same physical consequences,[7] it follows that if the symmetry-breaking condition (6) leads to the appearance of massless excitations,[9] such excitations are also present in the case of the sharp-particle number representation. It is thus convenient to find an analog of condition (6) for the latter case. Such an analog may be achieved by use of the translationally invariant state $|\Omega_2\rangle$, defined by[7]

$$|\Omega_2\rangle=[\varphi_0^*(z)]^{-1}\lim_{V\to\infty}\frac{1}{V}\int_V \psi^\dagger(x)\psi^\dagger(x+z)d^3x|\Omega\rangle. \quad (10)$$

Here, $|\Omega_2\rangle$ is orthogonal to $|\Omega\rangle$, and may be equally acceptable as a ground state. The representation of the sharp-particle number is a reducible representation of the operator algebra.[7] Here,

$$\langle\Omega|[N,\psi(x)\psi(x+z)]|\Omega_2\rangle=-2\langle\Omega|\psi(0)\psi(z)|\Omega_2\rangle\neq 0. \quad (11)$$

Equation (11) now replaces the usual symmetry-breaking condition (6) in proofs of the Goldstone theorem.[10,11]

Fruitful discussions with Professor A. Katz and comments on the manuscript by Dr. S. Brodsky are gratefully acknowledged. The author would also like to thank the U. S. Educational Foundation in Israel for a Fulbright travel grant.

[9] For additional conditions needed to guarantee their appearance, see Ref. 3. Let us mention here the condition $[N,H]=0$, where H is the Hamiltonian. This does not necessarily follow from the invariance of H under finite phase transformations.

[10] Note that although N is the generator of phase transformations on field operators, the states $|\Omega_{2n}\rangle$, $n=1,2,\cdots$ are not eigenstates of N. See Ref. 7.

[11] Reducible representations of the operator algebra were considered in the literature in relation with the Goldstone theorem. However, only ground-state expectation values of the form of Eq. (4) were discussed. The other translationally invariant states appeared only as intermediate states via the completeness relation. See, e.g., A. Klein and B. W. Lee, Phys. Rev. Letters **12**, 266 (1964); W. Gilbert, *ibid.* **12**, 713 (1964); A. Katz and Y. Frishman, Nuovo Cimento **42**, A1009 (1966).

PHYSICAL REVIEW VOLUME 156, NUMBER 5 25 APRIL 1967

S-Matrix Approach to Internal Symmetries*

R. BLANKENBECLER AND D. D. COON†
University of California, Santa Barbara, California

AND

S. M. ROY
University of California, San Diego, La Jolla, California

(Received 1 September 1966)

A criterion for internal symmetry in an S-matrix framework is formulated and a method to determine permitted internal symmetries by a rigorous application of unitarity and crossing is developed. Applying this method to particular cases, one finds amplitude relations that are usually obtained by assuming invariance under certain Lie groups. For example, applications are made to isotopic-spin invariance in the coupled pion-kaon system and to $SU(3)$ invariance in the scattering of the 8 pseudoscalar mesons. Other higher symmetries are discussed. By clarifying the essential assumptions which go into dynamical derivations of symmetries, our method permits more general derivations from weaker assumptions. It is shown that the usual dynamical schemes used to derive internal symmetries essentially assume most of their results and in addition contain many inessential assumptions. For example, the bootstrap approach requires the existence of poles and the bootstrap conditions, whereas our apporach requires neither.

I. INTRODUCTION

THERE have been several attempts to derive internal symmetries within the S matrix framework.[1–7] A common feature of these derivations is that they use unitarity, crossing relations and some highly questionable approximations. In addition, the existence of vector and (pseudo)scalar mesons with the experimental quantum numbers is assumed. These derivations then proceed to show that the PPV coupling constants are in the ratios predicted by an internal symmetry. Therefore, they do *not* derive the most important relations given by such a symmetry which are the energy-

* Based in part on a thesis submitted to Princeton University by one of us (S.M.R.) in partial fulfillment of the requirements for a Ph.D. degree. A substantial part of this work was done while the authors were at Palmer Physical Laboratory, Princeton University, Princeton, New Jersey.

† National Science Foundation predoctoral fellow.

[1] E. Abers, C. Zemach and F. Zachariasen, Phys. Rev. **132**, 1831 (1963); R. H. Capps, Phys. Rev. Letters **10**, 312 (1963); Phys. Rev. **132**, 2749 (1963).

[2] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963); R. E. Cutkosky and P. Tarjanne, *ibid.* **132**, 1354 (1963).

[3] H. M. Chan, P. C. Decelles, and J. E. Paton, Phys. Rev. Letters **11**, 521 (1963); Nuovo Cimento **33**, 70 (1964); H. M. Chan, K. Dietz, and C. Wilkin, *ibid.* **34**, 250 (1964); H. M. Chan, Phys. Letters **11**, 269 (1964).

[4] L. F. Cook and J. E. Paton, Phys. Rev. **137**, B1267 (1965).

[5] J. Franklin, Phys. Rev. **138**, B1202 (1965).

[6] R. C. Hwa and S. H. Patil, Phys. Rev. **140**, B1586 (1965).

[7] D. Y. Wong, *Dispersion Relations and their Connection with Causality* (Academic Press Inc., New York, 1964), p. 68.

independent linear relations among the various scattering amplitudes.

In this paper, our first object is to formulate a criterion for internal symmetry in a purely S-matrix framework and explore its rigorous consequences. In the cases we have examined, we find that this internal symmetry criterion via unitarity and crossing yields the predictions conventionally obtained by assuming invariance under certain Lie groups. This rather striking property of crossing and unitarity may make knowledge of Lie groups inessential (but perhaps convenient) in dealing with strong interactions. Also, the necessity for invoking the bootstrap hypothesis to pick out internal symmetries will be removed.

As we shall see, most of the approximations made in bootstrap derivations of symmetries are not needed and have nothing to do with the result. Our second purpose is therefore to clarify the previous approximate calculations. For example, how could such poor calculations derive *exact* relations among coupling constants? This happy accident will be explained.

We find the consequences of unitarity and crossing for possible internal symmetries without making any approximations. That is, the conservation of charge and strangeness will be assumed and then the possible linear relations among scattering amplitudes will be derived. For example, by considering all two-particle scattering amplitudes between π^+, π^-, and π^0, it will be shown below that the only possible nontrivial internal symmetry is the one corresponding to isotopic spin. A similar situation obtains for the coupled π, K system. Then, the more interesting case of higher internal symmetries will be examined.

In the physically interesting case of $SU(3)$, it will be shown that the essential elements in a dynamical model are the number of (equal-mass) pseudoscalar mesons and the form of the force. The existence of vector mesons is not required.

In Sec. II, the definition of an internal symmetry is proposed for S matrix theory and its consequences are examined. Some theorems necessary for the later discussion are proven. In Sec. III, the example of isotopic spin is given in some detail to explain our approach. In Sec. IV, unitary symmetry is discussed and in Sec. V, the possibility of other higher symmetries is briefly discussed. In Sec. VI, the relation of our approach to the bootstrap calculations is considered in detail. Finally, some concluding remarks are made in Sec. VII.

II. MEANING OF INTERNAL SYMMETRY IN S-MATRIX THEORY

As a prelude, we briefly mention the consequences of an internal symmetry in field theory for the S-matrix elements. An internal symmetry in conventional field theory means that there exist one or more operators acting in the internal quantum number space (like charge, hypercharge, etc.) which commute with the Hamiltonian. It then follows that these operators also commute with the S matrix. Hence, in a basis of the simultaneous eigenfunctions of a complete set of commuting operators chosen from these, the S matrix will be diagonal. The transformation needed to go to this basis is a matrix in the internal quantum-number space and hence independent of the space-time variables. Further, since the Hamiltonian will have the same eigenvalue for every state in a multiplet of such an internal symmetry, the particles corresponding to these states will have the same mass.

We shall now formulate a criterion for internal symmetry based entirely on physically measurable S matrix elements with no reference to field operators. The S matrix for a multichannel scattering process of the type

$$p_1+p_2\rightarrow p_3+p_4 \tag{2.1}$$

for particles with masses m_1, m_2, m_3, and m_4 can be written in the form

$$S=1+2i\rho^{1/2}M\rho^{1/2}, \tag{2.2}$$

where M is the invariant-amplitude matrix, and ρ a diagonal kinematic matrix with diagonal elements

$$\rho_i=q_i/(32\pi^2\sqrt{s}) \tag{2.3}$$

and we have defined

$$s=-(p_1+p_2)^2,\quad t=-(p_1-p_3)^2,\quad u=-(p_1-p_4)^2 \tag{2.4}$$

and q_i is the c.m. momentum in the ith channel. Since the matrix S is unitary and symmetric (assuming time-reversal invariance), there exists a real orthogonal matrix that diagonalizes it.[8] We then see from Eq. (2.2) that the same matrix must diagonalize $\rho^{1/2}M\rho^{1/2}$. *A scattering system will be said to possess an internal symmetry if the real orthogonal matrix diagonalizing the S matrix is independent of energy and angle of scattering.* This is not meant to be a precise or complete definition. For convenience of reference, a symmetry in which none of the amplitudes introduced in the beginning are found to vanish identically will be called "nontrivial." In some problems no nontrivial symmetry exists. A symmetry in which the least number of amplitudes is required to vanish will be called "least-trivial."

