If one argues that the gauge vector fields become massive as a result of vacuum polarization,[6] one may equally well start with the PCTC relation $\partial_\nu T_{\mu\nu}{}^\alpha(x) = -ma'(-\Box^2)\Box^2 j_\mu{}^\alpha(x)/(-\Box^2+m^2)$ as fundamental, m being the average mass of the vector-meson octet in exact $SU(3)$. In this case, a breakdown of $SU(3)$ will only split the masses, leading to exactly the same results.

Note added in proof. After this work was completed, the author came to learn that the idea of PCTC has also been independently proposed in different contexts by R. F. Dashen and M. Gell-Mann [in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, 1966; Caltech report (unpublished)] and by S. Fubini, G. Segrè, and J. D. Walecka [Ann. Phys. (N. Y.) (to be published)].

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Amount of Four-Particle Production Required in S-Matrix Theory

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It is shown that a knowledge of the two-particle scattering amplitude below inelastic threshold places restrictions on the amount of four-particle production at high angular momenta. Conversely, if there is no four-particle production, there is no elastic scattering.

INTRODUCTION

An elastic-scattering amplitude which has the analytic structure given by the Mandelstam representation and which satisfies the unitarity condition and crossing relations is severely restricted[1]: It cannot grow too rapidly with increasing energy,[2] poles in coupled scattering amplitudes are related,[3] and singularities must occur at production thresholds.[4]

In this note we explore the last restriction in greater detail. We shall show that the elastic two-particle scattering amplitude is directly related, even *below* inelastic thresholds, to the cross section for multiparticle production processes at high angular momenta. Thus, two-particle scattering depends on multiparticle production.

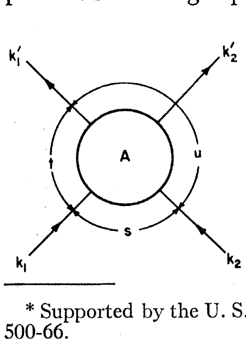

FIG. 1. Scattering-amplitude kinematics.

Conversely, if the two-particle scattering amplitude is known below inelastic thresholds, restrictions are placed on the amount of production at high angular momenta.

For simplicity, we shall consider a theory in which there is only one kind of particle having mass m and spin zero. We furthermore assume a theory of the ϕ^4 variety so that the S matrix does not connect states having an even number of particles with states having an odd number. The extension to the general case involves only algebraic complications provided the masses are not too different.

Section 1 discusses the unitarity equations for two-particle partial-wave amplitudes above the first inelastic threshold. Section 2 relates the partial-wave-production cross section defined in Sec. 1 to an integral over the double spectral function. This double spectral function is calculated in Sec. 3 using elastic unitarity in the crossed channel. The conclusions of Secs. 2 and 3 are combined in Sec. 4 to obtain a result for the production cross section at high angular momenta. Various implications of the result are discussed in a final section.

1. INELASTIC UNITARITY

Let $A(s,t,u)$ be the plane-wave scattering amplitude depicted in Fig. 1. It is defined by the relations

$$\langle k_1'k_2'|T|k_1k_2\rangle = \delta_4(k_1+k_2-k_1'-k_2')A(s,t,u)\,, \quad (1.1)$$

$$S=1+2iT\,, \quad (1.2)$$

where S is the scattering matrix and $|k_1k_2\rangle$ is a two-par-

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[1] M. Froissart, Phys. Rev. 123, 1053 (1961); A. Martin, Phys. Rev. Letters 9, 410 (1962).
[2] A. Martin, Nuovo Cimento 42A, 901 (1966); O. W. Greenberg and F. E. Low, Phys. Rev. 124, 2047 (1961).
[3] A. J. Dragt and R. Karplus, J. Math. Phys. 5, 120 (1964).
[4] S. Ø. Aks, J. Math. Phys. 6, 516 (1965); F. F. Cheung, University of Maryland Technical Report No. 534, 1965 (unpublished).

ticle linear-momentum state normalized by the rule

$$\langle k_1'k_2'|k_1k_2\rangle=\omega_1\omega_2\delta_3(\mathbf{k}_1-\mathbf{k}_1')\delta_3(\mathbf{k}_2-\mathbf{k}_2'). \quad (1.3)$$

