) to the form

$$-\{[E-(a^2/2\mu)]^2-[E+(a^2/2\mu)]^2\}^{1/2}\Gamma_0+ae^2=0,$$

or $E=-\mu e^4/2n^2$, $n=1, 2, \cdots$. Similarly, if $E>0$, then another rotation in the (0,4) plane brings (II 22) to the form

$$\{[E+(a^2/2\mu)]^2-[E-(a^2/2\mu)]^2\}^{1/2}\Gamma_1+ae^2=0,$$

or $E=-\mu e^4/2n^2$, $n^2<0$. These are the scattering states.

The electric-dipole interaction may easily be included. Adding the term $e\mathbf{A}\cdot\mathbf{q}/\mu$ to the wave operator in (II 23) one obtains, instead of (II 22),

$$\left[\left(E-\frac{a^2}{2\mu}\right)\Gamma_0-\left(E+\frac{a^2}{2\mu}\right)\Gamma_4+ae^2+\frac{ae}{\mu}\mathbf{A}\cdot\boldsymbol{\Gamma}\right]\psi=0. \qquad \text{(II 28)}$$

[35] Notation: The letters $r$ ($r_i$ or $\mathbf{r}$) and $q$ ($q_i$ or $\mathbf{q}$) always denote internal variables. The letters $x$ (or $x_\mu$) and $p$ (or $p_\mu$) always denote external variables. The letter $k$ is used to denote momentum transfer.

[36] It is evident from the definition and the calculation of the $\Gamma$ matrices in Sec. I.2, that $\Gamma_A{}^{-1}\Gamma_B$ commutes with $\Gamma_C{}^{-1}\Gamma_D$.

Thus it is seen that in this formalism the components of the electric dipole current are the matrix elements of generators of the group $SO(4,2)$! This fact was first discovered by Barut and Kleinert (private communication).

Finally it may be noted that the inner product

$$(\varphi,\varphi')_{\text{phys}}=\int \varphi^*(q)\,\varphi'(q)d^3q \qquad \text{(II 29)}$$

takes the form[36]

$$(\varphi,\varphi')_{\text{phys}}=\int \hat{\varphi}^*(\xi)\Gamma_0^{-1}(\Gamma_0-\Gamma_4)\,\hat{\varphi}'(\xi)d\Omega_\xi$$

$$=\int \hat{\varphi}^*(\xi)(1-\xi_4)\,\hat{\varphi}'(\xi)d\Omega_\xi, \qquad \text{(II 30)}$$

where the volume element on the unit four-sphere is

$$d\Omega_\xi=2\delta(\xi^2-1)d^4\xi.$$

This is the physical inner product, which can be interpreted as a probability amplitude, and with respect to which the Hamiltonian is a self-adjoint operator. It is *not* the same as the $SO(4,1)$ group-invariant inner product, which is given by (I 15) and with respect to which the $\Gamma$ matrices are Hermitian. The relationship is

$$(\varphi,\varphi')_{\text{phys}}=(\psi,r\psi)_{\text{inv}}. \qquad \text{(II 31)}$$

#### 4. Galilei Transformations: Locality

The foregoing treatment of the hydrogen atom in its rest system must be generalized to moving reference frames. From now on the internal degree of freedom will be denoted by an index $\sigma$ (often suppressed), and the argument $p$ in $\psi_\sigma(p)$ will mean the external, or total, energy-momentum four-vector of the entire atom. In particular, $\psi_\sigma$ is the same as $\psi_\sigma(p)$ when $\mathbf{p}=0$ and $p_0=m_p+m_e+E$. Thus, in the absence of any external field $\psi_\sigma$ satisfies (II 22); the problem is to find the equation for $\psi_\sigma(p)$ when $\mathbf{p}\neq 0$.

Consider a Galilei transformation that has the effect of changing the total momentum[35] from $\mathbf{p}$ to $\mathbf{p}+\mathbf{k}$. This transformation of coordinates is represented by a transformation of the wave function

$$\psi_\sigma(p)\rightarrow U_{\sigma\sigma'}{}^{-1}\psi_{\sigma'}(p+k). \qquad \text{(II 32)}$$

The operators $U(k)$ satisfy the structure relations of the Galilei group, e.g., $U(k)U(k')=U(k+k')$, but are otherwise arbitrary. In order to obtain a local field theory it is crucial to choose these operators correctly.