It may be noted that while this definition is very close to the field-theory result stated above, it has in its own right the direct physical meaning of a conservation law. We shall also note here another point of similarity with internal symmetry in the field-theory approach. For the examples considered in this paper, the different channels in the scattering matrices which are required to exhibit internal symmetry can be proved to be kinematically identical by using analyticity, crossing, and unitarity. This is analogous to the field-theory

[8] See, for example E. P. Wigner, *Group Theory* (Academic Press Inc., New York, 1959), p. 29.

result that particles in the same multiplet have the same mass.

The conditions that an $n\times n$ scattering matrix $\rho^{1/2}M\rho^{1/2}$ be diagonalized by a constant real orthogonal matrix U yield $\frac{1}{2}n(n-1)$ relations between the elements of M and the parameters of the transformation. For a 2×2 matrix, a convenient parametrization of U is

$$U=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}. \tag{2.5}$$

There is no loss of generality in taking $\det U=1$, because for every diagonalizing U with $\det U=-1$, there is a diagonalizing U with $\det U=+1$ which simply interchanges the eigenstates. The diagonalization condition is

$$M_{12}(st_{12}u_{12})=x[h(s)M_{11}(st_1u_1)-h^{-1}(s)M_{22}(st_2u_2)], \tag{2.6}$$

where

$$h^2(s)=\rho_1(s)/\rho_2(s) \tag{2.7}$$

and

$$x=\tfrac{1}{2}\tan 2\theta. \tag{2.8}$$

Equation (2.6) is a relation between amplitudes evaluated at the same s and scattering angle. When the two channels are kinematically identical the three t's and the three u's are respectively equal and $h(s)=1$. For higher-dimensional matrices we do not yet have any convenient form of the diagonalization conditions like (2.6); however, we will describe in Sec. IV a relatively simple way of handling the case of a 3×3 matrix which can be generalized to any number of channels.

Internal Symmetry and Mass Degeneracy

We shall prove here that for a 2×2 scattering matrix, Eq. (2.6), unitarity, crossing, real analyticity of the amplitudes, and the requirement that M_{12} be nonzero ($M_{12}=0$ means that the matrix is already diagonal) imply that the two channels must be kinematically identical so that $h(s)=1$.

The expression for $h(s)$ is

$$h(s)=\left[\frac{(s-s_1)(s-\Delta_1)}{(s-s_2)(s-\Delta_2)}\right]^{1/4}, \tag{2.9}$$

where

$$s_1=(m_1+m_2)^2,\quad \Delta_1=(m_1-m_2)^2, \tag{2.10}$$
$$s_2=(m_3+m_4)^2,\quad \Delta_2=(m_3-m_4)^2,$$

and particle labels are chosen so that $m_1\geq m_2$ and $m_3\geq m_4$. If the amplitudes M_{11}, M_{12}, and M_{22} are real analytic functions of s, then unitarity implies that they have second-order branch points at s_1. Continuing the amplitudes (for a fixed scattering angle) and $h(s)$ twice around their branch points at s_1 without circling any other branch points of the amplitudes or $h(s)$, leaves the amplitudes unchanged whereas

$$h(s)\to -h(s).$$

This, together with Eq. (2.6), would imply ($s_1\neq s_2$)

$$M_{12}(st_{12}u_{12})=0$$

and

$$x=0, \tag{2.11}$$

or

$$M_{12}(st_{12}u_{12})=0$$

and

$$\rho_1(s)M_{11}(st_1u_1)=\rho_2(s)M_{22}(st_2u_2), \tag{2.12}$$

or

$$1/x=0$$

and

$$\rho_1(s)M_{11}(st_1u_1)=\rho_2(s)M_{22}(st_2u_2). \tag{2.13}$$

The only way to avoid having $M_{12}=0$ when $1/x\neq 0$ is to have $s_1=s_2$. Similar considerations can be applied at the point $s=\Delta_1$. We can then conclude that Δ_1 must equal Δ_2 for $1/x\neq 0$ in order that $M_{12}\neq 0$.

For $1/x=0$, we examine Eq. (2.13) noting that

$$\frac{t_1}{(s-s_1)(s-\Delta_1)}=\frac{t_2}{(s-s_2)(s-\Delta_2)}. \tag{2.14}$$

As t_1 circles any given branch point of M_{11} associated with a threshold in the t channel, s can be chosen so that t_2 will not circle a branch point of M_{22} except when $s_1=s_2$ and $\Delta_1=\Delta_2$. Therefore, if we are to have a symmetry with $1/x=0$ and not forbid t-channel thresholds we must have $s_1=s_2$ and $\Delta_1=\Delta_2$ as was the case for $1/x\neq 0$. Thus, we conclude that

$$m_1=m_3,\quad m_2=m_4, \tag{2.15}$$

and

$$h(s)=1. \tag{2.16}$$

Eigenamplitude Theorems

We now state two theorems which have application, in virtually every problem where the symmetry postulate is applied, and which will be referred to as eigenamplitude theorems.

If three eigenamplitudes A_i satisfy the linear relation:

$$\sum_{i=1}^{3} c_iA_i=0, \tag{2.17}$$

where

$$\sum_{i=1}^{3} c_i=0 \tag{2.18}$$

and the c_i are nonzero constants, then the three eigenamplitudes are equal. The theorem is proved by expressing A_1 in terms of A_2 and A_3 using (2.17) and substituting this expression in the (elastic) unitarity equation for A_1. It is then seen that (2.18) plus unitarity for A_2 and A_3 imply $A_2=A_3$. Using (2.17) and unitarity again, one sees that $A_1=A_2=A_3$.

If n eigenamplitudes A_i satisfy the linear relation:

$$\sum_{i=1}^{n} c_iA_i=0, \tag{2.19}$$

where

$$\sum_{i=1}^{n} c_i = 0 \tag{2.20}$$

and the c_i are nonzero constants, and if all the c_i except one have the same sign, then all the A_i are equal. Any other possibility for the signs of the c_i will not force all the A_i to be equal, even though it is obvious that having all A_i equal will always satisfy Eqs. (2.19) and (2.20).

III. ISOTOPIC-SPIN EXAMPLE

In this section, we examine the possible symmetries in π-π elastic scattering as a simple example of our general method and then extend the problem to include scattering of pions and kaons. Before applying the symmetry postulate, we will demonstrate that the consequences of Bose statistics, charge-conjugation invariance and time-reversal invariance for π-π elastic amplitudes are already contained in the crossing relations for these amplitudes. Table I contains a list of the crossing relations. Amplitudes which are odd(even) under exchange of the last two kinematic invariants appearing as arguments have odd(even) indices.

Identical particle factors of $2(\sqrt{2})$ are introduced when particles in both (either) the incoming and (or) outgoing states are identical. In summing over intermediate states in unitarity relations, we must sum only over distinct physical states and this introduces factors of $\frac{1}{2}$ into the angular integration over intermediate states with identical particles. The amplitudes A_i are defined so that they will satisfy the usual two-particle unitarity equations with no extra factors of $\frac{1}{2}$.

Consider the first t-channel crossing relation in the table. The amplitude for the process $\pi^+\pi^+ \rightarrow \pi^+\pi^+$ is related by crossing to the amplitude for the process $\pi^+\pi^- \rightarrow \pi^-\pi^+$, which in turn is related by a 180° rotation of the outgoing π^- and π^+ to the amplitude for the process $\pi^+\pi^- \rightarrow \pi^+\pi^-$. The 180° rotation is responsible for the minus sign in front of the odd part (A_5) of the amplitude in the t channel. Similar considerations apply to the rest of the table.

The first u-channel crossing relation reads

$$2(A_1+A_2)(stu) = (A_5+A_6)(uts). \tag{3.1}$$

Exchanging t and u yields

$$2(A_1+A_2)(sut) = (A_5+A_6)(tus), \tag{3.2}$$

and by definition of odd and even amplitudes

$$2(-A_1+A_2)(stu) = (-A_5+A_6)(tsu). \tag{3.3}$$

However, the first t-channel crossing relation says that

$$2(A_1+A_2)(stu) = (-A_5+A_6)(tsu). \tag{3.4}$$

Therefore, $A_1=0$ as it should, since the pions are bosons. From the t-channel crossing relations, we see by inspection that

$$A_1+A_2 = A_{13}+A_{14} \tag{3.5}$$

TABLE I. Amplitudes and crossing relations for the elastic scattering of pions. Factors of 2 and $\sqrt{2}$ are included so that all the A_i will satisfy the usual two-particle unitarity relations. The last three processes are related to the others by charge conjugation and time reversal. It is pointed out in the text that the crossing relations contain all the consequences of Bose statistics and invariance under charge conjugation and time reversal.