The rule assures that Lorentz transformations realized by the relation

$$U(L)|k_1k_2\rangle=|Lk_1,Lk_2\rangle \quad (1.4)$$

are unitary transformations. The variables s, t, and u are defined by the usual equations

$$\begin{aligned} s&=(k_1+k_2)^2,\\ t&=(k_1-k_1')^2,\\ u&=(k_2-k_1')^2. \end{aligned} \quad (1.5)$$

We shall also use the variables k and z, the relative momentum and scattering-angle cosine, defined by the equations

$$s=4(k^2+m^2), \quad (1.6)$$

$$t=-2k^2(1-z), \quad (1.7)$$

$$u=-2k^2(1+z). \quad (1.8)$$

The unitarity condition

$$S^\dagger S=1 \quad (1.9)$$

is most simply expressed with the aid of states having a definite angular momentum. We define a two-particle angular-momentum state at rest in the center-of-momentum frame in terms of linear-momentum states by the equation

$$|Q^0JJ_3\rangle=[4k(2J+1)]^{1/2}s^{-1/4} \times\int dR\, D_{J_3,0}{}^{J*}(R)U(R)|k_1{}^0k_2{}^0\rangle. \quad (1.10)$$

Here dR indicates Haar integration over the rotation group, and $k_1{}^0$, $k_2{}^0$, and Q^0 are the 4-vectors

$$\begin{aligned} k_1{}^0&=(k\mathbf{e}_3,\omega_1),\\ k_2{}^0&=(-k\mathbf{e}_3,\,\omega_2),\\ Q^0&=(0,0,0,M), \end{aligned} \quad (1.11)$$

with

$$M=\omega_1+\omega_2=s^{1/2}.$$

Angular-momentum states with an arbitrary total 4-momentum Q are obtained by writing

$$|QJJ_3\rangle=U(L)|Q^0JJ_3\rangle, \quad (1.12)$$

where

$$Q=LQ^0, \quad (1.13)$$

and L is a pure velocity transfomation. From the definitions (1.10), (1.12), and the normalization rule (1.3), one obtains the analogous rule

$$\langle Q'J'J_3'|QJJ_3\rangle=\delta_4(Q'-Q)\delta_{J'J}\delta_{J_3'J_3}. \quad (1.14)$$

Equation (1.10) can also be inverted to express linear-momentum states in terms of angular-momentum states. One finds

$$U(R)|k_1{}^0k_2{}^0\rangle=M^{1/2}(4\pi k)^{-1/2} \times\sum_{JJ_3} D_{J_3,0}{}^J(R)(2J+1)^{1/2}|Q^0JJ_3\rangle. \quad (1.15)$$

The partial-wave scattering amplitude $A_J(s)$ is defined in analogy to Eq. (1.1) using angular-momentum states,

$$\langle Q'J'J_3'|T|QJJ_3\rangle=\delta_{J'J}\delta_{J_3'J_3}\delta_4(Q'-Q)A_J(s). \quad (1.16)$$

The relationship between the plane-wave and partial-wave scattering amplitude follows from Eqs. (1.10) and (1.15). One obtains the familiar formulas

$$A(s,t,u)=M(4\pi k)^{-1}\sum(2J+1)A_J(s)P_J(z), \quad (1.17)$$

and

$$A_J(s)=2\pi kM^{-1}\int_{-1}^{1} A(s,t,u)P_J(z)dz. \quad (1.18)$$

In terms of T, the unitarity condition (1.9) reads

$$(2i)^{-1}(T-T^\dagger)=T^\dagger T. \quad (1.19)$$

Taking matrix elements with two-particle angular-momentum states gives

$$\delta_4(Q'-Q)\ \mathrm{Im}A_J(s)=\langle Q'JJ_3|T^\dagger T|QJJ_3\rangle. \quad (1.20)$$

