at large q^2, the high W states, which require a large E to be excited, must make a much more important contribution to the sum rules than they do at $q^2=0$. The calculations of this paper shed no light on the important question of how rapidly $E(q^2,\delta)$ increases with q^2, but only serve to indicate at what energies E it may pay to begin the experimental study of the local current algebra.

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Low-Energy Λ-p Scattering and the Hypertriton Binding Energy with Sums of Separable Potentials*†

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The $_{\Lambda}H^3$ binding energy is calculated by finding the pole in the Λ-d doublet S-wave scattering amplitude. This amplitude is obtained from a multiple-scattering formalism of the Faddeev type. Each of the two-body interactions was represented as a sum of an attractive and a repulsive S-wave nonlocal separable potential of the Yamaguchi form. The Λ-N potentials were chosen to fit the scattering lengths and effective ranges obtained by Herndon, Tang, and Schmid (HTS) in a variation calculation. The N-N potential was adjusted to fit the deuteron binding energy and the triplet scattering length. Calculations with several sets of values for the free parameters are carried out. By making the Λ-N repulsive strengths infinite, fixing all three repulsive potential ranges at 0.2 F, and varying the range of the N-N attractive potential, a value for the $_{\Lambda}H^3$ binding energy that agrees with that of the HTS calculation is obtained.

I. INTRODUCTION

IN a previous paper,[1] low-energy Λ-d scattering and the hypertriton were treated using a Faddeev[2] type of multiple-scattering formalism. In that work the nucleons were treated as identical particles and the Λ-N and N-N interactions were represented by purely attractive, spin-dependent, S-wave, nonlocal separable (NLS) potentials. The purpose of this paper is to improve the $_{\Lambda}H^3$ binding-energy calculation by the inclusion of repulsive cores in the two-body potentials.

In HS3 the Λ-N potential parameters were determined by matching the low-energy, S-wave, singlet- and triplet-scattering amplitudes to the amplitudes obtained from two types of local potentials. The first type was the purely attractive Dalitz-Downs (DD) potential[3,4] whose parameters had been obtained from a variational treatment of the light hypernuclei.[5] The second type was the "hard core plus attractive exponential" Herdon-Tang-Schmid (HTS) potential[6] whose parameters also had been obtained from a variational treatment of light hypernuclei. The results of HS3 were that, if the Λ-N potential parameters were chosen so as to give agreement with the DD amplitudes, a value for B_{Λ} (the binding energy of the Λ in the hypertriton) in agreement with experiment was obtained. The values obtained for the low-energy Λ-p elastic cross section, however, were consistently smaller than the experimental[7,8] values. This discrepancy could be traced to the

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[1] J. H. Hetherington and L. H. Schick, Phys. Rev. **139**, B1164 (1965). Hereafter referred to as HS3.

[2] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]; Dokl. Akad. Nauk SSSR **138**, 561 (1961); **145**, 301 (1962) [English transls.: Soviet Phys.—Doklady **6**, 384 (1961); **7**, 600 (1963)].

[3] R. H. Dalitz and B. W. Downs, Phys. Rev. **111**, 967 (1958).

[4] B. W. Downs and R. H. Dalitz, Phys. Rev. **114**, 593 (1959).

[5] The scattering lengths and effective ranges of the Dalitz and Downs potentials used in HS3 were taken from J. J. de Swart and C. Dullemond, Ann. Phys. (N. Y.) **19**, 458 (1962).

[6] R. C. Herndon, Y. C. Tang, and E. W. Schmid, Phys. Rev. **137**, B294 (1965). For some other hypertriton calculations using hard cores, see B. Ram and B. W. Downs, Phys. Rev. **133**, B420 (1964) and references cited there.

[7] B. Sechi-Zorn, R. A. Burstein, T. B. Day, B. Kehoe, and G. A. Snow, Phys. Rev. Letters **13**, 282 (1964); G. Alexander, U. Karshon, A. Shapira, G. Yekutieli, R. Englemann, H. Filthuth, A. Fridman, and A. Minguzzi-Ranzi, *ibid.* **13**, 484 (1964).

[8] It has been brought to our attention that more recent experimental results for Λ-p cross sections than those used here have been published by G. Alexander, O. Benary, U. Karshon, A. Shapira, G. Yekutieli, R. Englemann, H. Filthuth, A. Fridman, and B. Schilby [Phys. Letters **19**, 715 (1966)]. These cross sections are larger than those used in this work so that the use of purely attractive NLS potentials may still be inadequate. In any case, the main result of this work, that NLS potentials which fit low-energy Λ-N scattering parameters obtained from local potentials also can give the same B_{Λ} as these local potentials, still holds. This result can be checked for the correct values of the low-energy Λ-N scattering parameters only after a calculation of B_{Λ} incorporating these parameters in carried out with local potentials.

fact that values of the singlet and/or triplet Λ-N scattering lengths were too small. On the other hand, if the parameters were chosen so as to give agreement with the HTS amplitudes—which in turn were then in good agreement with the Λ-p scattering data[8]—both scattering lengths were now so large as to give a value of 0.9 MeV for B_Λ. This is in marked disagreement with the value

$$B_\Lambda = 0.31 \pm 0.15 \text{ MeV} \tag{1}$$

used by HTS. These results indicated that if in each spin state the HTS Λ-N potential were more realistically represented by a sum of two NLS potentials—one repulsive and one attractive—agreement with the experimental values for both B_Λ and the low-energy Λ-p cross section could be obtained. It is shown in this paper that with such a potential for each Λ-N spin state (as well as for the N-N interaction) and with no more free parameters than in the case of a local potential, the HTS values for the hypertriton binding energy and Λ-p cross section may be duplicated.

The inclusion of a repulsive term in the two-body potentials is only one of several improvements that need to be made in the calculations carried out in HS3. For example, the effects of three-body forces and the $\Lambda N \leftrightarrow \Sigma N$ process, each of which has been investigated to some extent in different types of calculations by other authors,[9,10] must eventually be accounted for as must tensor forces and interactions in other than relative S states. Nevertheless, it seems most reasonable to investigate the ability of sums of simple two-body NLS potentials to reproduce the results of local potentials with repulsive cores before inquiring into the relative merits of more esoteric NLS and local potentials.