	(stu)		(tsu)		(uts)
$\pi^+\pi^+ \rightarrow \pi^+\pi^+$	$2(A_1+A_2)$	$=$	$(-A_5+A_6)$	$=$	(A_5+A_6)
$\pi^+\pi^0 \rightarrow \pi^+\pi^0$	(A_3+A_4)	$=$	$\sqrt{2}(A_7+A_8)$	$=$	(A_3+A_4)
$\pi^+\pi^- \rightarrow \pi^+\pi^-$	(A_5+A_6)	$=$	(A_5+A_6)	$=$	$2(A_1+A_2)$
$\pi^+\pi^- \rightarrow \pi^0\pi^0$	$\sqrt{2}(A_7+A_8)$	$=$	(A_3+A_4)	$=$	$(-A_3+A_4)$
$\pi^0\pi^0 \rightarrow \pi^0\pi^0$	$2(A_9+A_{10})$	$=$	$2(A_9+A_{10})$	$=$	$2(A_9+A_{10})$
$\pi^-\pi^0 \rightarrow \pi^-\pi^0$	$(A_{11}+A_{12})$	$=$	$\sqrt{2}(-A_7+A_8)$	$=$	$(A_{11}+A_{12})$
$\pi^-\pi^- \rightarrow \pi^-\pi^-$	$2(A_{13}+A_{14})$	$=$	$(-A_5+A_6)$	$=$	(A_5+A_6)
$\pi^0\pi^0 \rightarrow \pi^+\pi^-$	$\sqrt{2}(A_{15}+A_{16})$	$=$	$(A_{11}+A_{12})$	$=$	$(-A_3+A_4)$

or, since odd and even parts must be separately equal,

$$A_1 = A_{13}$$

and

$$A_2 = A_{14}, \tag{3.6}$$

which is an expression of invariance under charge conjugation. Proceeding similarly, one can find all the consequences of Bose statistics, charge-conjugation invariance, and time-reversal invariance for these reactions.

The crossing relations for the remaining independent amplitudes are

$$2A_2(stu) = (-A_5+A_6)(tsu), \tag{3.7}$$

$$(A_3+A_4)(stu) = \sqrt{2}A_8(tsu) = (A_3+A_4)(uts), \tag{3.8}$$

$$(A_5+A_6)(stu) = (A_5+A_6)(tsu) = 2A_2(uts), \tag{3.9}$$

$$\sqrt{2}A_8(stu) = (A_3+A_4)(tsu), \tag{3.10}$$

$$2A_{10}(stu) = 2A_{10}(tsu). \tag{3.11}$$

The u-channel relations which provide no extra information have been omitted.

To search for possible symmetries, we apply the symmetry postulate (2.6) to the zero-charge, even-parity, 2×2 matrix involving A_6, A_8, and A_{10}. For a nontrivial internal symmetry, the masses of the π^+ and the π^0 must be the same. The diagonalization condition (2.6) therefore becomes

$$A_8(stu) = x(A_6-A_{10})(stu). \tag{3.12}$$

Using the same manipulations as in Eqs. (3.1)–(3.4), we find

$$A_6(stu) = (\tfrac{1}{2}A_5+\tfrac{1}{2}A_6+A_2)(tsu), \tag{3.13}$$

$$A_8(stu) = \tfrac{1}{2}\sqrt{2}(A_3+A_4)(tsu), \tag{3.14}$$

and

$$A_{10}(stu) = A_{10}(tsu). \tag{3.15}$$

Substituting (3.13)–(3.15) in Eq. (3.12), exchanging s and t, and separating odd and even parts yields

$$A_3(stu) = (x/\sqrt{2})A_5(stu), \tag{3.16}$$

$$A_4(stu) = x\sqrt{2}(\tfrac{1}{2}A_6+A_2-A_{10})(stu). \tag{3.17}$$

Equation (3.16) relates two amplitudes which satisfy elastic unitarity in a finite energy region. If neither amplitude is to be zero, then the constant relating them must be unity. This leads to $x=\sqrt{2}$, and is the first of four possibilities:

$$x=\sqrt{2},\quad A_3=A_5\neq 0, \tag{3.18}$$

$$x=0,\quad A_3=0,\quad A_5\neq 0, \tag{3.19}$$

$$1/x=0,\quad A_5=0,\quad A_3\neq 0, \tag{3.20}$$

$$x \text{ undetermined},\quad A_3=A_5=0. \tag{3.21}$$

Thus, we see that crossing and unitarity have fixed the value of x for the least trivial cases.

In the first or nontrivial case in which $x=\sqrt{2}$, Eq. (3.17) becomes

$$A_4=A_6+2A_2-2A_{10}. \tag{3.22}$$

Since a knowledge of x is equivalent to a knowledge of the matrix which diagonalizes the scattering matrix, the eigenamplitudes are easily seen to be

$$\lambda_6\equiv 2A_6-A_{10}, \tag{3.23}$$

$$\lambda_{10}\equiv 2A_{10}-A_6. \tag{3.24}$$

Equation (3.22) now becomes

$$2A_2=A_4+\lambda_{10}, \tag{3.25}$$

which meets the requirements of the three eigenamplitude theorem (Sec. II). Therefore,

$$A_2=A_4=\lambda_{10}. \tag{3.26}$$

This equality of eigenamplitudes is just that which results from invariance under rotations in isotopic-spin space. The three amplitudes in Eq. (3.26) are the $I=2$, $I_z=2$, 1, 0 amplitudes. Also, from Eq. (3.18), we have the equality of the $I=1$, $I_z=1$, 0 amplitudes. The eigenamplitude λ_6 which appeared in the same 2×2 matrix with λ_{10} is the $I=0$ amplitude, and we find no relation between this eigenamplitude and any of the others. Furthermore, since we know the diagonalizing matrix (to within certain phases), we know what states form the basis in which the scattering matrix is diagonal. This is equivalent to knowing the isotopic-spin Wigner coefficients.

Thus, all the predictions of invariance under rotations in isotopic-spin space for elastic scattering of pions have been found.

It should be noted that in order to fix x, all we needed was that the elastic unitarity relations apply in some finite energy region near threshold. Because of this, the calculation performed here remains the same so long as any additional inelastic channels have thresholds above the elastic threshold. Since amplitudes are analytic functions, the amplitude relations calculated in the elastic region can be continued and hence will be the same above the inelastic thresholds.

If we look for amplitude relations corresponding to a nontrivial internal symmetry in the elastic scattering of kaons, neglecting coupling to other particles, we get only a few of the isotopic-spin predictions. However, kaons are coupled to pions (as well as other particles), and since the pion is less massive, the unitarity relations for $K\bar{K}$ amplitudes should include $\pi\pi$ channels below the $K\bar{K}$ threshold. It will be assumed, for simplicity, that $K\bar{K}$ is the first inelastic channel to open up in $\pi\pi$ scattering, and that Bose statistics, charge conjugation invariance and time-reversal invariance are valid. Then possible symmetries in the pion-kaon system will be found. In addition to the seven $\pi\pi\to\pi\pi$ amplitudes, we now have the $\pi\pi\to K\bar{K}$, $KK\to KK$, $K\bar{K}\to K\bar{K}$, and the $K\pi\to K\pi$ amplitudes plus their crossing relations as listed in Table II. The equality of pion masses among themselves and the equality of kaon masses follow immediately from the diagonalization postulate via the arguments in Sec. II.

We will now show that the existence of an internal symmetry in the $\pi\pi$-$\pi\pi$ and $K\pi$-$K\pi$ amplitudes leads to amplitude relations given by isotopic spin symmetry for the whole πK system.[9] We will see that the symmetries in the $\pi\pi$ and $K\pi$ channels fix the symmetry in the KK and $K\bar{K}$ channels without further symmetry postulates. The secret, of course, is crossing, and the unitarity relations involving coupled-channel amplitudes.

The internal symmetry or diagonalization conditions we assume are

$$\pi\pi\text{-}\pi\pi:\quad A_8=x_1(A_6-A_{10}), \tag{3.27}$$

$$K\pi\text{-}K\pi:\quad D_5=x_2(D_3-D_7), \tag{3.28}$$

$$D_6=x_3(D_4-D_8), \tag{3.29}$$

$$D_{11}=x_4(D_9-D_{13}), \tag{3.30}$$

$$D_{12}=x_5(D_{10}-D_{14}). \tag{3.31}$$

The implications of Eq. (3.27) have already been found. We note that for $m_K\neq m_\pi$ the amplitudes with odd (even) indices are odd (even) with respect to $\cos\theta\to-\cos\theta$ but not with respect to $t\leftrightarrow u$. Equation (3.28) yields via the crossing relations

$$\begin{aligned}&[-B_1+B_2-x_2(\sqrt{2}B_6+B_7-B_8)](stu)\\&=[D_{11}+D_{12}\\&\quad-x_2(D_3+D_4-D_{15}-D_{16})]\left(s+\frac{s_0\Delta}{t},\,u-\frac{s_0\Delta}{t},\,t\right),\end{aligned} \tag{3.32}$$

where

$$\Delta=(m_K-m_\pi)^2 \quad\text{and}\quad s_0=(m_K+m_\pi)^2.$$

The discontinuity with respect to s (for any given value of t) of each side of the equation must be separately equal to zero because of the unequal-mass factors appearing in the arguments. From the odd partial-wave projection of the discontinuity of the left-hand side of

[9] The only reservation is the difficulty of separation of Eq. (3.64) discussed at the end of this section.

TABLE II. Amplitudes and necessary crossing relations for the π, K system.