Now suppose that $16m^2<s<36m^2$ so that T connects two-particle states to two- and four-particle states. Then the two- and four-particle identities, $\mathbf{1}^{(2)}$ and $\mathbf{1}^{(4)}$, form a complete set of states for insertion between $T^\dagger$ and T on the right-hand side of Eq. (1.20). From the normalization (1.14) and a four-particle state normalization analogous to (1.3), we have

$$\mathbf{1}^{(2)}=\int d^4Q\ \theta(Q_0)\theta(Q^2-4m^2)\sum_{J,J_3}|QJJ_3\rangle\langle QJJ_3|, \quad (1.21)$$

and

$$\mathbf{1}^{(4)}=\int\prod_{i=1}^{4}(\omega_i{}^{-1}d^3k_i)|k_1k_2k_3k_4\rangle\langle k_1k_2k_3k_4|. \quad (1.22)$$

Inserting this into Eq. (1.20) gives

$$\mathrm{Im}\,A_J=|A_J|^2+k^2(4\pi)^{-1}\sigma_J(2\rightarrow4), \quad (1.23)$$

where

$$\sigma_J(2\rightarrow4)=4\pi k^{-2} \times\int\prod_{i=1}^{4}(\omega_i{}^{-1}d^3k_i)\delta_4(\sum k_i-Q)|B_J|^2, \quad (1.24)$$

and

$$\langle k_1k_2k_3k_4|T|QJJ_z\rangle=\delta_4(\sum k_i-Q)B_J(k_1k_2k_3k_4). \quad (1.25)$$

The inelastic term can be identified with the total partial-wave production cross section using the optical theorem. For $s<16m^2$, the inelastic term is absent, and

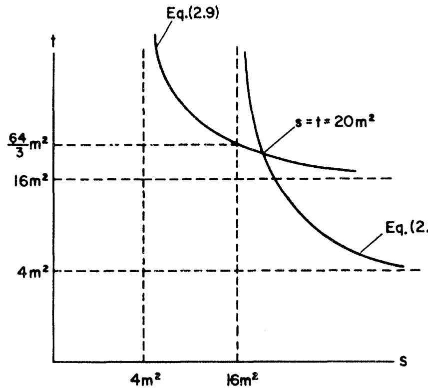

FIG. 2. The nearest set of Landau curves.

Eq. (1.23) has the solution

$$A_J=e^{i\delta_J}\sin\delta_J, \tag{1.26}$$

with δ_J real. Then writing

$$\sigma(2\to 2)=\sum(2J+1)\sigma_J(2\to 2), \tag{1.27}$$

gives

$$\sigma_J(2\to 2)=4\pi k^{-2}|A_J|^2. \tag{1.28}$$

Thus Eq. (1.23) is properly normalized provided we write

$$\sigma(2\to 4)=\sum(2J+1)\sigma_J(2\to 4). \tag{1.29}$$

2. RELATION TO THE SPECTRAL FUNCTION

We assume that $A(s,t,u)$ enjoys the analytic properties of the Mandelstam representation. Thus it is assumed analytic in the topological products of the s, t, and u planes cut along the real axis from $4m^2$ to $+\infty$. It then follows that $A(s,t,u)$ when viewed as a function of z for fixed positive s is analytic in the z plane with cuts running from z_0 to ∞ and $-\infty$ to $-z_0$, where

$$z_0(s)=1+8m^2(s-4m^2)^{-1}. \tag{2.1}$$

The z cuts are the image of the u and t cuts under the mappings (1.7) and (1.8).

With this information we can get an estimate on $|A_J(s)|$ for large J. Using the discontinuity relationship

$$P_J(z)=i\pi^{-1}[Q(z+i\epsilon)-Q(z-i\epsilon)],\quad z\in[-1,1] \tag{2.2}$$

allows Eq. (1.18) to be written as a contour integral,

$$A_J(s)=-2ikM^{-1}\int_C A(s,z)Q_J(z)dz, \tag{2.3}$$

where C is any contour encircling the interval $[-1,1]$ counterclockwise but avoiding the z cuts. We take C to be the ellipse C_0 with foci ±1 which just touches $\pm z_0$. We then have the inequality[5]