Sums of separable potentials have been used previously by Tabakin[11] to represent the N-N interaction. In a calculation of the triton binding energy Tabakin used two different sums of NLS potentials. In the first, the repulsive potential was chosen to differ from a local hard core in that it led to a smooth two-body wave function. In the second, the repulsive potential was designed to more closely simulate a local hard core, but the shape of this potential was not as convenient a form as that discussed below. Neither of the Tabakin potentials, therefore, was used in the present work. With a simplified model of the triton, Tabakin did find that of two N-N potentials that fit the same scattering data the one which was more repulsive gave the smaller value for the triton binding energy. This is just the effect looked for here. However, the $_\Lambda H^3$ binding energy is much smaller than that of H^3, the Λ-N S-wave amplitudes are not as well known as the correspondlng N-N amplitudes, and no spin averaging of the total Λ-N potential will be carried out. These factors make the $_\Lambda H^3$ problem more complex than the spin-averaged triton discussed by Tabakin. The emphasis here, then, is on obtaining a good result for B_Λ with a particular type of potential, rather than on investigating the sensitivity of B_Λ to various types of potentials.

Although many types of potentials have been used to represent the Λ-N interaction, to the authors' knowledge a sum of NLS potentials is not among them. With wide open freedom of choice for the shape of each two-body NLS potential, simplicity became the deciding factor.

The type of NLS potentials employed in the present work consisted of a sum of two Yamaguchi[12] potentials for each independent two-body interaction. The parameters in these potentials were determined so as to fit the appropriate two-body low-energy data. In the spirit of paralleling the HTS calculations, no attempt was made to fit high-energy N-N phase shifts. A detailed description of the potentials and the general method used to fit the two-body data is presented in Sec. IIA. Explicit application to the Λ-N singlet and triplet amplitudes, including the limit of an infinite repulsive potential, is made in Sec. IIB. The N-N triplet potential is dealt with in Sec. IIC.

In Sec. III the results of *exact* calculations of B_Λ using the potentials described in Sec. II are presented and discussed. Included are the results for B_Λ when a repulsive core is present in only the N-N potential, only the Λ-N potentials, in none of the potentials, and in all of the two-body potentials.

II. TWO-BODY POTENTIALS

A. General Considerations

Of the three particles present in the hypertriton, the Λ is an isospin singlet while the nucleons are members of an isospin doublet. Each of the three particles is a spin $\frac{1}{2}$ fermion. The two-body potentials between pairs of particles are taken to be S-wave spin-dependent potentials. The Λ-N force is assumed to be more attractive in the singlet state than in the triplet state.[13] Since the N-N $^3S_1$ force is more attractive than the $^1S_0$ force, the hypertriton has isospin zero and spin $\frac{1}{2}$.

Because the nucleons are identical, there are a total of three different pairs of particles to consider: $N+N$ in a $^3S_1$ state, $\Lambda+N$ in a $^3S_1$ state, and $\Lambda+N$ in a $^1S_0$ state. For any of these pairs the momentum-space matrix element of the two-body potential-energy operator V is taken to be

$$\langle \mathbf{p} | V | \mathbf{p}' \rangle = \lambda_1 v_1(p) v_1(p') + \lambda_2 v_2(p) v_2(p') , \tag{2}$$

where $\mathbf{p}$ is the relative momentum vector of the two

[9] For calculations which include three-body forces see A. R. Bodmer and S. Sampanthar, Nucl. Phys. **31**, 251 (1962).

[10] For calculations which include $\Lambda N \leftrightarrow \Sigma N$, see G. Rajasekaran and S. N. Biswas [Phys. Rev. **122**, 712 (1961)], as well as the article cited in Ref. 5.

[11] F. Tabakin, Phys. Rev. **137**, B75 (1965). For other than S-wave N-N potentials see F. Tabakin, Ann. Phys. (N. Y.) **30**, 51 (1964).

[12] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).

[13] See, for example, R. H. Dalitz and L. Liu, Phys. Rev. **116**, 1312 (1959); J. W. Cronin and O. E. Overseth, *ibid.* **129**, 1795 (1963).

particles and $p=|\mathbf{p}|$. All the potential shapes are chosen to have the form originally used by Yamaguchi.[12]

$$v_i(p)=[p^2+\beta_i^2]^{-1}, \quad i=1,2. \tag{3}$$

For each pair of particles, then, there are four potential parameters, λ_1, λ_2, β_1, β_2. These parameters, which vary from pair to pair, are to be determined so as to fit both the low-energy Λ-p scattering data[8] and the $_\Lambda H^3$ binding energy, as well as, of course, the low-energy N-N $^3S_1$ data. As 12 free parameters are far too many to work with sensibly, some restrictions are necessary. These restrictions are discussed below.

Low-energy two-body data involve the scattering length, the effective range, and, in the case of a two-body bound state, the binding energy. All of these may be obtained from a calculation of $k\cot\delta$, where k is the relative momentum and δ is the S-wave phase shift for the two-body potential in question. It is easily shown that the potential given in Eq. (2) yields

$$k\cot\delta=N/D, \tag{4a}$$

where

$$N=(k\cot\delta_1)(k\cot\delta_2)-W^2, \tag{4b}$$

$$D=k\cot\delta_1+k\cot\delta_2+2W, \tag{4c}$$

and

$$W=\frac{1}{\pi v_1(k)v_2(k)}P\int_{-\infty}^{\infty}\frac{v_1(q)v_2(q)q^2dq}{q^2-k^2}. \tag{5}$$

In Eqs. (4), $k\cot\delta_j$ is the value of $k\cot\delta$ when potential j alone is present (i.e., $\lambda_i=0$, $i\neq j$), and in Eq. (5), P stands for the principal value.