The interaction of the atom with an external electromagnetic radiation field, in the dipole approximation, is given by the term $(ae/\mu)\mathbf{A}\cdot\boldsymbol{\Gamma}$ in (II 28). The formal similarity of this interaction with the Dirac theory suggests that it represents the dipole part of the local, nonderivative interaction

$$(ae/\mu)\psi^*(x)\boldsymbol{\Gamma}\cdot\mathbf{A}(x)\psi(x), \qquad \text{(II 33)}$$

where $\psi_\sigma(x)$ is the Fourier transform of $\psi_\sigma(p)$ and $x_\mu$ is the external position coordinate (center-of-mass system) of the atom. In momentum space this is

$$(ae/\mu)\psi^*(p)\boldsymbol{\Gamma}\cdot\mathbf{A}(k)\psi(p+k). \qquad \text{(II 34)}$$

Let $\mathbf{p}=0$; then (II 32) means that $\psi(p+k)=U(k)\psi$, and (II 34) becomes

$$(ae/\mu)\psi_\sigma{}^*[\boldsymbol{\Gamma}\cdot\mathbf{A}(k)U(k)]_{\sigma\sigma'}\psi_{\sigma'}. \qquad \text{(II 35)}$$

Naturally this gives the right dipole transitions, for in the dipole approximation $U(k)$ is replaced by 1. In order that (II 35) agree exactly with the Schrödinger theory, including all the higher multipoles, it is necessary to put

$$U(k)=e^{-i\mathbf{k}\cdot\mathbf{r}}=e^{-(i/a)k_j(s_{j0}-s_{j4})}. \qquad \text{(II 36)}$$

The internal part (spin part) of the Galilei group is therefore seen to be generated by a subalgebra of $SO(4,1)$, namely, the subalgebra spanned by $s_{ij}$, $i, j=1, 2, 3$ and

$$r_i=(1/a)(s_{i0}-s_{i4}). \qquad \text{(II 37)}$$

*A priori*, there were two reasonable definitions of $U(k)$. If $r$ is interpreted as the distance between the proton and the electron, then $\mathbf{r}$ and $\mathbf{q}$ are invariant under Galilei transformations, and $U(k)=1$. On the other hand, if $\mathbf{r}$ is interpreted as the absolute position coordinate of the electron, which comes to the same so long as only one frame of reference is considered, then under a Galilei transformation $\mathbf{q}\rightarrow\mathbf{q}+\mathbf{k}$ and $U(k)=e^{k_j(\partial/\partial q_j)}=e^{-i\mathbf{k}\cdot\mathbf{r}}$. The first picture, in which $\mathbf{q}$ is invariant, may be compared to Wigner's invariant definition of spin.[37] If the index on the wave function corresponds to this quantity, then one is in the Foldy-Wouthuysen representation,[38] and the electromagnetic interaction is nonlocal and very complicated. The second picture, in which $\mathbf{q}\rightarrow\mathbf{q}+\mathbf{k}$ under Galilei transformations, may be compared with Dirac's[39] definition of the spin. The quantity is directly related to the index on the Dirac four-spinor. In the Dirac representation the electromagnetic interaction is local, nonderivative, and very simple. It is clear that Wigner's definition of the spin is the more convenient phenomenological parameter, and perhaps the interpretation of $r$ as the interatomic distance is the more physical one, but these are not the variables in terms of which local field theory is local.

From (II 32) and (II 36), we can now obtain the wave equation of $\psi_\sigma(p)$. An intermediary step in the calculation is the determination of the transformation law of

[37] E. P. Wigner, Ann. Math. **40**, 149 (1939).
[38] L. Foldy and S. Wouthuysen, Phys. Rev. **78**, 29 (1950).
[39] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, England, 1947).

the $\Gamma$ matrices:

$$e^{-ip\cdot r}\Gamma_i e^{ip\cdot r}=\Gamma_i+(1/a)p_i(\Gamma_0-\Gamma_4), \quad \text{(II 38)}$$

$$e^{-ip\cdot r}\Gamma_4 e^{ip\cdot r}=\Gamma_4+(1/a)\mathbf{p}\cdot\boldsymbol{\Gamma}+(p^2/2a^2)(\Gamma_0-\Gamma_4), \quad \text{(II 39)}$$

$$e^{-ip\cdot r}\Gamma_0 e^{ip\cdot r}=\Gamma_0+(1/a)\mathbf{p}\cdot\boldsymbol{\Gamma}+(p^2/2a^2)(\Gamma_0-\Gamma_4). \quad \text{(II 40)}$$