$$|Q_J(z)|\leq(\pi/J)^{1/2}\lambda_0^{-J}(\lambda_0^2-1)^{-1/2},\quad z\in C_0 \tag{2.4}$$

where

$$\lambda_0=z_0+(z_0^2-1)^{1/2}. \tag{2.5}$$

Inserting the inequality into (2.3) gives

$$|A_J(s)|\leq 4\pi W_0 z_0 kM^{-1}(\pi/J)^{1/2}\lambda_0^{-J}(\lambda_0^2-1)^{-1/2}, \tag{2.6}$$

where $W_0(s)$ is the maximum of $|A(s,z)|$ for z on C_0.

A similar calulation can be carried out for $\mathrm{Im}A_J$,

$$\mathrm{Im}\,A_J=-2ikM^{-1}\int_C \mathrm{Im}\,A(s,z)Q_J(z)dz. \tag{2.7}$$

Since A is real analytic,

$$\begin{aligned}\mathrm{Im}\,A(s,z)&=(2i)^{-1}[A(s+i\epsilon,z)-A(s-i\epsilon,z)]\\&=(2i)^{-1}D_sA(s,z),\end{aligned} \tag{2.8}$$

where D_s denotes the discontinuity is the s channel. Now D_sA as a function of z enjoys a larger region of analyticity in z than A itself. Its first singularities appear on the Landau curves which mark the boundary of the support of the double spectral function. The first set of Landau curves for $A(s,t,u)$ in the s and t variables are shown in Fig. 2. They have the equations

$$(s-4m^2)(t-16m^2)=64m^4, \tag{2.9}$$

$$(s-16m^2)(t-4m^2)=64m^4. \tag{2.10}$$

Analogous curves occur for other variable pairs by crossing symmetry. Thus for $20m^2\leq s\leq 36m^2$, D_sA is analytic in the z plane cut from $-\infty$ to $-z_1$ and z_1 to ∞, where

$$z_1(s)=1+8m^2s(s-4m^2)^{-1}(s-16m^2)^{-1}. \tag{2.11}$$

The cuts begin at the image of Eq. (2.10) and its partner in the pair s, u under the mappings (1.7) and (1.8). We now expand the contour of Eq. (2.7) to an ellipse C_1 which just touches $\pm z_1$. This gives the estimate

$$|\mathrm{Im}\,A_J|\leq 4\pi W_1 z_1 kM^{-1}(\pi/J)^{1/2}\lambda_1^{-J}(\lambda_1^2-1)^{-1/2}, \tag{2.12}$$

where λ_1 is defined by the analog of Eq. (2.5) and $W_1(s)$ is the maximum of $|A(s,z)|$ for z on C_1. The absolute-value sign on $\mathrm{Im}\,A_J$ is actually unnecessary since it is always positive by Eq. (1.23).

Comparing $\mathrm{Im}\,A_J$ and $|A_J|^2$ for large J, we find from our estimates (2.6) and (2.12) that $|A_J|^2$ becomes negligible compared to $\mathrm{Im}\,A_J$ for large J if $s>20m^2$ since then $\lambda_0^2>\lambda_1$.[6] Thus

$$\mathrm{Im}\,A_J\simeq k^2(4\pi)^{-1}\sigma_J(2\to 4) \tag{2.13}$$

for large J and $s>20m^2$. This is also evident from Fig. 2 which shows that Eq. (2.9) lies above Eq. (2.10), provided $s>20m^2$ In fact, the Landau curve (2.9) and its s, u partner come from the $|A_J|^2$ terms when Eq. (1.23) is substituted into the analog of Eq. (1.17) for $\mathrm{Im}\,A$. The other curve, Eq. (2.10) must consequently arise from the $k^2(4\pi)^{-1}\sigma_J(2\to 4)$ terms.

[5] E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics* (Cambridge University Press, Cambridge, England, 1931), p. 61.

[6] We assume here and show in Sec. 4 that Im A_J essentially takes on its upper bound for large J.