In particular, for the shape given in Eq. (3)

$$k\cot\delta_j=-(1/a_j)+\tfrac{1}{2}r_{0j}k^2+P_jk^4, \quad j=1,2 \tag{6}$$

where the jth scattering length, effective range, and shape parameter, respectively, are given by

$$a_j=(2/\beta_j)(1+4\pi\beta_j^3/\mu\lambda_j)^{-1}, \tag{7}$$

$$r_{0j}=(1/\beta_j)(1-8\pi\beta_j^3/\mu\lambda_j), \tag{8}$$

$$P_j=-2\pi/\mu\lambda_j, \tag{9}$$

and

$$W=(\beta_1\beta_2-k^2)/(\beta_1+\beta_2). \tag{10}$$

In these expressions, μ is the reduced mass of the pair of particles in question. For the purpose of fitting the experimental data it is convenient to introduce

$$\alpha_j\equiv a_j^{-1}=\tfrac{1}{2}\beta_j(1+4\pi\beta_j^3/\mu\lambda_j), \tag{11}$$

so that

$$r_{0j}=(3\beta_j-4\alpha_j)/\beta_j^2, \tag{12}$$

$$P_j=(\beta_j-2\alpha_j)/(2\beta_j^4). \tag{13}$$

With these relations, $k\cot\delta_j$ may be expressed in terms of the two parameters α_j, β_j, and it is a *linear* function of α_j.

The expansion of Eq. (4a) in terms of the two-body scattering length a and effective range r_0,

$$k\cot\delta=-(1/a)+\tfrac{1}{2}r_0k^2+\cdots, \tag{14}$$

may be made explicit with the aid of the above equations for $k\cot\delta_j$. After some algebra, there result two equations:

$$a\alpha_1\alpha_2-\alpha_1-\alpha_2=(a\beta-2)\beta, \tag{15}$$

$$y_{12}\alpha_1\alpha_2+y_1\alpha_1+y_2\alpha_2=y_0, \tag{16}$$

where

$$\begin{aligned} y_{12}&=4a(\beta_1^{-2}+\beta_2^{-2}),\\ y_1&=ar_0-3a\beta_2^{-1}-4\beta_1^{-2},\\ y_2&=ar_0-3a\beta_1^{-1}-4\beta_2^{-1},\\ y_0&=2\beta ar_0+(\beta_1\beta_2)^{-1}[4\beta(1-a\beta)-3(\beta_1+\beta_2)], \end{aligned} \tag{17}$$

and

$$\beta=\beta_1\beta_2/(\beta_1+\beta_2).$$

Equations (15) and (16) are used to impose two conditions on each of the two sets of Λ-N potential. By construction, once values of a and r_0 are chosen, any set of parameters λ_1, λ_2, β_1, β_2 that satisfies these equations defines a sum of NLS potentials that reproduces the chosen scattering length and effective range.

For the N-N potential, the experimental parameters to be fit are the scattering length and the deuteron binding energy. The condition for the existence of a bound state of the given pair of particles with binding energy

$$\epsilon\equiv\alpha^2/(2\mu), \tag{18}$$

is

$$k\cot\delta=ik=-\alpha. \tag{19}$$

This condition and Eqs. (4), (5), and (10)–(13) yield

$$z_{12}\alpha_1\alpha_2+z_1\alpha_1+z_2\alpha_2=z_0, \tag{20}$$

where

$$\begin{aligned} z_{12}&=[1-(\alpha/\beta_1)^2]^2[1-(\alpha/\beta_2)^2]^2,\\ z_1&=[1-(\alpha/\beta_1)^2]^2[\alpha^2(3\beta_2^2-\alpha^2)-2\beta_2^3\alpha]/(2\beta_2^3),\\ z_2&=[1-(\alpha/\beta_2)^2]^2[\alpha^2(3\beta_1^2-\alpha^2)-2\beta_1^3\alpha]/(2\beta_1^3),\\ z_0&=(W'-\alpha)^2-z_1z_2/z_{12}, \end{aligned} \tag{21}$$

and

$$W'=W \quad \text{with} \quad k\to i\alpha.$$

Once values of a and ϵ are chosen, any set of N-N potential parameters that satisfy Eqs. (15) and (20) defines a sum of NLS potentials that reproduces these chosen values.

The subscript $j=2$ on $v_j(r)$, $k\cot\delta_j$, etc., is chosen to refer to the repulsive part of the potential. Further, this repulsive part is taken to be of shorter range than the attractive part of the potential, so that

$$\lambda_2>0, \quad \lambda_1<0, \quad \beta_2>\beta_1. \tag{22a}$$

From Eqs. (2) and (3), the momentum-space matrix element of V is

$$\langle \mathbf{p}|V|\mathbf{p}'\rangle = \frac{\lambda_1}{(p^2+\beta_1^2)(p'^2+\beta_1^2)} + \frac{\lambda_2}{(p^2+\beta_2^2)(p'^2+\beta_2^2)},$$

where $\mathbf{p}$ ($\mathbf{p}'$) is the relative momentum of the two particles. Since the Born amplitude for scattering at energy $E=p^2/(2\mu)$ is proportional to this matrix element with $p'=p$, it is clear that if V is to act as a repulsive potential at high energies (which is one effect of a hard core inserted into a local attractive potential) it is necessary that

$$\lambda_2 > |\lambda_1|. \quad (22b)$$

Equations (22) are used to descriminate against "unphysical" fits to the two-body amplitudes.

B. Λ-N Parameters

To determine the parameters in the Λ-N amplitudes, the scattering lengths and effective ranges were set equal to the values obtained by HTS. These values are

$$a=-2.89\text{ F}, \quad r_0=1.94\text{ F} \quad (23a)$$

in the singlet state and

$$a=-0.71\text{ F}, \quad r_0=3.75\text{ F} \quad (23b)$$

in the triplet state. For each spin channel, the two free parameters were chosen to be the range parameters β_1 and β_2, since for fixed values of these inverse ranges Eqs. (15) and (16) become simple quadratic equations for the variables α_1 and α_2.

The values assumed by the parameters when $\lambda_2=0$ (i.e., no core) are shown in Table I. As the range of the attractive part of the potential is expected to shrink from its "no core" value,[14] β_1^{-1} was assumed ≤ 0.6 F for each spin channel. From Eq. (22a), only values of $\beta_2^{-1} \leq \beta_1^{-1}$ were used.