	(*stu*)	(*tsu*)	(*uts*)
$K^+\bar{K}^0 \to \pi^+\pi^0$	B_1+B_2		
$K^+K^- \to \pi^+\pi^-$	B_3+B_4		
$K^+K^- \to \pi^0\pi^0$	$\surd 2 B_6$		
$K^0\bar{K}^0 \to \pi^+\pi^-$	B_7+B_8		
$K^0\bar{K}^0 \to \pi^0\pi^0$	$\surd 2 B_{10}$		
$K^+K^+ \to K^+K^+$	$2C_2$	$= -C_9+C_{10} =$	C_9+C_{10}
$K^+K^0 \to K^+K^0$	C_3+C_4	$= -C_{11}+C_{12} =$	C_7+C_8
$K^0K^0 \to K^0K^0$	$2C_6$	$= -C_{13}+C_{14} =$	$C_{13}+C_{14}$
$K^+\bar{K}^0 \to K^+\bar{K}^0$	C_7+C_8	$= C_{11}+C_{12} =$	C_3+C_4
$K^+K^- \to K^+K^-$	C_9+C_{10}	$= C_9+C_{10} =$	$2C_2$
$K^+K^- \to K^0\bar{K}^0$	$C_{11}+C_{12}$	$= C_7+C_8 =$	$-C_3+C_4$
$K^0\bar{K}^0 \to K^0\bar{K}^0$	$C_{13}+C_{14}$	$= C_{13}+C_{14} =$	$2C_6$
$K^+\pi^+ \to K^+\pi^+$	D_1+D_2	$= -B_3+B_4 =$	D_9+D_{10}
$K^+\pi^0 \to K^+\pi^0$	D_3+D_4	$= \surd 2 B_6 =$	D_3+D_4
$K^+\pi^0 \to K^0\pi^+$	D_5+D_6	$= -B_1+B_2 =$	$D_{11}+D_{12}$
$K^0\pi^+ \to K^0\pi^+$	D_7+D_8	$= -B_7+B_8 =$	$D_{15}+D_{16}$
$K^+\pi^- \to K^+\pi^-$	D_9+D_{10}	$= B_3+B_4 =$	D_1+D_2
$K^+\pi^- \to K^0\pi^0$	$D_{11}+D_{12}$	$= B_1+B_2 =$	D_5+D_6
$K^0\pi^0 \to K^0\pi^0$	$D_{13}+D_{14}$	$= \surd 2 B_{10} =$	$D_{13}+D_{14}$
$K^0\pi^- \to K^0\pi^-$	$D_{15}+D_{16}$	$= B_7+B_8 =$	D_7+D_8

Eq. (3.32), we obtain

$$\rho_1(b_1{}^*a_3+x_2b_7{}^*a_5)=0. \quad (3.33)$$

But we have already derived the relation $a_3=a_5$, and since $a_3\neq 0$, we obtain

$$B_1+x_2B_7=0. \quad (3.34)$$

We repeat that we frequently use analyticity to continue any relation between analytic functions valid over a finite region of the right-hand cut into the whole complex plane. Using crossing on the diagonalization Eq. (3.29), we obtain

$$[-B_1+B_2-x_3(\surd 2B_6+B_7-B_8)](stu)$$
$$=-[D_{11}+D_{12}$$
$$-x_3(D_3+D_4-D_{15}-D_{16})]\left(s+\frac{s_0\Delta}{t},\, u-\frac{s_0\Delta}{t},\, t\right). \quad (3.35)$$

Using the same argument which led to Eq. (3.34), we find that

$$B_1+x_3B_7=0. \quad (3.36)$$

Assuming $B_7\neq 0$, Eq. (3.34) and (3.36) yield

$$x_2=x_3. \quad (3.37)$$

But then, the left-hand sides of the Eqs. (3.32) and (3.35) become identical, and the right-hand sides become negatives of each other. Hence, both sides of the two equations are equal to zero. Setting odd and even parts separately equal to zero we obtain

$$D_{11}=x_2(D_3-D_{15}), \quad (3.38)$$

$$B_2=x_2(\surd 2B_6-B_8), \quad (3.39)$$

$$D_{12}=x_2(D_4-D_{16}). \quad (3.40)$$

By entirely analogous arguments we obtain from Eqs. (3.30) and (3.31), if $B_3\neq 0$,

$$B_1=x_4B_3, \quad (3.41)$$

$$D_5=x_4(D_1-D_{13}), \quad (3.42)$$

$$x_5=x_4, \quad (3.43)$$

$$B_2=x_4(B_4-\surd 2B_{10}), \quad (3.44)$$

$$D_6=x_4(D_2-D_{14}). \quad (3.45)$$

The unitarity relations for the $\pi^+\pi^+\to\pi^+\pi^+$ and $\pi^+\pi^0\to\pi^+\pi^0$ amplitudes give, using $A_2=A_4$,

$$B_2=0. \quad (3.46)$$

We will now find a solution with *no other* amplitude equal to zero. Equations (3.39), (3.44), and (3.46) then imply

$$B_8=\surd 2B_6, \quad (3.47)$$

$$B_4=\surd 2B_{10}. \quad (3.48)$$

Assuming the previously derived internal symmetry relations between the $\pi\pi\to\pi\pi$ amplitudes and assuming $a_8\neq 0$, the unitarity relations for the partial-wave amplitudes b_6 and b_8 applied below the $K\bar{K}$ threshold yield

$$B_4=B_8. \quad (3.49)$$

Applying the crossing relations to Eqs. (3.46)–(3.49), we deduce that

$$D_3=D_{13},\quad D_4=D_{14},\quad D_3=\tfrac{1}{2}(D_7+D_{15}),$$
$$D_4=\tfrac{1}{2}(D_8+D_{16}),\quad D_3=\tfrac{1}{2}(D_1+D_9),$$

and

$$D_4=\tfrac{1}{2}(D_2+D_{10}),\quad D_5=-D_{11},\quad D_6=-D_{12}. \quad (3.50)$$

Use of Eqs. (3.28) and (3.42) and the relation $D_3=D_{13}$ leads to

$$(x_2+x_4)D_3=x_4D_1+x_2D_7. \quad (3.51)$$

Defining $D_3{}^{(e)}$ and $D_7{}^{(e)}$ to be the eigenamplitudes obtained by diagonalizing the amplitude matrix with elements D_3, D_5, and D_7, and defining θ_2 to be the mixing angle of the diagonalizing matrix we have

$$(x_2+x_4)(D_3{}^{(e)}\cos^2\theta_2+D_7{}^{(e)}\sin^2\theta_2)$$
$$=x_4D_1+x_2(D_3{}^{(e)}\sin^2\theta_2+D_7{}^{(e)}\cos^2\theta_2). \quad (3.52)$$

We note that the conditions for validity of the eigenamplitude theorem are satisfied. Ruling out the alternative $D_3{}^{(e)}=D_7{}^{(e)}=D_1$ which requires $D_5=0$, we conclude that the coefficients of at least one of the eigenamplitudes $D_3{}^{(e)}$ and $D_7{}^{(e)}$ must be zero when Eq. (3.52) is written in the standard form of the eigenamplitude theorem. Hence,

$$(x_2+x_4)\cos^2\theta_2-x_2\sin^2\theta_2=0,\quad D_7{}^{(e)}=D_1 \quad (3.53)$$

or

$$(x_2+x_4)\sin^2\theta_2-x_2\cos^2\theta_2=0,\quad D_3{}^{(e)}=D_1. \quad (3.54)$$

From (3.50) we also have

$$D_{15}=2D_3-D_7$$
$$=2[D_3^{(e)}\cos^2\theta_2+D_7^{(e)}\sin^2\theta_2]$$
$$-[D_3^{(e)}\sin^2\theta_2+D_7^{(e)}\cos^2\theta_2]. \quad (3.55)$$

Using the eigenamplitude theorem and $D_5\neq 0$, we obtain

$$x_2=\pm\sqrt{2}. \quad (3.56)$$

Similarly, from

$$D_1=2D_3-D_9$$
$$=2D_{13}-D_9$$
$$=2[D_9^{(e)}\sin^2\theta_4+D_{13}^{(e)}\cos^2\theta_4]$$
$$-[D_9^{(e)}\cos^2\theta_4+D_{13}^{(e)}\sin^2\theta_4], \quad (3.57)$$

we obtain

$$x_4=\pm\sqrt{2}. \quad (3.58)$$

From (3.56), $\tan^2\theta_2$ can be 2 or $\frac{1}{2}$. Hence, from (3.53) and (3.54), $(x_2+x_4)/x_2$ can be 2 or $\frac{1}{2}$; but for (3.58) also to be true, we must have

$$x_2=x_4=\pm\sqrt{2}. \quad (3.59)$$

The ambiguity in sign in Eq. (3.59) corresponds to the fact that we have not specified the phase of the $K^+\pi^0$ state with respect to the $K^0\pi^+$ state and not to any real ambiguity in the physical predictions. We now turn to derive the relations between the KK-KK and the $K\bar{K}$-$K\bar{K}$ amplitudes. Looking at the unitarity relations for the $\pi\pi$-$K\bar{K}$ amplitudes above the $K\bar{K}$ threshold and imposing our previous relations (3.36), (3.41), (3.46)–(3.49) we deduce that

$$C_7=C_{13}-C_{11}, \quad (3.60)$$

$$C_7=C_9-C_{11}, \quad (3.61)$$

$$C_{10}=C_{14}. \quad (3.62)$$

Using the crossing relations on the equation $C_9=C_{13}$ obtained from (3.60) and (3.61) we obtain

$$C_2=C_6. \quad (3.63)$$

Equation (3.61) and the crossing relations yield

$$2(C_2-C_4)=(C_{10}-C_{12})-C_8. \quad (3.64)$$

This is a relation between four eigenamplitudes C_2, C_4, C_8, and $(C_{10}-C_{12})$. That the last one of these is also an eigenamplitude can be easily verified by using the unitarity relations. The left-hand side of Eq. (3.64) has only strangeness equal to two amplitudes, and the right-hand side only strangeness equal to zero amplitudes. There are many channels of strangeness equal to two (for example, ΞN) corresponding to which there are no zero-strangeness channels of equal mass, and conversely there are many channels of zero strangeness (for example, multipion states, nucleon-antinucleon states, etc.) corresponding to which there are no channels of strangeness equal to two; since the two sides must have the same branch points, all the branch

TABLE III. Amplitude relations which result from the symmetry postulates made in the text. The two relations in heavy brackets ([ ]) are the only isotopic-spin relations which do not follow rigorously from the postulates. It is reasonable to expect that inclusion of more channels will pick out the [relations] for reasons stated in the text.