The estimate (2.12) can be improved to give an expression for $\operatorname{Im} A_J$ in the limit of large J. We first expand C_1 to a somewhat larger ellipse C_2 passing through the points $\pm(z_1+\epsilon)$. In doing so, we obtain small indented contours from $z_1+\epsilon-i\delta$ to $z_1+\epsilon+i\delta$ etc. running above and below the cuts (see Fig. 3). Now in the limit of high J, the bound (2.4) on Q_J says that the integral over C_2 is less than the integral over the indented contours since λ on C_2 exceeds the value of λ on the indented contours. Therefore,

$$\operatorname{Im} A_J \simeq -kM^{-1}\left[\int_{z_1}^{z_1+\epsilon} dz\, Q_J D_t D_s A - \int_{-z_1-\epsilon}^{-z_1} dz\, Q_J D_u D_s A\right] \quad (2.14)$$

for large J.

Defining the spectral function

$$\rho(s,t,u)=D_t D_s A(s,t,u) \quad (2.15)$$

finally gives

$$\operatorname{Im} A_J \simeq -2kM^{-1}\int_{z_1}^{z_1+\epsilon} \rho Q_J(z)dz\,, \quad (2.16)$$

for large even J. Here we have used the fact that A is an even function of z for identical spin-zero particles.

3. CALCULATION OF THE SPECTRAL FUNCTION

The use of Eq. (2.16) requires a knowledge of ρ for $s>20m^2$ and t near the boundary (2.10). By crossing symmetry, ρ is even under the interchange of s and t so that it suffices to know ρ for $t>20m^2$ and s near the boundary (2.9). Inspection of Fig. 2 reveals that this is the region for elastic scattering in the s channel provided $t>(64/3)m^2$. Consequently ρ can be obtained in this region entirely in terms of the elastic two-particle scattering amplitude using standard techniques.

Employing plane-wave states and setting $4m^2<s<16m^2$, the unitarity condition (1.19) along with real analyticity gives

$$D_s A(s,z)=2ikM^{-1}\int d\mathbf{\Omega}\, A^{\mathrm{II}}(s,z')A(s,z'')\,, \quad (3.1)$$

where $d\mathbf{\Omega}$ indicates integration over the solid angle of the two-particle intermediate state and $A^{\mathrm{II}}(=A^*)$ is the continuation of A across the cut beginning at $s=4m^2$. We next employ the representations

$$A(s,z)=(2\pi i)^{-1}\oint_E A(s,\eta)(\eta-z)^{-1}d\eta, \quad \text{etc.,} \quad (3.2)$$

where E is some counterclockwise contour about $[-1,1]$. The integral over $d\mathbf{\Omega}$ can then be done to give

$$D_s A(s,z)=k(2\pi^2 iM)^{-1}\oint_{E_1}\oint_{E_2} d\eta_1 d\eta_2 \times A^{\mathrm{II}}(s,\eta_1)A(s,\eta_2)K(\eta_1,\eta_1,z)\,. \quad (3.3)$$

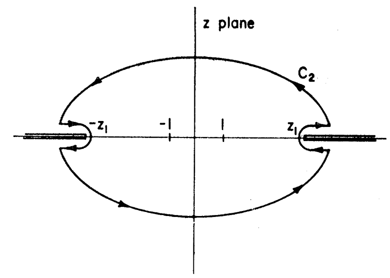

FIG. 3. The expanded ellipse C_2.

The kernel K has the representation

$$K(\eta_1\eta_2 z)=4\pi\int_{\eta_+}^{\infty} d\eta\, (\eta-z)^{-1} \times[(\eta-\eta_+)(\eta-\eta_-)]^{-1/2}, \quad (3.4)$$

with

$$\eta\pm=\eta_1\eta_2\pm[(\eta_1{}^2-1)(\eta_2{}^2-1)]^{1/2}. \quad (3.5)$$

We now continue Eq. (3.3) in z or t to find $D_t D_s A$. In doing so, it is necessary to expand the contours E_1 and E_2 in the η_1 and η_2 planes. A singularity in z occurs by way of the structure of K when the expansions of E_1 and E_2 are terminated by singularities in A^{II} and A. The amplitudes A^{II} and A are first singular when t_i or $u_i=4m^2$, where t_i and u_i are the images of η_i under the mappings (1.7) and (1.8) with z replaced by η_i.