For fixed β_1^{-1}, β_2^{-1} in the ranges just described, the other parameters were determined by reducing Eqs. (15) and (16) to a linear relation between α_1 and α_2 and a quadratic equation for α_1 alone. The parameters λ_1 and λ_2 were obtained from the solutions to these equations in conjunction with Eq. (11). The equation for α_1 being quadratic, two sets of solutions were obtained. For one set the strengths had signs opposite to those expected; i.e., $\lambda_1>0$ and $\lambda_2<0$ which means a short-range attraction inside a longer-range repulsion. More important, however, for this set $|\lambda_2|>\lambda_1$ so that at high energies the potential appears attractive rather than repulsive. This set of solutions was discarded. For the other set of solutions, values of λ_1, λ_2 could be obtained such that Eqs. (22) were satisfied. Even for this set, however, with a fixed β_2^{-1}, there is a limited range of values of β_1^{-1} for which Eqs. (22) are satisfied.[15] Some typical results including the range of β_1^{-1} for which Eqs. (22) are satisfied are shown in Figs. 1 and 2.

[14] By analogy with the case of a hard core inserted into a two-parameter local potential well. This behavior is also present in the NLS potentials used in the first article of Ref. 11.

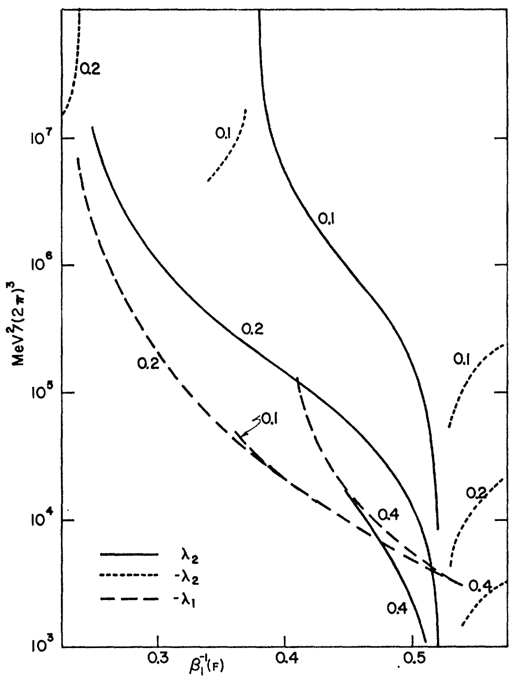

FIG. 1. Λ-N $^1S_0$ potential parameters. The attractive and repulsive potential strengths λ_1 and λ_2 are plotted against the attractive range β_1^{-1} for three different values of the repulsive range, $\beta_2^{-1}=0.1$, 0.2, and 0.4 F. For a given β_1, β_2, the values of λ_1 and λ_2 plotted have been adjusted to give the HTS singlet scattering length and effective range.

From Figs. 1 and 2 it appeared that $\lambda_2\to\infty$ was an acceptable limit; i.e., physically reasonable values of λ_1, β_1, β_2 such that the HTS values of a and r_0 could be fit still existed in this limit. This limit is of interest since with $\lambda_2\to\infty$ the core potential simulates a local hard core in that the Born approximation is never valid. The fact that in this limit the potential is *everywhere* infinite is of no consequence. The NLS potential has no physical significance other than as a parametrization of the interaction. The physical quantities of interest such as the scattering amplitude are all well behaved in this limit.

TABLE I. Values of the attractive potential parameters when no repulsive core potentials are present. These parameters have been adjusted so as to give the low-energy scattering parameters described in the text.

	β_1^{-1} (F)	λ_1 [MeV²/(2π)³]
N-N ($^3S_1$)	0.6978	-3.292×10^3
Λ-N ($^1S_0$)	0.5213	-3.961×10^3
Λ-N ($^3S_1$)	0.5920	-1.380×10^3

[15] From a plot in the α_1, α_2 plane of the hyperbola described by Eq. (15) and application of $\lambda_2>0$ and $\lambda_1<0$ but large (i.e., α_1, $\alpha_2>0$) it becomes clear that solutions exist only for limited ranges of the variables.

TABLE II. Λ-N potential parameters for $\lambda_2=\infty$. For a given β_2^{-1} the other parameters have been adjusted so as to give the HTS scattering lengths and effective ranges.

	β_2^{-1} (F)	β_1^{-1} (F)	λ_1 [$\mathrm{MeV}^2/(2\pi)^3$]
Λ-N ($^1S_0$)	0.1000	0.3789	-3.325×10^4
	0.2000	0.2403	-5.718×10^6
Λ-N ($^3S_1$)	0.1000	0.4095	-1.585×10^4
	0.2000	0.2544	-2.215×10^6

A determination of the other Λ-N parameters in the $\lambda_2\rightarrow\infty$ limit was carried out next. In this limit it follows from Eq. (11) that $\alpha_2\rightarrow\beta_2/2$: This value of α_2 was substituted into Eqs. (15) and (16) to yield two equations which were written

$$\alpha_1=\alpha_1(\beta_2,\beta_1)\,,\quad \beta_1=\beta_1(\beta_2,\alpha_1)\,.$$

For several fixed values of β_2 these equations were iterated to obtain β_1 and α_1. The strength λ_1 was then obtained as above. Some typical results are shown in Table II.

Out of the eight free Λ-N potential parameters, four parameters (two in each spin channel) have now been determined by fixing the scattering length and effective range in each spin channel at the HTS values. In the local potential variational calculations—such as HTS—it is usual to fix the range of the attractive potential by requiring the intrinsic range b in each channel to correspond to a two pion exchange mechanism; namely $b=1.5$ F.[16] Because of the more complicated form of the potential used in the present work, this was not done here.[17] This is a defect of the calculation in that it is a breakdown in the parallelism with the HTS work. It is not viewed as a serious defect. It is felt that the two-body t-matrix elements are the most physically significant quantities and these by construction are the same (at least on the energy shell at low energies) as for the HTS local potential calculation. Further, it is possible to duplicate some other features of the HTS potential. The ranges of the repulsive potentials in each channel may be set equal and the repulsive strength in each channel may be made infinite. With this choice there remains one free parameter—the common core range—just as in the local potential calculations.[18]