The final result is[34,35]

$$\{[E-(p^2/2\mu)](\Gamma_0-\Gamma_4)-(a^2/2\mu)(\Gamma_0+\Gamma_4)-(a/\mu)\mathbf{p}\cdot\boldsymbol{\Gamma} +ae^2+(e/\mu)\mathbf{A}\cdot[a\boldsymbol{\Gamma}+\mathbf{p}(\Gamma_0-\Gamma_4)]\}\psi(p)=0.$$

If the quadratic term $(e^2/2\mu)\mathbf{A}^2$ is included, then this may be rearranged to

$$\{[E-(1/2\mu)(\mathbf{p}-e\mathbf{A})^2](\Gamma_0-\Gamma_4)-(a^2/2\mu)(\Gamma_0+\Gamma_4) -(a/\mu)(\mathbf{p}-e\mathbf{A})\boldsymbol{\Gamma}+ae^2\}\psi(p)=0. \quad \text{(II 41)}$$

This equation is the final result of our metamorphosis of the Schrödinger theory. Nothing has been added; Eq. (II 41) is *precisely* equivalent to the Schrödinger equation.

## III. SIGNIFICANCE FOR ELEMENTARY PARTICLES

### 1. Infinite Multiplets Versus Bound States

It has been seen that the theory of the hydrogen atom can be recast into the form of a theory of an infinite-component field $\psi_\sigma(x)$, where $x$ is the center-of-mass coordinate and $\sigma=1, 2, \cdots$ enumerates the "internal" states. This field satisfies the field equation (II 41) or, in the absence of external electromagnetic fields[35]:

$$\left\{\left(E-\frac{p^2}{2\mu}\right)(\Gamma_0-\Gamma_4)-\frac{a^2}{2\mu}(\Gamma_0+\Gamma_4)-\frac{a}{\mu}\mathbf{p}\cdot\boldsymbol{\Gamma}+ae^2\right\} \times\psi(p)=0, \quad \text{(III 1)}$$

where $p$ is the *total* momentum of the hydrogen atom, $\mu$ is the reduced mass, and $a$ is an arbitrary constant. It is expected that (III 1) represents an approximation to a relativistic field equation. Attempts to construct a relativistic $SU(6)$ theory of elementary particles[11] also lead to the description of physical states by the components of infinite-component fields, with field equations and local interactions.[19,20] Under what circumstances can the formalism of multicomponent local fields be interpreted as a formal description of a bound-states system?[40]

Consider a system of two particles; presumably, it can be described by a bilocal field $\psi(x_1,x_2)$. Introduce the center-of-mass coordinate $x$ and the relative coordinate $y$. The internal degree of freedom, represented by the coordinate $y$, can obviously be replaced by a discrete index by introducing a basis

$$\psi(x,y)=\sum_\sigma \psi_\sigma(x)f_\sigma(y), \quad \text{(III 2)}$$

where the $f_\sigma(y)$ are a fixed set of basis functions. Therefore, a system of bound states can always be described as a multicomponent field. The generators of the Lorentz group

$$L_{\mu\nu}=i\left(x_\mu\frac{\partial}{\partial x^\nu}-x_\nu\frac{\partial}{\partial x^\mu}\right)+i\left(y_\mu\frac{\partial}{\partial y^\nu}-y_\nu\frac{\partial}{\partial y^\mu}\right)$$

take the form

$$L_{\mu\nu}=i[x_\mu(\partial/\partial x^\nu)-x_\nu(\partial/\partial x^\mu)]+s_{\mu\nu}, \quad \text{(III 3)}$$

where $s_{\mu\nu}$ is a set of constant matrices. Conversely, let a multicomponent field $\psi_\sigma(x)$ be given that forms a basis for a representation of $P^x\otimes S$, where $P^x$ is the orbital part of the Poincaré group and $S$ contains $SL(2,C)$ as a subgroup. Then, if $S$ is not too large, it is possible to represent the generators of $S$ as differential operators on a space of functions of a real variable $y$. Each component of $\psi_0(x)$ becomes associated with a function of $y$, and every linear superposition of $\psi_\sigma(x)$ is represented by a bilocal field $\psi(x,y)$.