At these points A^{II} and A have "square-root" singularities.[7] In fact, we may write for $\operatorname{Re} t>0$

$$A(s,t)=\phi(s,t)+i(s-4m^2)^{1/2}G(s,t) + i(t-4m^2)^{1/2}G(t,s)\,, \quad (3.6)$$

where ϕ and G are free of singularities in the region of interest except for possible compensating poles. The behavior for $\operatorname{Re} u>0$ is the same by crossing symmetry. For A^{II} we have

$$A^{\mathrm{II}}(s,t)=\phi(s,t)-i(s-4m^2)^{1/2}G(s,t) + i(t-4m^2)^{1/2}G(t,s) \quad (3.7)$$

by continuation around $s=4m^2$. The quantity G is defined by the equation

$$A(s,t)=F(s,t)+i(s-4m^2)^{1/2}G(s,t)\,, \quad (3.8)$$

revealing that $G(s,t)$ is the absorptive part of A is the s channel. Thus the discontinuities that appear when we are forced to indent E_1 and E_2 are familiar from elastic unitarity.

We now have sufficient information to calculate $D_t D_s A$. One finds

$$\rho=-32kM^{-1}\int_{z_0}^{\infty}\int_{z_0}^{\infty} d\eta_1 d\eta_2\, [(t_1-4m^2)(t_2-4m^2)]^{1/2} \times G(t_1,s)G(t_2,s)[(z-\eta_+)(z-\eta_-)]^{-1/2}\theta(z-\eta_+)\,. \quad (3.9)$$

Here we have again used the fact that A is even in z. The quantities z, t_i, and z_0 are those previously defined

[7] W. Zimmerman, Nuovo Cimento **21**, 249 (1961).

in Eqs. (1.7) and (2.1). Although the integral is formally over an infinite range, it is actually cut off due to the theta function. In fact, the region of integration shrinks to zero as we approach the boundary of the Landau curve. According to Eq. (2.16), we are only interested in values of ρ near the boundary. Consequently, we may replace the nonvanishing terms in the integrand by their values at the lower limit[8] to obtain

$$\rho \simeq -32kM^{-1}G^2(4m^2,s)(\eta_+-\eta_-)^{-1/2}\int\int d\eta_1 d\eta_2 \times[(t_1-4m^2)(t_2-4m^2)]^{1/2}(z-\eta_+)^{-1/2}\theta(z-\eta_+). \quad (3.10)$$

The integrals can now be evaluated near the boundary in terms of elementary functions with the aid of the substitution

$$\eta_i = \cosh\alpha_i, \quad (3.11)$$

and repeated application of the rule that nonvanishing terms in the integrand can be replaced by their values at the lower limit. One finds

$$\rho(s,t) = -(2\pi/15)G^2(4m^2,s)(s-4m^2)^{1/2}(\mathrm{ms})^{-1}(s+4m^2)^{-3} \times[(s-4m^2)(t-16m^2)-64m^4]^{5/2}+O([\ ]^{7/2}). \quad (3.12)$$

The above expression gives the leading term in ρ for $4m^2<s<16m^2$ and t near the Landau curve (2.9).[9] The remaining terms vanish as least as a $\frac{7}{2}$ power at the boundary.

4. THE PRODUCTION CROSS SECTION

We now have the necessary tools to exploit Eqs. (2.13) and (2.16), for from crossing symmetry the value of ρ for $s>(64/3)m^2$ and t near the curve (2.10) is given by interchanging s and t in Eq. (3.12).

In evaluating the integral (2.16), it is again permissible to take nonvanishing terms outside the integrand. Using this fact and Eq. (2.11) then gives

$$\operatorname{Im} A_J(s) \simeq f(s)\int_{z_1}^{z_1+\epsilon}(z-z_1)^{5/2}Q_J(z)dz, \quad (4.1)$$

where

$$f(s) = (\pi/\sqrt{2})G^2(4m^2,t)k(t-4m^2)^{1/2}(mMt)^{-1}(t+4m^2)^{-3} \times[(s-4m^2)(s-16m^2)]^{5/2}. \quad (4.2)$$

The value of t occuring in the expression for $f(s)$ is that given by Eq. (2.10).