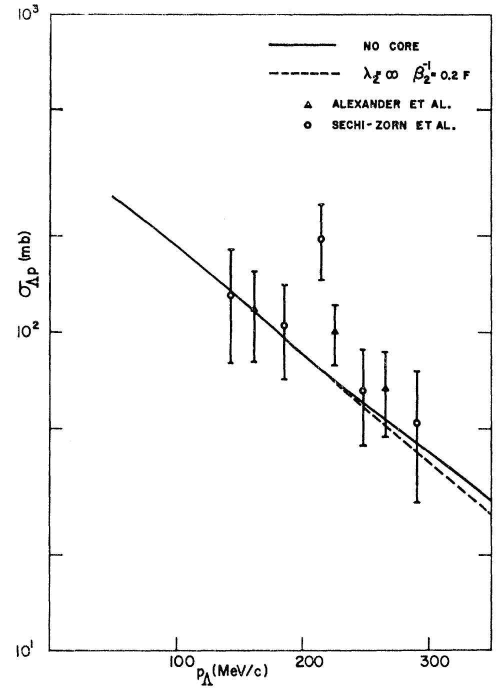

FIG. 3. Low-energy Λ-p elastic-scattering cross section versus Λ laboratory momentum. The solid curve is calculated with purely attractive Λ-N potentials; i.e., no repulsive cores. The dashed curve is calculated with the Λ-N potentials given in the last row of Table IV. The experimental points are from Ref. 7. See Ref. 8 for more recent experimental results.

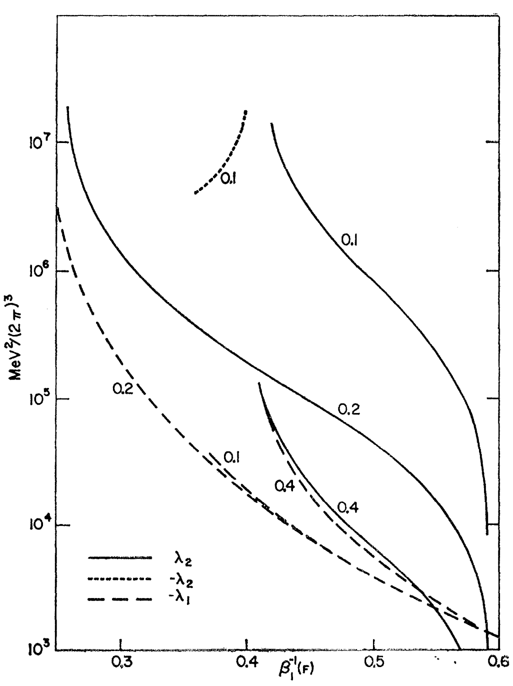

FIG. 2. Λ-N $^3S_1$ potential parameters. The attractive and repulsive potential strengths λ_1 and λ_2 are plotted against the attractive range β_1^{-1} for three different values of the repulsive range, $\beta_2^{-1}=0.1$, 0.2, and 0.4 F. For a given β_1, β_2 the values of λ_1 and λ_2 plotted have been adjusted to give the HTS triplet scattering length and effective range.

[16] For a discussion of how b was related to the other local potential parameters, see B. W. Downs, D. R. Smith, and T. N. Truong, Phys. Rev. **129**, 2730 (1963).

[17] Calculations which include consideration of the intrinsic range are in progress.

[18] In HTS, a hard-core range $r_c=0.4$ F was used. The relation of β_2^{-1} to this local range is not clear. It is easily verified that in the limit $\lambda_2\rightarrow\infty$ the repulsive NLS potential used here yields the same low-energy scattering amplitude as a local hard core of radius $2\beta_2^{-1}$; i.e., $r_c=0.4$ F corresponds to $\beta_2^{-1}=0.2$ F, which is exactly the value fixed on in Sec. III. No great significance is attached to this value. For a general discussion of the range parameter of an NLS potential, see A. N. Mitra, Phys. Rev. **123**, 1892 (1961).

In Fig. 3 the low-energy Λ-p elastic cross section is shown as a function of Λ lab momentum. Curves obtained from calculations using the NLS potentials discussed above (both with and without repulsive cores) are shown. For the momentum range shown, the scattering lengths and effective ranges dominate the amplitudes, the core effects becoming distinguishable only at the upper end of this range. For all practical purposes the fit to the low-energy data shown[8] is independent of the core parameters.

C. N-N Parameters

To determine the parameters in the N-N $^3S_1$ potential, the scattering length and deuteron binding energy were fixed at the experimental values[19]

$$a=5.39\text{ F},\quad \epsilon=2.225\text{ MeV}.$$

Equations (15), (20), and (11) were then used to determine λ_1 and λ_2 for given β_1 and β_2. The general features of the calculation are the same as those of the Λ-N calculation described above. Results which include the range of β_1^{-1} for which Eqs. (22) are satisfied are shown in Fig. 4.

Here, also, no attempt was made to incorporate a fit to the intrinsic range into the determination of the potential parameters. Further, the effective range for all values of β_2^{-1} shown in Fig. 4 varied over 1.730–1.745 F for those λ's that satisfied Eqs. (22). The experimental value of the effective range is 1.704 F.[20] As expected, this parameter is well-fit already and is insensitive to the potential parameters as long as a and ϵ are kept at their experimental values.

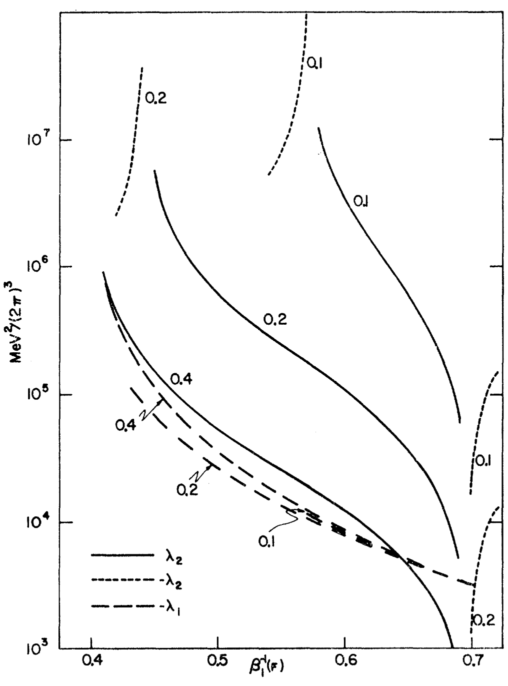

FIG. 4. N-N $^3S_1$ potential parameters. The attractive and repulsive potential strengths λ_1 and λ_2 are plotted against the attractive range β_1^{-1} for three different values of the repulsive range, $\beta_1^{-1}=0.1$, 0.2, and 0.4 F. For a given β_1, β_2 the values of λ_1 and λ_2 plotted have been adjusted to give the experimental values of the triplet scattering length and deuteron binding energy.