$A_2=A_4=2A_{10}-A_6$
$A_3=A_5$
$A_8=\frac{1}{3}\sqrt{2}[(2A_6-A_{10})-(2A_{10}-A_6)]$
$B_7=-B_3=\mp\frac{1}{2}\sqrt{2}B_1$
$B_2=0$
$B_8=B_4=\sqrt{2}B_6=\sqrt{2}B_{10}$
$C_{11}=C_9-C_7$
$C_{13}=C_9$
$C_6=[C_2=C_4]$
$C_{14}=C_{10}$
$[C_{12}=C_{10}-C_8]$
$D_{11}=-D_5=\pm\frac{1}{2}\sqrt{2}(D_1-D_3)$
$D_9=D_7=2D_3-D_1$
$D_{13}=D_3$
$D_{15}=D_1$
$D_{12}=-D_6\pm\frac{1}{2}\sqrt{2}(D_2-D_4)$
$D_{10}=D_8=2D_4-D_2$
$D_{14}=D_4$
$D_{16}=D_2$

points corresponding to the thresholds of the aforementioned types of channels must cancel. An obviously consistent possibility is

$$C_2-C_4=C_{10}-C_{12}-C_8=0. \quad (3.65)$$

Inclusion of further inelastic channels might help in picking out the above possibility which is the isotopic-spin prediction.

However, with the number of channels included here, we cannot go further than Eq. (3.64), which is consistent with, but a little less than, the isotopic-spin prediction. Except for Eq. (3.65), we find all the predictions of isotopic-spin symmetry.

Finally, we remark that the amplitude relations (Table III) are independent of whether we assume the mass of the pion to be greater than or less than the mass of the kaon.

IV. UNITARY SYMMETRY

We have already seen that isotopic spin emerges as the least trivial internal symmetry when we consider diagonalization of scattering matrices with channels of various charge states. In the present section, we will therefore accept the fact that the particles break up into isotopic-spin multiplets and derive the more interesting relations corresponding to higher symmetries. We will discuss the scattering of the eight pseudoscalar (P) mesons (π,η,K) from each other and allow the masses to be arbitrary but nonzero. If we require that *each* of the scattering matrices listed in Table IV be diagonalized by a constant real orthogonal matrix U, we conclude that the mesons must have equal mass and that the only nontrivial amplitude relations are precisely those given by SU_3. The results of truncating the diagonalization assumptions will be discussed in Sec. V.

In Table V, we have listed the t-channel crossing relations and those u-channel crossing relations which give additional information. Identical particle factors of 2 and $\sqrt{2}$ multiply some of the amplitudes so that they satisfy the usual two-particle unitarity equations. Using both the t- and u-channel crossing relations, it is straightforward to express each of the $A_i(stu)$ in terms of the $A_i(tsu)$. Table IV shows the four 2×2 amplitude matrices and one 3×3 amplitude matrix in the problem. The diagonalization conditions for the four 2×2 matrices can be expressed as

$$A_3(stu)=x_1[h_1(s)A_2(stu)-h_1^{-1}(s)A_4(stu)], \qquad (4.1)$$

$$A_6(stu)=x_2[h_2(s)A_5(stu)-h_2^{-1}(s)A_7(stu)], \qquad (4.2)$$

$$A_{18}(stu)=x_3[h_3(s)A_{17}(stu)-h_3^{-1}(s)A_{19}(stu)], \qquad (4.3)$$

$$A_{22}(stu)=x_4[h_2(s)A_{21}(stu)-h_2^{-1}(s)A_{23}(stu)], \qquad (4.4)$$

where none of the constant x_i's are zero.

The expressions for $A_i(stu)$ obtained from crossing can now be substituted into these equations and then the interchange of t and s will yield s-channel relations. The physical cut structure of the amplitudes in these relations imply, by arguments mentioned in Sec. II, that for all amplitudes to be nonzero, the masses of the π, η, and K must be equal. Therefore, all the h_i are unity. The odd-amplitude relations derived as above are

$$A_1(stu)=3A_2(stu)-2A_9(stu), \qquad (4.5)$$

$$A_3=\tfrac{2}{3}\sqrt{2}(A_5-A_8), \qquad (4.6)$$

$$A_6=x_3(\sqrt{3}/2\sqrt{2})(A_2+A_{10}-2A_9), \qquad (4.7)$$

$$A_6=x_4\tfrac{1}{3}(8A_8-3A_7-5A_5). \qquad (4.8)$$

Unlike most of the attempts to derive SU_3 from a dynamical framework, we will not assume anything about the existence or otherwise of resonances in the P-P amplitudes corresponding to the eight known vector mesons. However, we remark that if we do make this assumption, the relations between the amplitudes can be deduced very simply already at this primitive stage of our actual calculation.

We must now consider the consequences of the diagonalization conditions on the 3×3 matrix M:

$$M=\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{13} & A_{15} \\ A_{16} & A_{15} & A_{14} \end{bmatrix}. \qquad (4.9)$$

TABLE IV. List of the 25 odd- and even-parity pseudoscalar amplitudes. Subscripts denote isotopic spins. Note that there are four 2×2 and one 3×3 amplitude matrices indicated by brackets. The numbers correspond to subscripts for amplitudes used in the text.

Odd amplitudes	Even amplitudes
1. $\langle K\bar{K}\|K\bar{K}\rangle_0$	11. $\langle K\bar{K}\|K\bar{K}\rangle_0$ }
2. $\langle K\bar{K}\|K\bar{K}\rangle_1$ }	12. $\langle K\bar{K}\|\pi\pi\rangle_0$ }
3. $\langle K\bar{K}\|\pi\pi\rangle_1$ }	13. $\langle \pi\pi\|\pi\pi\rangle_0$ }
4. $\langle \pi\pi\|\pi\pi\rangle_1$ }	14. $\langle \eta\eta\|\eta\eta\rangle_0$ }
5. $\langle \pi K\|\pi K\rangle_{1/2}$ }	15. $\langle \eta\eta\|\pi\pi\rangle_0$ }
6. $\langle \pi K\|\eta K\rangle_{1/2}$ }	16. $\langle \eta\eta\|K\bar{K}\rangle_0$ }
7. $\langle \eta K\|\eta K\rangle_{1/2}$ }	17. $\langle K\bar{K}\|K\bar{K}\rangle_1$ }
8. $\langle \pi K\|\pi K\rangle_{3/2}$	18. $\langle K\bar{K}\|\eta\pi\rangle_1$ }
9. $\langle KK\|KK\rangle_0$	19. $\langle \eta\pi\|\eta\pi\rangle_1$ }
10. $\langle \eta\pi\|\eta\pi\rangle_1$	20. $\langle KK\|KK\rangle_1$
	21. $\langle \pi K\|\pi K\rangle_{1/2}$ }
	22. $\langle \pi K\|\eta K\rangle_{1/2}$ }
	23. $\langle \eta K\|\eta K\rangle_{1/2}$ }
	24. $\langle \pi K\|\pi K\rangle_{3/2}$
	25. $\langle \pi\pi\|\pi\pi\rangle_2$

TABLE V. The t- and u-channel crossing relations for the pseudoscalar-pseudoscalar amplitudes used in Secs. IV and V. Identical particle factors of $\sqrt{2}$ and 2 multiply some of the amplitudes.

$$\begin{bmatrix} A_2+A_{17} \\ A_1+A_{11} \end{bmatrix}(stu)=\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} A_2+A_{17} \\ A_1+A_{11} \end{bmatrix}(tsu) =\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 2A_{20} \\ 2A_9 \end{bmatrix}(uts)$$

$$\begin{bmatrix} \sqrt{2}A_3 \\ \sqrt{2}A_{12} \end{bmatrix}(stu)=\begin{bmatrix} -\frac{2}{3} & \frac{2}{3} \\ 4/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}\begin{bmatrix} A_8+A_{24} \\ A_5+A_{21} \end{bmatrix}(tsu)$$

$$\begin{bmatrix} 2A_{13} \\ 2A_4 \\ 2A_{25} \end{bmatrix}(stu)=\begin{bmatrix} \frac{1}{3} & 1 & 5/3 \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} 2A_{13} \\ 2A_4 \\ 2A_{25} \end{bmatrix}(tsu)$$

$$\begin{bmatrix} A_8+A_{24} \\ A_5+A_{21} \end{bmatrix}(stu)=\begin{bmatrix} -\frac{1}{2} & \frac{1}{6}\sqrt{6} \\ 1 & \frac{1}{6}\sqrt{6} \end{bmatrix}\begin{bmatrix} \sqrt{2}A_3 \\ \sqrt{2}A_{12} \end{bmatrix}(tsu) =\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{4}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} A_8+A_{24} \\ A_5+A_{21} \end{bmatrix}(uts)$$

$$(A_6+A_{22})(stu)=(\sqrt{\tfrac{3}{2}})A_{18}(tsu) =(A_6+A_{22})(uts)$$

$$(A_7+A_{23})(stu)=-\tfrac{1}{2}\sqrt{2}(\sqrt{2}A_{16})(tsu) =(A_7+A_{23})(uts)$$