The evaluation of the remaining integral can be accomplished by the following considerations: First of all, the upper limit can be replaced by ∞ for large J by the estimate (2.4). We are thus interested in calculating

$$g_J = \int_{z_1}^{\infty}(z-z_1)^{5/2}Q_J(z)dz \quad (4.3)$$

in the limit of large J. Next consider the function g defined by

$$g(z) = \pi i(z-z_1)^{5/2}, \quad (4.4)$$

with the root chosen to make $g(0)$ negative. Expand g in a Legendre series,

$$g(z) = \sum(2J+1)a_J P_J(z). \quad (4.5)$$

For a_J we have the formula

$$a_J = (2\pi i)^{-1}\oint_C g(z)Q_J(z)dz, \quad (4.6)$$

where C is some contour around the interval $[-1, 1]$. If we now expand the contour, we get the integral (4.3). Thus, $g_J = a_J$, and we only need to find the expansion coefficients in Eq. (4.5) for large J. This is a simple task. The formula for the generating function of the P_J can be rewritten to read[10]

$$(z_1-z)^{-1/2} = (2/\lambda_1)^{1/2}\sum\lambda_1^{-J}P_J(z). \quad (4.7)$$

Integrating Eq. (4.7) three times and employing the recurrence formulas for the P_J gives the result

$$g_J = (15\pi/16)(2/\lambda_1)^{1/2} \times[(\lambda_1-\lambda_1^{-1})/2]^3\lambda_1^{-J}J^{-4}[1+O(J^{-1})]. \quad (4.8)$$

The estimates given in Eqs. (2.13), (4.1), and (4.8) become equalities in the limit of large J. Combining them, we obtain the exact statement

$$\lim_{J\to\infty}\lambda_1^J J^4\sigma_J(2\to4) = h(s)G^2(4m^2,t), \quad (4.9)$$

where

$$h(s) = (3840)\pi^3m^4t^{-1}(t+4m^2)^{-3}\lambda_1^{-1/2} \times s(s-8m^2)^3(s-4m^2)^{-1}(s-16m^2)^{-1}, \quad (4.10)$$

and t is given as a function of s by Eq. (2.10).

5. DISCUSSION

If there is no four-particle production for a range of s in the inelastic region, then $G(4m^2,t)$ must vanish identically since it is analytic in t and t varies with s by Eq. (2.10). Then our derivation of Eq. (3.10) is incorrect, and we shall have to expand G in a power series about $4m^2$ with respect to its first argument and retain the first nonzero term. This process will again lead to a result similar to Eq. (4.9) except for higher powers of J and a different h, and we shall be forced to conclude that the term retained is also zero. Consequently, G must vanish identically as a function of both its arguments. But

[8] Here we assume $G(4m^2,s)\neq0$. If G did vanish, we would expand it in a power series about $t=4m^2$ and retain the first nonzero term. In this case, the term in square brackets of Eq. (3.12) has a higher power.

[9] L. Streit, Nuovo Cimento **23**, 934 (1962).

[10] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, England, 1952), p. 302.

then the elastic-scattering amplitude A must vanish identically by Eq. (1.23) since it has no imaginary part. The same conclusion can be reached if $\lambda_1{}^J\sigma_J(2\to 4)$ falls off with increasing J faster than any inverse power. Thus if there is to be any scattering (and production) at all, the four-particle production amplitude cannot fall off too fast with increasing J. A similar conclusion holds for A_J by Eq. (2.13).