TABLE III. Values of B_Λ for various N-N potentials and both Λ-N potentials purely attractive. The Λ-N no core potential parameters used here are given in Table I.

	N-N ($^3S_1$)			
β_1^{-1} (F)	λ_1 [MeV^2/$(2\pi)^3$]	β_2^{-1} (F)	λ_2 [MeV^2/$(2\pi)^3$]	B_Λ (MeV)
0.6978	-3.292×10^3	...	...	0.880
0.6400	-5.345×10^3	0.2000	4.930×10^4	0.845
0.5500	-1.363×10^4	0.2000	2.459×10^5	0.765
0.4600	-5.440×10^4	0.2000	2.510×10^6	0.630
0.5500	-1.630×10^4	0.4000	2.600×10^4	0.825
0.4600	-8.625×10^4	0.4000	1.171×10^5	0.775

The remaining N-N parameters could be determined by again paralleling the local hard-core potential calculations; i.e., the core range could be set equal to that used in the Λ-N potentials and the limit $\lambda_2\to\infty$ could be taken. Within the framework of two-body NLS potential the aim of this work was to show the feasibility of fitting B_Λ and the low-energy Λ-p cross section rather than to produce the most economical way to obtain these fits. Thus this procedure was not carried out. Instead, after some preliminary calculations of B_Λ described below the core range β_2^{-1} was fixed and the N-N core strength was varied (by varying β_1^{-1}) until a fit to the "known" value of B_Λ was obtained.

III. $_\Lambda H^3$ BINDING ENERGY

The hypertriton binding energy was determined by the use of a Faddeev type of multiple-scattering analysis which yielded a set of coupled integral equations for the Λ-d doublet scattering amplitude. The three-body bound-state pole in this amplitude was found by varying the energy until the Fredholm determinant for the set of integral equations vanished. It is important to note that, unlike the calculation of the scattering amplitude itself, in the (negative) energy region of interest all quantities that appear in the calculation are real. There is a corresponding reduction by a factor of two in the computer memory size needed relative to the space needed in the scattering-amplitude calculation. It is this factor of two that makes the present work, which involves up to *six* coupled integral equations, feasible.[21]

The HTS Λ-N scattering lengths and effective ranges used above were obtained from a variational calculation

[19] J. L. Gammel and R. H. Thaler, Progr. Elem. Particle Cosmic Ray Phys. **5**, 99 (1960). The potential used here gives an exact fit to these parameters, while the potential used in HTS gives $a=5.35$ F and $\epsilon=2.253$ MeV.

[20] M. H. MacGregor, M. J. Moravosik, and H. P. Stapp, Ann. Rev. Nucl. Sci. **10**, 291 (1960).

[21] The numerical calculations were carried out on the University of Southern California Computer Sciences Laboratory's Honeywell 800, a machine with a 32K-word memory.

TABLE IV. Values of B_Λ for various Λ-N potentials and purely attractive N-N potentials. The N-N no-core potential parameters used here are given in Table I.

Λ-N ($^1S_0$) β_1^{-1} (F)	λ_1 [MeV^2/$(2\pi)^3$]	β_2^{-1} (F)	λ_2 [MeV^2/$(2\pi)^3$]	Λ-N ($^3S_1$) β_1^{-1} (F)	λ_1 [MeV^2/$(2\pi)^3$]	β_2^{-1} (F)	λ_2 [MeV^2/$(2\pi)^3$]	B_Λ (MeV)
0.5000	-5.089×10^3	0.2000	1.272×10^4	0.5000	-3.975×10^3	0.2000	4.481×10^4	0.815
0.2600	-1.143×10^6	0.2000	5.226×10^6	0.2600	-1.422×10^6	0.2000	1.862×10^7	0.425
0.2403	-5.718×10^6	0.2000	∞	0.2544	-2.215×10^6	0.2000	∞	0.360

in which B_Λ as given in Eq. (1) was inserted as a known parameter. The central value of Eq. (1) was therefore used here as the experimental value. The three-body calculations were carried out until good agreement with this value was obtained.

The equations used to calculate B_Λ when each two-body interaction is a single S-wave NLS potential have been described in HS3. The adaption of these equations to the case at hand is straightforward. The resulting equations are given in the Appendix. It is merely noted here that calculations were carried out with the core strength equal to zero (i.e., no core) in all, some, and none of the two-body potentials, so that programs containing from three to six coupled integral equations were used.

The "no-core" result is shown in the first row of Table III.[22] The rest of Table III shows the effect of inserting a repulsive core in the N-N potential only. For rows 2 through 4 the N-N core radius was set at $\beta_2^{-1}=0.2$ F and values of β_1^{-1} (with corresponding λ_1 and λ_2) were chosen from Fig. 4 such as to sample the range of finite λ_1, λ_2 that satisfy Eqs. (22). For rows 5 and 6 β_2^{-1} was set at 0.4 F and values of β_1^{-1} that duplicated those in rows 3 and 4 were used. It is seen from this table (and Tables IV and V as well) that, as expected, the insertion of a repulsive core causes the attractive potential to shrink in radius and increase in depth. From a comparison of row 3 with row 5 and row 4 with row 6, it is seen that for fixed β_1 a smaller β_2^{-1} implies a smaller B_Λ. This is so because in such a case a smaller β_2^{-1} implies a larger λ_2 and hence, according to the discussion at the end of Sec. IIA, a larger effective repulsion. Rows 2, 3, and 4 demonstrate (as does Fig. 4) that for fixed β_2^{-1}, since λ_2 increases much faster with decreasing β_1^{-1} than does $|\lambda_1|$, the smaller the attractive range the smaller the value of B_Λ. It appears from this table that without going to very small values (<0.1 F) of the core range it is impossible to reduce B_Λ to agreement with experiment by the use of a repulsive N-N core only. From a comparison of Figs. 1 and 2 with Fig. 4, it appears that to have any success at all in reducing B_Λ sufficiently, even with the addition of Λ-N cores, a core range $\beta_2^{-1}\leq 0.2$ F is needed. Within the limits of the computer, a core range of 0.2 F is about the smallest that could be used and still obtain trustworthy numerical results for B_Λ. Smaller values of β_2^{-1} were therefore not used.