A most important question is that of disassociation. A bilocal field $\psi(x,y)$, that transforms according to a unitary representation of the Lorentz group, always gives rise to an infinite-component field, since the $s_{\mu\nu}$ in (III 3) must be Hermitian and hence infinite. This does not necessarily imply that there is an infinite number of discrete bound states. If the dynamics of the theory, including the mass spectrum, is provided by a field equation, then it is possible to envisage three disparate cases.

A. The mass operator has a discrete spectrum with a finite number of states. Completeness requires the existence of a continuum, which can only be interpreted as multiparticle states, and disassociation must occur.

B. The mass spectrum has a discrete part with an infinite number of states, as well as a continuum. This is realized by the hydrogen atom.

C. The spectrum of the mass operator consists of an infinite set of discrete points, and there is no continuum. In this case disassociation is impossible; in fact, it is meaningless to inquire into the nature of the constituents of the bound system. A remark with similar content is often made: if the potential that binds "quarks" is similar to the potential of a harmonic oscillator, then these objects can never be produced in collisions between ordinary particles, and there is no need to postulate that they are very massive.

If quarks exist, then the formalism of infinite-component local fields offers a description of elementary particles as strongly bound systems, without the need to introduce any nonrelativistic notions or other

[40] Ideas that are very close to those expressed here have been put forward in a series of papers by T. Takabayashi [Progr. Theoret. Phys. (Kyoto) **32**, 981 (1964); **36**, 185 (1966); **36**, 187 (1966), and other references given there].

unjustifiable approximations. If they don't exist, then the possibility of a formulation of elementary-particle physics in terms of infinite-component local fields remains, and the close relationship of this formulation to the bound-state picture helps to explain why the "quark picture" has been successful in accounting for some properties of strong interactions.

### 2. The Mass Spectrum

In view of the apparent contrast between the mass spectra of elementary particles (Gell-Mann–Okubo formula) and that of hydrogen, it is pertinent to recall a successful attempt to describe the mass spectrum of the 35 negative-parity mesons with the aid of a relativistic wave equation.[41] Suppose that these mesons are described by a 35-component field $\phi(x)$, that satisfies a second-order wave equation:

$$(\Gamma^{\mu\nu}p_\mu p_\nu - m^2)\psi = 0. \qquad \text{(III 4)}$$

Here $m^2$ is a constant and the matrices $\Gamma^{\mu\nu}$ transform among themselves under Lorentz transformations in the manner implied by the vector indices. This gives the mass spectrum $p^2=(\Gamma^{00})^{-1}m^2$. Furthermore, suppose that $\Gamma^{00}$ is a function of the generators of $SU(6)$. The physical reason for this postulate is to guarantee that the mass operator contain no mixing between different $SU(6)$ multiplets. Within the framework of static $SU(6)$ theory, there is no way to determine whether a function of $SU(6)$ generators is the (0,0) component of a relativistic tensor, and for this reason it is necessary to appeal to a relativistic generalization of $SU(6)$. Assuming that $SU(6)$ is embedded in the group $SL(6,C)$, which is the smallest group that contains both $SU(6)$ and the spin-group $SL(2,C)$, but making no assumption concerning the representation of $SL(2,C)$ to which the mesons belong, it was found that the most general form of $\Gamma^{00}$ contains five arbitrary parameters. Since there are eight masses to account for—when splittings of isospin multiplets are ignored—this leads to three sum rules. If $\Gamma^{00}$ is expanded as a sum of irreducible $SU(6)$ tensors in the manner of Bég and Singh,[42] then the requirement of relativistic invariance is that the three phenomenological parameters associated with the $(405)_{1,8,27}$ tensors must be equal to the parameters of the $(189)_{1,8,27}$ tensors. The experimental values of the three ratios are $R_1=1.001$, $R_8=1.007$, $R_{27}=0.950$. We feel that this agreement is excellent and that the success of this approach is an encouragement to further exploration of the idea that masses are determined by relativistic wave equations.

