Broken-Symmetry Sum Rules and the Algebra of Currents*

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A general method is developed for obtaining first-order broken-symmetry sum rules for form factors using the algebra of currents. It is found that these sum rules are the same as the corresponding ones derived group-theoretically provided that the same or analogous physical assumptions are made in both methods. This is illustrated by a derivation of the Muraskin-Glashow sum rules for strong baryon-meson coupling constants.

1. INTRODUCTION

THIS paper describes an investigation of the implications of having an octet of currents or densities forming part or all of a closed algebra with the generators of $SU(3)$. Using a method which is a natural extension of that used by Faustov[1] we obtain broken-symmetry sum rules for the form factors corresponding to these currents or densities. This method is sufficiently general that it could be applied to any form of amplitude and to any broken symmetry provided that the breaking mechanism is known.

Contrary to expectation, we find that the algebra of currents will produce the *same* broken-symmetry sum rules as earlier group-theoretical methods provided that the *same or analogous* physical assumptions are made. In particular, we show in Secs. II and III that the spin-$\frac{1}{2}^+$ baryon matrix elements of scalar and pseudoscalar densities satisfy five broken-$SU(3)$ sum rules which are identical with the sum rules for strong-interaction coupling constants derived by Muraskin and Glashow[2] from a standard group-theoretical technique.

The virtue of the current-algebra method is evidenced, however, in its ability to incorporate different physical assumptions and exhibit their effects. For example, we show in Sec. III that in the case that the nonzero divergences of the vector current[3] belong to the same octet as the scalar densities we obtain one additional sum rule (to the five above) involving K-type densities only. In Sec. IV we show the effect that different dynamical assumptions, such as the hypothesis of partially conserved axial-vector current (PCAC), have on sum rules for strong-interaction coupling constants.

In addition, we use our method to analyze the results of two other investigations[4] of strong-coupling constants and comment on the recent applications of scalar and pseudoscalar densities in nonleptonic baryon decays.

Finally, we point out further applications of our method which are of physical interest and, in Sec. V, discuss our results.

2. APPLICATION OF THE FUBINI-FURLAN TECHNIQUE

In our analysis we shall concentrate, without loss of generality, entirely on scalar and pseudoscalar densities, $J^{(\alpha)}(x)$, where α is a spherical $SU(3)$ index. We assume that the densities transform like an eight-vector even in the presence of $SU(3)$ breaking. This will be the case, for example, when these operators form part or all of a closed algebra with the generators of $SU(3)$ and may be mathematically expressed by

$$[Q^{(\alpha)}(t),J^{(\beta)}(x)]_{x_0=t}=-\sqrt{3}\begin{pmatrix}8 & 8 & 8_a\\ \alpha & \beta & \gamma\end{pmatrix}J^{(\gamma)}(x), \quad (1)$$

where

$$Q^{(\alpha)}(t)=-i\int d^3x\,\mathfrak{F}_4{}^{(\alpha)}(\mathbf{x},t),$$

$\mathfrak{F}_\mu{}^{(\alpha)}$ being the F-spin vector current, and

$$\begin{pmatrix}8 & 8 & 8_a\\ \alpha & \beta & \gamma\end{pmatrix}$$

is an $SU(3)$ Clebsch-Gordan coefficient as defined by De Swart.[5] Defining

$$D^{(\alpha)}(x)=\partial_\mu\mathfrak{F}_\mu{}^{(\alpha)}(x),$$

we have formally[6]

$$Q^{(\alpha)}(0)=\int d^4x\,\theta(-x_0)D^{(\alpha)}(x).$$

We assume λ_8 breaking of $SU(3)$ and hence $D^{(\alpha)}(x)$

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[1] R. Faustov, Nuovo Cimento 45, 145 (1966).
[2] M. Muraskin and S. L. Glashow, Phys. Rev. 132, 482 (1963).
[3] This is the case in the quark model [M. Gell-Mann, Phys. Rev. 125, 1067 (1962)] or simply when the breaking Hamiltonian is proportional to a scalar density S_8.
[4] S. K. Bose and Y. Hara, Phys. Rev. Letters 17, 409 (1966); R. J. Rivers, Phys. Letters 22, 514 (1966).
[5] J. J. De Swart, Rev. Mod. Phys. 35, 949 (1963).
[6] Note that all matrix elements of $D^{(\alpha)}(x)$ will be taken between states with unequal energies.

will have the same $SU(3)$ transformation properties as $Q^{(\alpha)}$.

Taking matrix elements of (1) between physical baryon states we obtain

$$\int d^4z\,\theta(-z_0)\langle B_1(p_1)|[D^{(\alpha)}(z),J^{(\beta)}(0)]|B_2(p_2)\rangle = -\sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ \alpha & \beta & \gamma \end{pmatrix}\langle B_1(p_1)|J^{(\gamma)}(0)|B_2(p_2)\rangle, \quad (2)$$

where translational invariance has been invoked.