TABLE V. Values of B_Λ for three of the N-N potentials given in Table III combined with the Λ-N potentials given in the last row of Table IV.

N-N ($^3S_1$) β_1^{-1} (F)	λ_1 [MeV^2/$(2\pi)^3$]	β_2^{-1} (F)	λ_2 [MeV^2/$(2\pi)^3$]	B_Λ (MeV)
0.6400	-5.345×10^3	0.2000	4.930×10^4	0.355
0.5500	-1.363×10^4	0.2000	2.459×10^5	0.345
0.4600	-5.440×10^4	0.2000	2.510×10^6	0.295

The effect of inserting a repulsive core only in the Λ-N potentials is shown in Table IV. With the core range in both spin channels fixed at 0.2 F and, again as in HTS, the attractive ranges in the two channels set equal, values of β_1^{-1} were again chosen so as to sample the range of finite λ_1, λ_2 from Figs. 1 and 2 that satisfy Eqs. (22). In row 1, $\beta_1^{-1}=0.5$ F was chosen because this corresponds to the attractive part of the potential having an intrinsic range corresponding to a two-pion exchange; i.e., for a single NLS potential $b=3\beta^{-1}$. This value is too large for the core to have much effect on B_Λ. However, with a value of β_1^{-1} close to the smallest value allowed, it is seen from row 2 that λ_2 finally becomes large enough in both channels to significantly reduce B_Λ. By going to an infinite λ_2,[23] a value of B_Λ is obtained such as to indicate that the experimental value may be reached by the inclusion of an N-N core.

In Table V the results found for B_Λ when cores are inserted in all three two-body potentials are displayed. In all three potentials the core range has been fixed at $\beta_2^{-1}=0.2$ F. Both core strengths in the Λ-N potentials have been made infinite. The one free parameter left is the N-N attractive range. Results are given for the three sets of values of the N-N parameters used in rows 2, 3, and 4 of Table III. A good fit to the experimental value of B_Λ given in Eq. (22) has been obtained with the values of the parameters shown in the last row of Table V. This fit is so good that the calculations were not carried further. However, it is clear from the results presented that with the N-N core strength also infinite, the *only* free parameter for the entire calculation, the common core radius β_2^{-1}, could be varied until the

[22] The no-core result for B_Λ given here differs from that given in HS3 by 0.02 MeV, partly due to the use of slightly different input parameters and partly due to the limited accuracy with which the numerical work could be carried out. The error in any of the values of B_Λ shown in the tables is ≤ 0.01 MeV.

[23] In this case, with β_2^{-1} the same in each channel, the values of β_1^{-1} are necessarily not the same.

experimental value of B_Λ was obtained to as high an accuracy as desired.

It has been demonstrated that with each two-body interaction represented by a sum of NLS potentials adjusted to reproduce the two-body low-energy data, the same value of B_Λ may be obtained as in a calculation using local potentials with hard cores similarly adjusted. Whether this success can be repeated if the known behavior of the high-energy *N-N* *S*-wave phase shift is used to further restrict the two-body parameters remains to be seen. The sensitivity of the result to different types of NLS potentials needs to be investigated. The ability of sums of NLS potentials determined on the basis of two- and three-body calculations to produce the known values of the binding energies and mesonic decay rates of other light hypernuclei, as well as the binding energy of the Λ in nuclear matter, is also a subject for further study.

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APPENDIX

The two-body momentum-space *S*-wave *t*-matrix element at energy E for a single NLS potential, such as given in Eq. (2) with $\lambda_2=0$, may be written

$$\langle \mathbf{p}|t_E|\mathbf{p}'\rangle = t_E(p,p') = v_1(p)\tau_E v_1(p')\,, \qquad \text{(A1)}$$

where

$$\tau_E=\lambda_1[1-g_k{}^{11}\lambda_1]^{-1}\,, \qquad \text{(A2)}$$

$$g_k{}^{ij}=\int\frac{d\mathbf{q}}{(2\pi)^3}\frac{v_i(q)v_j(q)}{[E^+-(q^2/2\mu)]}\,, \qquad \text{(A3)}$$

where

$$E^+=E+i\eta\,,\quad \eta\to 0^+\,,$$
$$k=(2\mu E)^{1/2}\,,$$

and for the general off-energy-shell matrix element

$$k\neq p\,,\quad k\neq p'\,,\quad p\neq p'\,.$$

According to HS3,[24] with each two-particle interaction represented by such a potential, the elastic Λ-*d* scattering amplitude, in either the doublet or quartet spin state, from an initial state b to a final state a may be written as the partial-wave sum

$$M_{ab}=\textstyle\sum_l(2l+1)[\eta_l{}^{\rm IA}+\eta_l{}^{\rm MS}]P_l(\hat{q}_a\cdot\hat{q}_b)\,. \qquad \text{(A4)}$$

Here $\eta_l{}^{\rm IA}$ is the impulse approximation contribution, which is of no interest here, while $\eta_l{}^{\rm MS}$ is the multiple scattering contribution. As the $_\Lambda H^3$ is bound in an S state, only the $l=0$ part of Eq. (A4) is relevant. The subscript l will now be dropped, $l=0$ being understood.

The *S*-wave amplitude of interest is given by

$$\eta^{\rm MS}=\sum_{\alpha\beta\lambda\gamma}\int_0^\infty\int_0^\infty \Phi_{\alpha 2}(q,q_a)\tau^{\alpha\gamma}(q)R_{\gamma\lambda}(q,q')\tau^{\lambda\beta}(q') \times\Phi_{\beta 2}(q',q_a)(2\pi)^{-4}q^2q'^2dqdq'\,. \qquad \text{(A5)}$$