$$\begin{bmatrix} 2A_{20} \\ 2A_9 \end{bmatrix}(stu)=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -A_2+A_{17} \\ -A_1+A_{11} \end{bmatrix}(tsu)$$

$$(A_{10}+A_{19})(stu)=-\tfrac{1}{3}\sqrt{3}(2A_{15})(tsu) =(A_{10}+A_{19})(uts)$$

$$(2A_{14})(stu)=(2A_{14})(tsu)$$

$$(2A_{15})(stu)=-\sqrt{3}(A_{10}+A_{19})(tsu)$$

$$(\sqrt{2}A_{16})(stu)=-\sqrt{2}(A_7+A_{23})(tsu)$$

$$A_{18}(stu)=(\sqrt{\tfrac{2}{3}})(A_6+A_{22})(tsu)$$

Since the transformation U can be formed from the normalized eigenvectors of M, the eigenvectors themselves must have constant elements. We denote an energy-independent eigenvector of M by

$$\begin{pmatrix} e \\ f \\ g \end{pmatrix} . \tag{4.10}$$

For the case $e=0$, the eigenvalue condition on M and crossing yield

$$\sqrt{3}(g^2-f^2)A_{10}-2gfA_4=0. \tag{4.11}$$

It can be shown, using crossing and unitarity, that this condition requires either A_3 or A_9 to vanish identically. This case is therefore ruled out. For the case $e\neq 0$, let $y\equiv f/e$ and $z\equiv g/e$.

The eigenvalue condition on M and crossing yield for the odd amplitudes the two relations

$$y(\tfrac{1}{2}A_9+\tfrac{1}{4}A_1+\tfrac{3}{4}A_2)+y^2(\tfrac{2}{3}\sqrt{3}A_8+\tfrac{1}{3}\sqrt{3}A_5)-yzA_7 \\ +(\tfrac{1}{2}\sqrt{3}zA_{10}-\tfrac{2}{3}\sqrt{3}A_8-\tfrac{1}{3}\sqrt{3}A_5-yA_4)=0, \tag{4.12}$$

and

$$z(\tfrac{1}{2}A_9+\tfrac{1}{4}A_1+\tfrac{3}{4}A_2)+yz(\tfrac{2}{3}\sqrt{3}A_8+\tfrac{1}{3}\sqrt{3}A_5) \\ -z^2A_7+(A_7+\tfrac{1}{2}\sqrt{3}yA_{10})=0. \tag{4.13}$$

The object now is to determine the parameters x_1, x_2, x_3, and x_4 and the three eigenvectors. The restrictions due to unitarity are simple to express if we use the partial-wave projections of the A_i, denoted by $a_i(s)$. Using the previous equations, it is possible to express all the odd amplitudes in terms of a_1, a_2, a_3, and a_5. The unitarity relations for a_6, a_8, a_9, and a_{10} yield, respectively,

$$4\sqrt{2}x_4(1-r)^{-1}\,\mathrm{Re}[a_3^*(a_2-a_5)]=\rho|a_3|^2(1-r)^{-1} \\ \times[2\sqrt{2}(x_4/x_1)+8x_4r(1-r)^{-1}], \tag{4.14}$$

$$(3/\sqrt{2})\,\mathrm{Re}[a_3^*(a_2-a_5)]=\rho|a_3|^2 \\ \times[-9/8+3/(2\sqrt{2}x_1)+8x_4^2(1-r)^{-2}], \tag{4.15}$$

$$a_3\equiv\eta(s)(a_1-a_2)/\sqrt{2}, \quad |\eta|=1. \tag{4.16}$$

and

$$2+\mathrm{Re}\eta(16\sqrt{\tfrac{2}{3}})(x_4/x_3)(1-r)^{-1}+(64/3)(x_4/x_3)^2(1-r)^{-2} \\ -(8/\sqrt{3})(x_4/x_3)(1-r)^{-1}(1/x_1)=0, \tag{4.17}$$

where

$$r\equiv x_4/x_2. \tag{4.18}$$

The unitarity relation for a_8, Eq. (4.15), shows that $x_2\neq x_4$. Combining Eqs. (4.14) and (4.15), we obtain

$$r(1-r)^{-1}=-\tfrac{3}{8}+(\tfrac{8}{3})x_4^2(1-r)^{-2}. \tag{4.19}$$

Equation (4.17) implies that $\mathrm{Re}\eta(s)$ is energy-independent; hence $\mathrm{Im}\eta(s)$ is also. But since a_1, a_2, and a_3 are real analytic functions, $\eta=\pm 1$. Unitarity for a_3 then yields

$$x_1=-\sqrt{2}\eta. \tag{4.20}$$

Equation (4.17) is a quadratic equation which gives two solutions for the quantity

$$\beta\equiv(16/\sqrt{6})(x_4/x_3)\eta(1-r)^{-1}=1 \text{ or } 4. \tag{4.21}$$

We consider first the alternative $\beta=1$. Equations (4.12) and (4.13) now become

$$a_5(\sqrt{3}y^2-yz-\sqrt{3}) \\ =a_1\left[\tfrac{1}{2}\sqrt{3}\eta y^2-\frac{\sqrt{6}}{16}yz\frac{2x_3}{x_2}-\tfrac{1}{2}\sqrt{3}\eta+\frac{y}{2}+\tfrac{1}{4}\sqrt{3}z\right] \\ +a_2\left[-\tfrac{3}{2}y-\tfrac{1}{2}\sqrt{3}\eta y^2+yz\frac{2x_3}{x_2}\frac{\sqrt{6}}{16} \right. \\ \left. +\tfrac{1}{2}\sqrt{3}\eta+\tfrac{1}{2}y-\frac{3\sqrt{3}}{4}z\right] \tag{4.22}$$

and

$$a_5(\sqrt{3}yz-z^2+1)=a_1\left[\frac{\sqrt{6}}{16}\frac{2x_3}{x_2}+\tfrac{1}{4}\sqrt{3}y+\tfrac{1}{2}\sqrt{3}\eta yz-\frac{\sqrt{6}}{16}z^2\frac{2x_3}{x_2}\right] \\ +a_2\left[-\frac{\sqrt{6}}{16}\frac{2x_3}{x_2}-\tfrac{3}{4}\sqrt{3}y-\tfrac{3}{2}z-\tfrac{1}{2}\sqrt{3}\eta yz+z^2\frac{2x_3}{x_2}\frac{\sqrt{6}}{16}\right]. \tag{4.23}$$

We note first that $a_2=$ constant times $a_1(s)$ implies via unitarity, either $a_2=a_1$ or $a_2=a_1/3$ which correspond to $A_3=0$ or $A_9=0$, respectively. We therefore assume that $a_2(s)$ and $a_1(s)$ are linearly independent amplitudes in order to derive the least trivial symmetry. In Eqs. (4.22) and (4.23), the linear independence of a_1 and a_2 implies that either the coefficients of a_5, a_1, and a_2 are all zero, or we obtain two expressions for a_5 in terms of a_1 and a_2 in which the coefficients of a_1 and a_2 can be separately equated. Consider first the case $\eta=1$. For the coefficients of a_5 in Eqs. (4.22) and (4.23) to vanish, we must have

$$\sqrt{3}z=-y, \quad y=\pm\tfrac{1}{2}\sqrt{3}. \tag{4.24}$$

For $y=\frac{1}{2}\sqrt{3}$, the coefficients of a_1 and a_2 are found to be proportional to x_3/x_2, and hence x_3/x_2 must vanish. Equations (4.19) and (4.21) then yield the values of x_3 and x_4. The solution $y=-\frac{1}{2}\sqrt{3}$ is ruled out because it yields an imaginary value for x_4 ($x_4^2=-3/16$). The solutions with coefficients of a_5, a_1, and a_2 not equal to zero are similarly worked out.

We find that for $\beta=1$, $\eta=1$, and a nontrivial set of amplitude relations, the diagonalizing matrices are given by

$$x_1=-\sqrt{2}, \quad x_2=\infty, \quad x_3=\mp\sqrt{6}, \quad x_4=\pm\tfrac{3}{8}, \tag{4.25}$$

and the three normalized eigenvectors of M are

$$\begin{pmatrix} e \\ f \\ g \end{pmatrix} = \pm \begin{bmatrix} 1/\sqrt{2} \\ \sqrt{3}/(2\sqrt{2}) \\ -1/(2\sqrt{2}) \end{bmatrix}, \quad \pm \begin{bmatrix} \sqrt{3}/\sqrt{10} \\ -1/(2\sqrt{10}) \\ 3\sqrt{3}/(2\sqrt{10}) \end{bmatrix}, \quad \pm \begin{bmatrix} 1/\sqrt{5} \\ -3/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}. \tag{4.26}$$

A somewhat tedious calculation along the lines indicated above shows that for the cases $\beta=1$, $\eta=-1$, and $\beta=4$, $\eta=\pm1$ there are only trivial solutions.