### 3. Currents and Generators

If the stationary states of the hydrogen atom were labeled by the eigenvalue of the $SO(4,2)$ generator $\Gamma_0$; that is, if the eigenvalues of $\Gamma_0$ were identified with the principal quantum number $n$, then no $SO(4,2)$ generator could connect states with principal quantum numbers $n$, $n'$ differing by more than one unit. This would rule out the notion that the matrix elements of some of the generators coincide with the dipole transition matrix elements. However, if the states of the atom are defined by the solutions of the wave equation (II 22), which means that the principal quantum number is related to the operator (II 20), then it was found [see (II 28)] that the dipole current can be written

$$d_i=(\varphi,q_i\varphi')_{\text{phys}}=(\psi,a\Gamma_i\psi')_{\text{inv}}. \qquad \text{(III 5)}$$

That is, the dipole-current matrix elements are precisely the matrix elements of the $SO(4,2)$ generators $\Gamma_i$.

Thus it is seen that the simple suggestion of associating currents with generators needs amplification in two respects if it is to be valid for the hydrogen atom. As a direct consequence of the imperfect degeneracy, the physical states must be defined by diagonalizing the mass "operator," and not as eigenstates of some fixed operator such as $\Gamma_0$. If attention is limited to a small number of adjacent levels, then this "representation mixing" is small, especially in a region of fairly large principal quantum numbers. (If the constant $a$ is chosen so that $q_0/a=1$ for the level with principal quantum number $n_0$, then the nearby physical states are nearly exact eigenstates of $\Gamma_0$.) If representation mixing were the only symmetry-breaking effect, then the dipole currents would form a current algebra with commutation relations $[d_i, d_j]=-iL_{ij}$. However, there is another aspect to symmetry breaking, namely, the difference between the two inner products. The physical probability amplitude $(\phi,\phi')_{\text{phys}}$ is not invariant under the transformations of $SO(4,2)$. The representation of $SO(4,2)$ is therefore not unitary with respect to the inner product defined by the probability amplitude, although it is equivalent to a unitary representation. The physical normalization is given by

$$\sum_{\text{states}} \psi_\sigma[\psi^*(\Gamma_0-\Gamma_4)]_{\sigma'}=\delta_{\sigma\sigma'},$$
$$\sum_{\sigma} [\psi^*(\Gamma_0-\Gamma_4)]_\sigma\psi_\sigma=1, \qquad \text{(III 6)}$$

and the dipole currents commute[36]:

$$\langle\Gamma_i\Gamma_j\rangle=(\psi,\Gamma_i\textstyle\sum\psi\psi^*\Gamma_j\psi')$$
$$=(\psi,\ \Gamma_i(\Gamma_0-\Gamma_4)^{-1}\Gamma_j\psi')=\langle\Gamma_j\Gamma_i\rangle.$$

As a "current algebra" the dipole currents of hydrogen are a rather trivial case.

This description of the dipole currents of hydrogen finds a close analogy in the relativistic wave equations that we considered in Sec. I of this report. A simple and instructive example, though it is quite unphysical in some respects, is given by Eq. (I 20). An electromagnetic interaction may be introduced into this

[41] G. Bisiacchi and C. Fronsdal, Acta Phys. Austriaca, Suppl. 3 (1966).
[42] M. A. B. Bég and V. Singh, Phys. Rev. Letters 13, 418 (1964).

theory through the replacement of $p_\mu$ by $p_\mu - eA_\mu$, $\mu=0, 1, 2, 3$. The electric current is then $\psi^*\Gamma_\mu\psi$. This does not mean that transition amplitudes are precisely the matrix elements (I 18) of the $\Gamma$ matrices, which would have implied that $\Delta n=\pm 1$. First, transition amplitudes are defined as matrix elements evaluated between physical (i.e., stationary) states. While these form $SO(4)$ multiplets, each multiplet is defined relative to its own $SO(4)_n$ subgroup, exactly as in the nonrelativistic theory of hydrogen. Second, the physical inner product is not the invariant $\psi^*\psi$, but $\psi^*\Gamma_0\psi$. Here $\Gamma_0$ takes the place of its nonrelativistic analog $\Gamma_0-\Gamma_4$. The current operators, referred to a physical basis, are therefore not $\Gamma_\mu$ but $\Gamma_0^{-1}\Gamma_\mu$, and these commute with each other.[36]

Thus it is seen that, despite appearances, currents in atomic physics are closely related to generators of the algebras, and that the nature of the relationship is Lie, as in the relativistic infinite-component field theory. The application of this new insight to the theory of weak and electromagnetic interactions of the strongly interacting particles in the framework of relativistic $SU(6)$ theory is straightforward.