The term $\lambda_1{}^J$ is suggestive of the expected threshold behavior for a production amplitude. For, looking at the expression for λ_1,

$$\lambda_1=1+[8m^2s+4ms^{1/2}(s-8m^2)]\times[(s-4m^2)(s-16m^2)]^{-1}, \quad (5.1)$$

one sees that λ_1 grows at threshold ($s=16m^2$) as $k_f{}^{-2}$, where k_f is the final momenta of one of the produced particles. However, our result does not constitute a proof because Eq. (4.9) is strictly true only for $(64/3)m^2 \leq s \leq 36m^2$. To extend the result to $s\approx 16m^2$ requires a knowledge of the behavior of the Landau curves for $s\approx 16m^2$ and $t\to\infty$. Conversely, if one is able to prove, say from quantum field theory,[11] that endothermic *production* processes do have the expected threshold behavior, then one may draw conclusions about the behavior of Landau curves.[12]

The term J^4 is a consequence of the two-sheeted nature of A at the two-particle elastic threshold plus the assumption that A grows at the expected rate at threshold. If A grows more slowly $[G(4m^2,t)\equiv 0$, etc.$]$, then higher powers of J are required to get a nonvanishing limit.

If we assume, as seems likely, that the Landau curves for production processes always lie above the curve Eq. (2.10) for any fixed s including $s>36m^2$, then Eq. (4.9) is valid for all $s>(64/3)m^2$. This is equivalent to the statement that four-particle production dominates six and more particle production in the high-J limit. Taking the limit as $s\to\infty$ gives

$$\lim_{s\to\infty}\lim_{J\to\infty}\lambda_1{}^J J^4\sigma_J(2\to 4) \approx 15\pi^3(8m^4)^{-1}G^2(4m^2,4m^2)s^2. \quad (5.2)$$

The order in which the limits are taken cannot in general be interchanged since the production Landau curves are tangent to Eq. (2.10) at $s=\infty$.[13]

The high-energy limit is peculiar since, by unitarity, each σ_J must satisfy

$$k^2(4\pi)^{-1}\sigma_J\leq 1. \quad (5.3)$$

Evidently the introduction of the factor $\lambda_1{}^J J^4$ changes the asymptotic behavior. Note that $\lambda_1\to 1$ as $s\to\infty$. An example of a prescription satisfying both Eqs. (5.2) and (5.3) is

$$\sigma_J\sim(\lambda_1{}^J J^4)^{-1}(a+bJ^{-4}k^6)^{-1}k^4 \quad (5.4)$$

with suitably chosen constants a and b.

Note added in proof. The main result, given in Eq. (4.9), can be extended to give a direct relation between inelastic and elastic cross sections by the following considerations: The functions F and G defined in Eq. (3.8) are real for s real and near $4m^2$, and t physical.[7] Combining this result with Eqs. (1.17) and (1.23) gives the relation

$$G(s,t)=M(8\pi k^2)^{-1}\sum(2J+1)|A_J|^2P_J(z). \quad (5.5)$$

The relation can be continued in t to define G for t below the Landau curve given by Eq. (2.9). Now let s approach $4m^2$ while holding t fixed. As s approaches threshold, $A_J(s)$ vanishes according to the power law

$$A_J\approx k^{2J+1}. \quad (5.6)$$

This result follows directly from the known analytic properties of F and G when combined with Eqs. (1.7) and (1.18).[7,14] When t is fixed, $P_J(z)$ grows according to the inverse power law

$$P_J\approx k^{-2J}. \quad (5.7)$$

Comparison of Eqs. (5.6) and (5.7) shows that only the term with $J=0$ contributes to the sum in Eq. (5.5) at threshold. Thus $G(4m^2,t)$ is in fact independent of t. Using Eq. (1.28) gives the explicit formula

$$G(4m^2,t)=m\sigma_0(2\to 2;4m^2), \quad (5.8)$$

where σ_0 is the *elastic* s-wave scattering cross section evaluated at threshold. Combining Eqs. (5.8) and (4.9) gives a direct relation between elastic and inelastic cross sections.

[11] G. Roepstorff and J. L. Uretsky, Phys. Rev. **152**, 1213 (1966).

[12] See Appendix of Ref. 3.

[13] In the case $G(4m^2,t)\equiv 0$, the power of s on the right-hand side of Eq. (5.2) is increased. Of course, we also need a higher power of J on the left-hand side.

[14] Y. S. Jin, Phys. Rev. (to be published).