With the eigenvectors given by (4.26), the conditions such that U diagonalizes M now read

$$\sqrt{3}A_{11}-\tfrac{1}{4}\sqrt{3}A_{13}-\tfrac{3}{4}\sqrt{3}A_{14}+A_{12}+\tfrac{5}{2}A_{15}+\sqrt{3}A_{16}=0, \tag{4.27}$$

$$\sqrt{3}A_{11}+\tfrac{1}{2}\sqrt{3}A_{13}-\tfrac{3}{2}\sqrt{3}A_{14}-\tfrac{7}{2}A_{12} -4A_{15}+\tfrac{1}{2}\sqrt{3}A_{16}=0, \tag{4.28}$$

and

$$A_{11}-\tfrac{3}{2}A_{13}+\tfrac{1}{2}A_{14}-\tfrac{1}{2}\sqrt{3}A_{12}-\tfrac{3}{2}A_{16}=0. \tag{4.29}$$

Crossing these relations we obtain two new relations between the odd-parity amplitudes

$$A_3=2\sqrt{2}(A_4-A_5), \tag{4.30}$$

$$\sqrt{2}A_3=A_4-A_8. \tag{4.31}$$

Now all the odd-parity amplitudes can be expressed in terms of two of them (say A_1 and A_9). An entirely similar procedure enables us to express all the even-parity amplitudes in terms of three of them (say A_{11}, A_{12}, and A_{13}). These amplitude relations are listed in Table VI. The ($\pm$) signs in front of the amplitudes A_6, A_{18}, and A_{22} correspond to the fact that these amplitudes reverse sign if the relative phase of the states η and π is reversed, and do not therefore correspond to any physical ambiguity in our answer.

Finally we observe that the amplitude relations in Table VI are precisely those obtained by assigning the pseudoscalars to an octet representation of $SU(3)$. The two independent odd-parity eigenamplitudes and the three independent even-parity eigenamplitudes we find are, in group-theoretic language, the odd-parity amplitudes ($8_A \to 8_A$), ($10 \to 10$) and the even-parity amplitudes ($1 \to 1$), ($8_S \to 8_S$), ($27 \to 27$) in $SU(3)$ symmetry.

TABLE VI. Odd- and even-amplitude relations that follow under diagonalization assumptions for all the amplitude matrices in Table IV. These are identical with the SU_3 predictions.

Odd	Even
$A_2=\frac{1}{3}(A_1+2A_9)$	$A_{14}=2A_{11}-\frac{2}{3}\sqrt{3}A_{12}-A_{13}$
$A_3=\frac{1}{3}\sqrt{2}(A_1-A_9)$	$A_{15}=-\frac{1}{3}\sqrt{3}A_{11}-\frac{1}{3}A_{12}+\frac{1}{3}\sqrt{3}A_{13}$
$A_4=\frac{1}{3}(2A_1+A_9)$	$A_{16}=\frac{4}{3}A_{11}-(5/9)\sqrt{3}A_{12}-\frac{4}{3}A_{13}$
$A_5=\frac{1}{2}(A_1+A_9)$	$A_{17}=A_{11}-\frac{2}{3}\sqrt{3}A_{12}$
$\pm A_6=-\frac{1}{2}(A_1-A_9)$	$\mp A_{18}=-\frac{2}{3}(\sqrt{6})A_{11}+\frac{1}{3}\sqrt{2}A_{12}+\frac{2}{3}(\sqrt{6})A_{13}$
$A_7=A_5$	$A_{19}=5/3A_{11}-7/9\sqrt{3}A_{12}-\frac{2}{3}A_{13}$
$A_8=A_9=A_{10}$	$A_{20}=3A_{11}-\sqrt{3}A_{12}-2A_{13}$
	$A_{21}=-\frac{1}{2}\sqrt{3}A_{12}+A_{13}$
	$\mp A_{22}=A_{11}-\frac{1}{6}\sqrt{3}A_{12}-A_{13}$
	$A_{23}=8/3A_{11}-17/18\sqrt{3}A_{12}-5/3A_{13}$
	$A_{24}=A_{25}=A_{20}$

V. OTHER HIGHER SYMMETRIES

In this section, we look for possible higher symmetries[10] which allow one of the pseudoscalar masses (m_η, m_π, or m_K) to be different from the other two ($m_\pi=m_K$, $m_\eta=m_K$, or $m_\eta=m_\pi$). It is easy to see what kind of diagonalization postulates are needed. First, we take the case $m_\eta \neq m_\pi = m_K$. In the 3×3 matrix with even spatial parity

$$\left[\begin{array}{cc|c} K\bar{K}\to K\bar{K} & K\bar{K}\to\pi\pi & K\bar{K}\to\eta\eta \\ \pi\pi\to K\bar{K} & \pi\pi\to\pi\pi & \pi\pi\to\eta\eta \\ \hline \eta\eta\to K\bar{K} & \eta\eta\to\pi\pi & \eta\eta\to\eta\eta \end{array}\right] \equiv \left[\begin{array}{cc|c} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{13} & A_{15} \\ \hline A_{16} & A_{15} & A_{14} \end{array}\right], \tag{5.1}$$

we postulate diagonalization for that 2 by 2 submatrix which involves kinematically identical channels. Therefore,

$$A_{12}=y_1(A_{11}-A_{13}). \tag{5.2}$$

The only other matrix in which we invoke the symmetry postulate is the matrix involving A_2, A_3, and A_4. Hence, we write

$$A_3=x(A_2-A_4). \tag{5.3}$$

Any further diagonalization postulates would force all three masses to be equal and this case has already been treated.

It should be noted that $m_\pi=m_K$ is in accord with the assignment of pions and kaons to a seven-dimensional representation of G_2. The amplitude relations which follow from the symmetry postulates are consistent with the predictions of G_2. However, not all of the G_2 amplitude relations are found. This is similar to the situation concerning isotopic spin and the relation (3.64). In any event, it is found that three amplitudes must be zero and since G_2 predicts the same three amplitudes to be zero, G_2 must be one of the least trivial symmetries.

When all the techniques at our disposal are applied in working out the consequences of (5.2) and (5.3), we find four eigenamplitude relations that are not of the type which decompose via the eigenamplitude theorem (Sec. II) into further linear relations among amplitudes.

[10] R. E. Behrends and A. Sirlin, Phys. Rev. **121**, 324 (1961); R. E. Behrends, J. Dreitlein, C. Fronsdal, and W. Lee, Rev. Mod. Phys. **34**, 1 (1962); P. G. O. Freund, H. Ruegg, D. Speiser, and A. Morales, Nuovo Cimento **25**, 307 (1962); R. E. Behrends and L. F. Landovitz, Phys. Rev. Letters **11**, 296 (1963).

For example, one of the four relations is

$$5A_{25}=2(4A_{13}-3A_{11})+\left(1+\frac{6\sqrt{2}}{x}\right)A_{21}+\left(2-\frac{6\sqrt{2}}{x}\right)A_{24}, \quad (5.4)$$

where A_{25}, $4A_{13}-3A_{11}$, A_{21}, and A_{24} are eigenamplitudes and the coefficients obviously meet the requirement (2.20) of the eigenamplitude theorem. Therefore, values of x for which the coefficients on the right-hand side of (5.4) are all positive correspond to

$$A_{25}=4A_{13}-3A_{11}=A_{21}=A_{24}. \quad (5.5)$$

However, this equation via crossing and unitarity implies that $x=-\sqrt{2}$ which is not in the set of values of x being considered. Therefore, we exclude the values of x for which Eq. (5.4) implies Eq. (5.5).

It is worthwhile to note that with Eq. (5.4) and the unitarity relations for the eigenamplitudes we have somewhat more than an expression for one independent function A_{25} (or its eigen phase shift) in terms of three other functions and the parameter x. Using Eq. (5.4) and the unitarity relation for A_{25}, one can eliminate A_{25}. Thus, one obtains a nonlinear relation expressing one function (e.g., an eigen phase shift) in terms of two independent functions and the parameter x. This nonlinearity is allowed by our approach but certainly does not correspond to what is usually meant by an internal symmetry.

Even for those values of x for which the eigenamplitude theorem does not imply (5.5), it is obvious that (5.5) is *a* solution of Eq. (5.4) which is consistent with unitarity. When linear relations between eigenamplitudes, such as (5.5), are arbitrarily chosen for all the eigenamplitude relations in the problem, the results are precisely the amplitude relations of G_2. Inclusion of inelastic states frequently picks out such "obvious" linear relations. Indeed, inclusion of the ϕ, a singlet vector meson, does help pick out more G_2 relations, including $x=-\sqrt{2}$, but it still does not yield all the G_2 relations. It seems reasonable to expect that enlarging the problem to include more inelastic states would select G_2 as the least trivial symmetry.

Now consider the case $m_\pi\neq m_K=m_\eta$. Here, diagonalization of the appropriate 2×2 submatrix of the 3×3 matrix yields

$$A_{16}=y_2(A_{11}-A_{14}). \quad (5.6)$$

This case involves an eigenamplitude relation which decomposes so that one arrives at a full set of amplitude relations with no difficulty. It seems consistent to assign the K's and the η to the five-dimensional representation of C_2. We observe that the amplitude relations calculated by our procedure correspond to the $1\oplus10\oplus14$ reduction of the direct product of two five-dimensional representations of C_2.

The third possibility is $m_K\neq m_\pi=m_\eta$. If we take the diagonalization postulate

$$A_{15}=y_3(A_{13}-A_{14}), \quad (5.7)$$

we arrive at a set of amplitude relations with as much ease as in the $\pi\pi$ isotopic-spin treatment. This case corresponds to putting the pions and the η in the same multiplet. If one has invariance under the group $SU_2\times SU_2$, where the SU_2's act on quark and antiquark spaces, it is a simple consequence that the pions and the η, as nonstrange quark-antiquark states, will belong to the same (four-dimensional) multiplet. The relations among amplitudes involving π's and η which follow from Eq. (5.7) are the same as those which follow from invariance under $SU_2\times SU_2$.