### 4. Local Interactions, Form Factors, and the Anomalous Threshold

The exact current was found to be [Eq. (II 41)]:

$$J_i(x)=(e/\mu)\psi^*(x)[a\Gamma_i-\tfrac{1}{2}i\overleftrightarrow{\partial}_i(\Gamma_0-\Gamma_4) -eA_i(x)(\Gamma_0-\Gamma_4)]\psi(x). \quad \text{(III 7)}$$

The Yukawa interaction between the hydrogen atom and an external electromagnetic field is therefore a local interaction. That is, the nonrelativistic hydrogen atom is a living example of a type of field theory that has recently been investigated in connection with relativistic $SU(6)$ theory.[19] In fact, it is possible to apply the methods developed there to the calculation of the form factors of the hydrogen atom. To illustrate, we shall evaluate the "atomic factor," that is, the form factor that governs the Coulomb transitions from the ground state to an excited state. If $\mathbf{k}$ is the momentum transfer, then this function is

$$\epsilon_\alpha(k)=\psi_\alpha{}^*(k)^{A_1\cdots A_N}(\Gamma_0-\Gamma_4)\psi_0(0)_{A_1\cdots A_N}, \quad \text{(III 8)}$$

where $\psi_0(0)$ is the tensor for the state $n=1$, at rest, and $\Psi_\alpha(k)$ is the tensor for the state $(n,l,m)=\alpha$, with momentum $\mathbf{k}$. Using (I 11), with the tensor $\delta_A{}^0$ replaced by [see (II 21)]

$$n_A=(2aa_0)^{-1}(a_0{}^2+a^2, 0, 0, 0, a_0{}^2-a^2)$$

and (I 18), we easily obtain (leaving out terms that involve $g_{AB}$, since $\psi_a{}^*(k)^{A_1\cdots A_N}$ is traceless)

$$(\Gamma_0-\Gamma_4)\psi_0(0)_{A_1\cdots A_N} \sim\tilde{\psi}(0)S(\delta_{A_1}{}^0-\delta_{A_1}{}^4)n_{A_2}\cdots n_{A_N}. \quad \text{(III 9)}$$

The other tensor may be expanded with the aid of (I 12),

$$\psi_\alpha{}^*(k)^{A_1\cdots A_N} =2^{n/2}S\tilde{\psi}^{*A_1\cdots A_{n-1}}\sum_{i=n,n+2,\cdots}\frac{i!}{(i+n)!!(i-n)!!} \times(-\Theta^{A_nA_{n-1}})\cdots(-\Theta^{A_{i-2}A_{i-1}}) \times n(k)^{A_i}\cdots n(k)^{A_N}, \quad \text{(III 10)}$$

where the "boosted" spurion is [see (III 1)]

$$n(k)^A=(2aa_0)^{-1}(a_0{}^2+a^2+k^2, 2a\mathbf{k}, a^2-a_0{}^2-k^2). \quad \text{(III 11)}$$

Inserting (III 9) and (III 10) into (III 8) one obtains complete results for the atomic form factors.[43]

The final expression for $\epsilon_\alpha(k)$ depends, essentially, only on the inner product

$$n(k)^An_A\equiv z \quad \text{(say)} \quad \text{(III 12)}$$

of the boosted and the unboosted spurions. Thus, the form factor of the ground state is

$$\epsilon_1(k)=4(z+1)^{-2}. \quad \text{(III 13)}$$

From (III 11) and (III 12),

$$z=1+k^2/4\mu B, \quad \text{(III 14)}$$

where $B=\frac{1}{2}\mu e^4$ is the binding energy. Inserting this into (III 13), we notice the anomalous threshold singularity at $\mathbf{k}^2=-8\mu B$.

The same technique may be used to calculate form factors in the relativistic theories described by the wave equations of Sec. I. In fact, it is possible to take over the results of the nonrelativistic calculation, merely inserting the appropriate spurions. The spurion associated with (I 20) is

$$n(k)^A=(2MB)^{-1/2} \times(k_0, -\mathbf{k}, -(k_0{}^2-k^2-2MB)^{1/2}), \quad \text{(III 15)}$$

where $B=k_0-p_4$ is the "binding energy," and $M$ is the total mass of the system, defined by $2MB=k_0{}^2-k^2-p_4{}^2$ Thus, in this theory (III 14) is replaced by

$$z=\tfrac{1}{2}-t/4MB, \quad t=(k_0-M)^2-k^2\approx -k^2. \quad \text{(III 16)}$$

Inserting this into (III 13) we find a singularity at $t=8MB$,[44] instead of the correct value of $8\mu B$.