In this equation the Greek indices run 1 through 3, referring, respectively, to the Λ-*N* $S=0$, *N-N*, and Λ-*N* $S=1$ interactions. The functions $\Phi_{\alpha 2}$ is the zero-order partial-wave part of the product of the initial-state wave function with v_α. The matrix $\tau^{\alpha\beta}$ is given by

$$\tau^{\alpha\beta}=\begin{bmatrix}\tau_1 & 0 & 0\\ 0 & \tau_2 & 0\\ 0 & 0 & \tau_3\end{bmatrix}\,, \qquad \text{(A6)}$$

where again the subscripts 1, 2, 3 refer to the respective two-particle channels in the order just given. The matrix element $R_{\alpha\beta}(q,q')$ satisfies the integral equation

$$R_{\alpha\beta}(q,q')=K_{\alpha\beta}(q,q')+\sum_{\gamma\lambda}\int_0^\infty K_{\alpha\gamma}(q,q'')\tau^{\gamma\lambda}(q'') \times R_{\lambda\beta}(q'',q')(2\pi)^{-2}q''^2dq''\,, \qquad \text{(A7)}$$

with

$$K_{\alpha\beta}(q,q')=W_{\alpha\beta}\int_{-1}^{1}\frac{v_\alpha(|\mathbf{q}'+(m_\beta/\mathfrak{M}_\alpha)\mathbf{q}|)v_\beta(|\mathbf{q}+(m_\alpha/\mathfrak{M}_\beta)\mathbf{q}'|)d\omega}{E-(q^2/2m_\alpha)-(q'^2/2m_\beta)-(\mathbf{q}+\mathbf{q}')^2/(2m_0)}\,. \qquad \text{(A8)}$$

In Eq. (A8), ω is the cosine of the angle between $\mathbf{q}$ and $\mathbf{q}'$, m_β is the mass of the particle *not* present in the βth two-particle channel, $\mathfrak{M}_\alpha=m_\beta+m_0$, and $m_0=\mathfrak{M}-m_\alpha-m_\beta$, where $\mathfrak{M}$ is the total mass of all three particles. The matrix $[W_{\alpha\beta}]$ for the total spin state of interest, the doublet state, is again from HS3

$$[W_{\alpha\beta}]=\begin{bmatrix}1/2 & \sqrt{(3/2)} & -\sqrt{3}/2\\ \sqrt{(3/2)} & 0 & -1/\sqrt{2}\\ -\sqrt{3}/2 & -1/\sqrt{2} & -1/2\end{bmatrix}\,. \qquad \text{(A9)}$$

The integral equation (A7) may be written symbolically

$$R=K+K\tau R\,, \qquad \text{(A10)}$$

where the matrices R, K, τ are all parametrized by the energy E. To find the multiple-scattering contribution to the scattering amplitude, Eq. (A10) is solved numerically by converting it to a purely matrix equation

[24] The equations described here are those given in J. H. Hetherington and L. H. Schick [Phys. Rev. **137**, B935 (1965)] as modified in the Appendix of HS3.

(i.e., the continuous variables are replaced by discrete variables) and by using matrix inversion to obtain

$$R=[1-K\tau]^{-1}K \tag{A11}$$

explicitly. Substitution of this result into Eq. (A5) yields η^{MS}. To obtain the ${}_{\Lambda}H^3$ binding energy, it merely needs to be noted that at the bound-state energy η^{MS}, and hence R, is singular. From Eq. (A11), then, the bound state occurs at that energy for which

$$\det[1-K\tau]=0. \tag{A12}$$

With E_0 as the bound-state energy, B_Λ is given by

$$B_\Lambda=|E_0|-\epsilon. \tag{A13}$$

Equations (A6), (A8), (A9), (A12), and (A13) were used in HS3 to determine B_Λ. The modifications in these equations necessary to make them applicable to the present work follow directly from the modifications to Eqs. (A1) and (A2). The generalization of these last expressions to an arbitrary sum of NLS potentials is easily seen to be

$$\langle \mathbf{p}|t_E|\mathbf{p}'\rangle=t_E(p,p')=\tilde{V}(p)\tau_E V(p'),$$

where

$$\tilde{V}(p)=(v_1(p),v_2(p),\cdots)$$

is the transpose of the column vector $V(p)$, and the matrix τ_E is given by

$$\tau_E=\Lambda[1-G_E\Lambda]^{-1},$$

with

$$\Lambda=\begin{bmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots\end{bmatrix},\quad G_E=\begin{bmatrix}g_k^{11} & g_k^{12} & \cdots \\ g_k^{21} & \cdots & \\ \vdots & & \end{bmatrix}.$$

g_k^{ij} is given in Eq. (A3).

When each two-body interaction is a sum of two NLS potentials, the modifications to the two-body t-matrix elements can be propagated through the calculation of B_Λ. It is easily seen that Eqs. (A12) and (A13) can still be used provided Eqs. (A6), (A8), and (A9) are modified as follows:

Each zero in $[\tau^{\alpha\beta}]$ becomes a 2×2 zero matrix. Each element τ_α in $[\tau^{\alpha\beta}]$ becomes a 2×2 matrix which, suppressing the α index as well as the energy dependence, may be written

$$[\tau_\alpha{}^{ij}]=D^{-1}\begin{bmatrix}\lambda_1[1-g^{22}\lambda_2] & \lambda_1 g^{12}\lambda_2 \\ \lambda_2 g^{21}\lambda_1 & \lambda_2[1-g^{11}\lambda_1]\end{bmatrix},$$

with

$$D=(1-g^{11}\lambda_1)(1-g^{22}\lambda_2)-g^{12}\lambda_2 g^{21}\lambda_1.$$

Here the indices 1 and 2 are the same as those used in the body of the paper.

Each matrix element $W_{\alpha\beta}$ on the right-hand side of Eq. (A9) becomes a 2×2 matrix $[W_{\alpha\beta}{}^{ij}]$ given by

$$[W_{\alpha\beta}{}^{ij}]=\begin{bmatrix}W_{\alpha\beta} & W_{\alpha\beta}\\ W_{\alpha\beta} & W_{\alpha\beta}\end{bmatrix}.$$

For fixed q and q' each matrix element $K_{\alpha\beta}(q,q')$ becomes a 2×2 matrix. This matrix is found from the right-hand side of Eq. (A8) by letting

$$W_{\alpha\beta}\rightarrow[W_{\alpha\beta}{}^{ij}],$$
$$v_\alpha\rightarrow v_\alpha{}^i,$$
$$v_\beta\rightarrow v_\beta{}^j,$$

where both i and j run over 1 and 2. The potential $v_\alpha{}^i$ is for the αth two-body channel just what was called v_i in the body of the paper.

When only some of the two-body interactions are represented as a *sum* of NLS potentials, modifications similar to those just described must be made in Eqs. (A6), (A8), and (A9).