VI. RELATION TO BOOTSTRAP CALCULATIONS

There have been many calculations based on the bootstrap hypothesis and some questionable dynamical approximations that advocate "bootstrap" as the dynamical mechanism responsible for induction of symmetries in strong interactions, and then suggest that a particular symmetry like SU_3 is selected by a bootstrap calculation. A natural question to ask at this point concerns the relation between our assumptions and those made by bootstrappers.

The first case to be considered is the class of calculations which start with multiplets of equal-mass particles (e.g., eight pseudoscalars of equal mass and eight vectors of equal mass). The partial-wave scattering matrix is written in the form

$$M=ND^{-1}, \quad (6.1)$$

where the discontinuity of M across the left-hand cut is defined to be $\alpha=\alpha^t$, and

$$N=[\alpha D]_L, \quad (6.2)$$

$$D=1-[\rho N]_R, \quad (6.3)$$

where $[\ ]_{L,R}$ means a dispersion integral over the left- and right-hand cuts, respectively. The phase-space factor ρ is a multiple of the unit matrix because of the equal-mass assumption. For definiteness, we consider the scattering of pseudoscalars by pseudoscalars. In the odd partial waves, there are at most only two coupled channels. For these cases, one can always write

$$M_{12}=x(M_{11}-M_{22}), \quad (6.4)$$

where x in general depends on the energy $\sqrt{s}$. Using the previous expressions for M in terms of N over D, it is straightforward to disperse x and to see that

$$x=[\alpha_{12}d]_L/[(\alpha_{11}-\alpha_{22})d]_L, \quad (6.5)$$

where d is the determinant of D.

In our approach to internal symmetries as applied to partial-wave amplitudes, besides assuming some sacred conditions such as crossing, unitarity, etc., it is further assumed that x is independent of energy and angular momentum. The energy independence will follow if the dynamics forces the α's to obey the conditions

Case a, $\alpha_{11}(s)=\alpha_{22}(s)$. The mixing angle is then 45° for all energies.

Case b, $\alpha_{12}(s)$ is proportional to $(\alpha_{11}(s)-\alpha_{22}(s))$.

In our approach, it is further assumed that no reaction amplitudes are zero. In the case of the scattering of pions, kaons, and η's, the $SU(3)$ results then follow uniquely whether or not there are vector mesons.

Bootstrap calculations usually proceed under the following set of assumptions (or some equivalent set). The first assumption is to restrict the form of the force by assuming only certain exchanged particles contribute (for example, equal-mass vector mesons, nucleons, isobars, etc.). The second assumption is to invoke the bootstrap hypothesis and to require that the exchanged particles show up in the direct amplitude at the correct place with the correct residue (or slope). This second requirement is only an approximate and certainly crude statement of crossing. It will be incorporated correctly in any scheme which satisfies crossing exactly.

The first assumption usually corresponds to the statement that the left-hand cut has the structure

$$\alpha(s)=na(s)\,, \tag{6.6}$$

where $a(s)$ is a scalar function of the energy and n is a constant symmetric matrix with nonzero elements between states that have the same strangeness and isotopic spin. Bootstrap derivations of internal symmetry are usually confined to a single partial-wave amplitude. However, if the justification for the assumption of single-particle-exchange dominance of the force is taken to be a "nearest-singularity"-type argument, it should apply to all partial waves in a certain energy region. Then, an equation of the form (6.6), with the same matrix n, would be valid for all partial waves. Indeed, bootstrap calculations[3] would have to make such hypotheses to derive consequences of an internal symmetry for all partial waves. The mathematical statement (6.6) satisfies our criteria and is enough by itself to derive the symmetry by our method without making any further assumptions or approximations. A constant n corresponds to case b discussed above.

It is amusing to note that this is not the most general α which leads to an internal symmetry. For example, if α can be written in the form

$$\alpha(s)=\sum_{i=0}^{\infty} n^i a_i(s)\,, \tag{6.7}$$

where the term involving n to the ith power is due to the exchange of i particles, then an internal symmetry exists and is the same as that found by neglecting all terms except the one with $i=1$. This form for α will follow, for example, if one uses the model of the force which leads to Eq. (6.6), but instead uses it to define an effective relativistic potential. Iteration of this potential will then lead to the form (6.7).

The pion-nucleon bootstrap models of Abers, Zachariasen, and Zemach[1] and Franklin[5] correspond to case a and hence are different in details from most other calculations. However, all of these approaches require the presence of poles which we see are not really necessary.

The bootstrap hypothesis does serve to connect models and symmetries. For example, in the case of the above scalar- and vector-meson models as applied to G_2, one can show that the force due to ρ and K^* exchange is repulsive in the $T=\frac{1}{2}$ state of $K\eta$ scattering and hence the K^* cannot bootstrap itself. However, other physically reasonable dynamical models will produce symmetries other than $SU(3)$.

The point we wish to stress here is that most bootstrap "derivations" of symmetries start off by assuming most of their results and the details of the calculation must be done only to see if the forces are attractive in the correct states. In order to illustrate this point, let us examine a nonbootstrap derivation of $SU(3)$. The model assumes that there is some sort of four-point interaction between the usual eight mesons which can be ignored in the calculation of the force but will be computed later from the complete amplitude. This is essentially a no-subtraction hypothesis. Equal masses will be assumed and the force will be taken to be given by the two-particle-exchange bubble diagram. The force input can be written in the form

$$\alpha^J(s)=na^J(s) \tag{6.8}$$

for odd J and a similar expression for even J. The symmetric matrix n has nonzero elements as before. Since the force can be diagonalized by an energy and J-independent transformation, the total scattering amplitude can also. Knowing this fact, one can proceed by our method and prove that the only symmetry possible is $SU(3)$. Note that we get the complete content of $SU(3)$ symmetry for these amplitudes. The usual assumptions involving vector mesons are not only unnecessary, they lead to the predictions that there is zero scattering in the 10 and 10* channels [which $SU(3)$ does not require at all]. See, however, Ref. 3 for attempts to rectify this inadequacy which unfortunately lead to the introduction of unconfirmed low-mass scalar mesons.

An alert reader at this juncture may ask himself how any bootstrap calculation which treats crossing so crudely can compute the *exact* relative values of the $SU(3)$ coupling constants. One reason for this wonderful accident can be seen from the dispersion relation for x, Eq. (6.5). It has been noted many times before that the left-hand cut due to the exchange of a particle is logarithmically infinite at the beginning of the cut. Since x is actually independent of the energy, one can compute it by evaluating it at some convenient value of the energy. If one evaluates it at the beginning of the left-hand cut, both numerator and denominator are infinite and are given exactly by one-particle exchange. Thus, x is computed exactly from the Born approximation. Now if x is also evaluated from its definition (6.4) at the position of the vector-meson poles, one gets an exact relation between coupling constants.

The second important property of bootstrap calculations is that there are few parameters. Thus, if the 8 vector-meson masses are forced to have the same value, then the various eigenamplitudes which have these poles are, in fact, identical. Since x is known exactly, all the $SU(3)$ coupling-constant ratios then follow even though the eigenamplitude itself may be grossly in error.

VII. CONCLUSIONS

The authors feel that the most important results of this approach are that unitarity and crossing without further dynamical assumptions are enough to determine possible internal symmetries and the consequent understanding of the amazing success of bootstrap as applied to the "derivation" of internal symmetries. First, we understand how such a poor approximation can yield exact results in the equal-mass limit. Second, we understand that the really essential elements of the derivation are the number of equal-mass scalar mesons and the form of the force. The imposition of the requirement that there be vector mesons and that they bootstrap themselves is completely irrelevant for the derivation of a symmetry.

Since the success of the bootstrap model is so accidental in the equal-mass case, it would be surprising indeed if it gave good results in the broken symmetry case. This more difficult case will be discussed in a later paper.

It should be noted that although the diagonalization postulate corresponds to an internal symmetry, all internal symmetries do not correspond to the diagonalization postulate. However, it is easy to find the postulates which are appropriate for other types of internal symmetry. One simply performs an energy-independent, real orthogonal transformation on the amplitude matrix and observes that various numbers of zeros as off-diagonal elements of the transformed matrix correspond to all the various possible types of conservation laws.

In all the examples considered in this paper, we have postulated complete diagonalization of all the amplitude matrices having kinematically identical channels. It happens that this leads to amplitude relations which correspond to well-known symmetries. However, there is no physical basis for diagonalizing all (and not just some) of the matrices in the problem. Thus, we state that one of many possible postulates leads to SU_3 relations and that other postulates will in general lead to other relations. The postulate which yields the SU_3 relations for the meson-meson system is unique only in the sense that it postulates the maximum number of selection rules, or "the maximal symmetry." However, SU_3 is clearly not the maximal symmetry in the baryon-meson system because there is an undiagonalized 2×2 scattering matrix with the channels 8_S, 8_A.

There are several deep questions about this S-matrix approach which come to mind that we cannot answer at this time. For example, what is the precise connection between our symmetry postulate and the standard Lie-group approach? Are they always equivalent? Other types of questions concern the derivation of symmetries, especially broken ones. One important question is whether the number of pseudoscalar mesons that are assumed to exist actually determine the symmetry if a particular dynamical scheme is used (such as the Mandelstam representation). We hope to return to these questions later.