Our final contribution is a relativistic wave equation that has both the mass spectrum and the form factors of the hydrogen atom, including the correct position of the anomalous threshold. The equation, a modifica-

[43] The calculation is tremendously shortened, and the form of the answer is more convenient, in comparison with the classical calculation. However, in this day of reliance on computers this is probably not interesting. For the classical calculation see H. S. W. Massey and C. B. O. Mohr, Proc. Roy. Soc. (London) **A132**, 605 (1931).

[44] This result was first obtained in Ref. 16.

tion of (I 24), is[28]

$$\left\{(\Gamma_\mu p_\mu-\mu e^2)\Gamma_4^{-1}(\Gamma_\mu p_\mu-\mu e^2)-\left(1+\frac{\mu}{M}\right)p^2\Gamma_4+\mu M\Gamma_4\right\}\psi=0. \quad \text{(III 17)}$$

Some properties of this equation are:

A. The mass spectrum is

$$\frac{1}{2M}(p^2-M^2)=-\frac{\mu e^4}{2n^2},$$

where $n$ has the discrete spectrum $n=1, 2, 3, \cdots$, and a continuous spectrum $0>n^2>-\infty$. This is precisely the spectrum of hydrogen, including bound states and scattering states, and there are no unphysical solutions.

B. The spurion is

$$n(k)^A=(2\mu B)^{-1/2}(k_0, -\mathbf{k}, -(k_0^2-k^2-2\mu B)^{1/2}).$$

Therefore $z=1-t/4\mu B$, and the anomalous threshold singularity appears at the correct position. We learn from this example that the spatial extension of the particle states is determined by the spurion, and that the latter is not fixed by the mass spectrum.

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# Shape of the $N^*(1236)$ Resonance

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Corrections to the Breit-Wigner shape of the $N^*(1236)$ resonance are calculated using analyticity and inelastic unitarity incorporating the $N\pi$ and $N^*\pi$ channels in a propagator formalism. This method, which includes only a right-hand cut and which evaluates the effect of bubble insertions in the propagator, is motivated by the rigorous results which have been proved for the nucleon. It is argued that even though background and left-hand cuts have been neglected, it is the inclusion of inelasticity that enables the $P_{33}$ phase-shift data to be reproduced, even at energies well above resonance. The structureless-vertex decay model with a Breit-Wigner propagator gives an $N^*$ shape too asymmetric, and inclusion of an inelastic channel with an analytic propagator serves to correct this. Assuming $N^*\pi$ as the only inelastic channel, the $N^{*++}N^{*++}\pi^0$ coupling is estimated as $170\pm50$, where the unknown behavior of the vertices far off the mass shell causes the uncertainty. This estimate can be compared with about 75 from relativistic $SU(6)$, and 136 using Adler-Weissberger techniques. The $P_{33}$ partial-wave amplitude constructed on this model has a left-hand pole which simulates the effect of the neglected nucleon-exchange short cut, and which tends to lie too far left and with too large a residue. The application of the method to other resonances and bound states is discussed.

## 1. INTRODUCTION

THE $N^*(1236)$ resonance is interpreted as the $J^P=\frac{3}{2}^+, I=\frac{3}{2}$ contribution to $\pi N$ scattering, so that the $P_{33}$ phase-shift analyses are the experimental source of data. This is assuming that there is no nonresonant background to the $N^*$ which has the same quantum numbers. The intention is not to calculate the resonance parameters, but to consider the detailed consequences of the resonant behavior of the partial wave from threshold to center-of-mass (c.m.) energies of 1500 MeV or higher. The unstable particle will be treated in a way motivated by the field-theoretic behavior of an off-mass-shell stable particle above threshold. Thus we will try to treat the resonance as a stable particle that has wandered above the threshold.

The general form for the partial-wave amplitude $a_{33}(E)$, deduced from unitarity and the requirement of a phase shift $\delta$ of 90° at the real resonance mass $m$, is

$$a_{33}(E)=\frac{\epsilon(E)}{\cot\delta(E)-i}=\frac{\frac{1}{2}\Gamma_1(E)}{G(E)(m-E)-\frac{1}{2}i\Gamma_T(E)}, \quad (1.1)$$

where $E$ is the c.m. energy, $\epsilon(E)$ is the elasticity, and by definition $G(m)=1$, so that the total width at