If we now introduce k_1 so that $k_1^2=0$ and define $\nu=k_1p_1/m_1=k_1p_2/m_1$ then, following the method of Fubini *et al.*,[7] the left-hand side of (2) can be written as $F(0,\Delta^2)$, where

$$F(\nu,\Delta^2)=\frac{1}{\pi}\int_0^\infty \frac{A_{\rm I}(\nu',\Delta^2)}{\nu'-\nu}d\nu'-\frac{1}{\pi}\int_0^\infty \frac{A_{\rm II}(\nu',\Delta^2)}{\nu'-\nu}d\nu', \quad (3)$$

$$A_{\rm I}=\frac{i}{2}\sum_n (2\pi)^4\delta(p_1+k_1-p_n) \times\langle B_1|D^{(\alpha)}(0)|n\rangle\langle n|J^{(\beta)}(0)|B_2\rangle, \quad (4)$$

$$A_{\rm II}=\frac{i}{2}\sum_{n'} (2\pi)^4\delta(p_2-k_1-p_n') \times\langle B_1|J^{(\beta)}(0)|n'\rangle\langle n'|D^{(\alpha)}(0)|B_2\rangle.$$

We define the following matrix elements, between spin-$\frac{1}{2}^+$ baryon states

$$\langle B_1|D^{(\alpha)}(0)|B_2\rangle = i(m_1-m_2)\sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ B_2 & \alpha & B_1 \end{pmatrix}F_{B_1B_2}{}^{(\alpha)}(\Delta^2)\bar{u}_1u_2$$

and

$$\langle B_1|J^{(\beta)}(0)|B_2\rangle=G_{B_1B_2}{}^{\beta}(\Delta^2)\bar{u}_1\Gamma u_2, \quad \Gamma=1 \text{ or } \gamma_5,$$

where $F_{B_1B_2}{}^{(\alpha)}(0)=r_{B_1B_2}{}^{\alpha}$, the renormalization ratio, and $G_{B_1B_2}{}^{\beta}(\Delta^2)$ is the form factor for $J^{(\beta)}$.

If we now set $\nu=0$, Eqs. (2), (3), and (4) lead to the following relation:

$$\sum_{n,n'(=B)}\left\{\sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ n & \alpha & B_1 \end{pmatrix}r_{B_1n}{}^{\alpha}G_{nB_2}{}^{\beta}(\Delta^2) -\sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ B_2 & \alpha & n' \end{pmatrix}r_{n'B_2}{}^{\alpha}G_{B_1n'}{}^{\beta}(\Delta^2)\right\} +\frac{1}{\pi}\int\frac{{\rm Im}A_{B_1B_2}{}^{\alpha\beta}(\nu')d\nu'}{\nu'} = -\sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ \alpha & \beta & \gamma \end{pmatrix}G_{B_1B_2}{}^{\gamma}(\Delta^2), \quad (5)$$

where $A(\nu,\Delta^2)$ is defined by the decomposition $F(\nu,\Delta^2) =\bar{u}_1\{A(\nu,\Delta^2)\Gamma+B(\nu,\Delta^2)\Gamma\gamma\cdot k_1\}u_2$.

3. DERIVATION OF THE FIRST-ORDER SUM RULES

We wish to obtain broken-$SU(3)$ sum rules for the form factors which will be valid to first order in the symmetry-breaking parameter λ. We can therefore ignore the ratios $r_{BC}{}^A$ which, by the Ademollo-Gatto theorem,[8] differ from 1 only by a term of order λ^2. Since ${\rm Im}A_{B_1B_2}{}^{\alpha\beta}$ is of[9] order λ, we will take suitable linear combinations of relations of the form (5) so that the resulting continuum contribution is of order λ^2, the single-particle contribution remaining of order λ.

Since $D^{(\alpha)}(x)$ transforms as the αth member of an octet we can consider it as the source current for a fictitious "scalar meson," ζ_α and, similarly, consider $J^{(\beta)}(x)$ as the source current for a fictitious "scalar or pseudoscalar meson" π_β for $\Gamma=1$ or γ_5, respectively. Thus we can set up an analogy with scattering amplitudes that will provide an easy determination of the suitable linear combinations. First, we note that A_I and A_{II} in (4) have a form analogous to the scattering amplitudes in the s and u channels, respectively, for the "reaction" $\pi_\beta+B_2\to\zeta_\alpha+B_1$. Thus, by changing the variable of integration in the second integral in (3) to $(-\nu')$ we see that the continuum contribution in (5) has the same $SU(3)$ properties as an integral over the *sum* of the s- and u-channel amplitudes.[10] The ${\rm Im}A_{B_1B_2}{}^{\alpha\beta}$ will therefore obey the same $SU(3)$ sum rules as this sum of amplitudes. When $SU(3)$ is broken such sum rules will equal a quantity of at least order λ. Therefore, if we choose our linear combinations to correspond to these amplitude sum rules then, since ${\rm Im}A_{B_1B_2}{}^{\alpha\beta}$ is already of order λ, the resulting continuum contribution will be of at least order λ^2.

The necessary amplitude sum rules were determined by expressing the amplitudes in the direct channel. For example, the matrix element $\langle N|[Q^{(K0)},J^{(K0)}]|N\rangle$ gives rise to the amplitudes

$$(NK^0|NK^0)=A_{27}{}^s$$

for the s channel, and

$$(N\bar{K}^0|N\bar{K}^0)=(7/40)A_{27}{}^u+(1/12)A_{10}{}^u+(1/12)A_{\overline{10}}{}^u +(1/5)A_{8s}{}^u+(1/3)A_{8a}{}^u+(1/8)A_1{}^u$$

for the u channel. By requiring that the form factors $G_{B_1B_2}{}^{\beta}(\Delta^2)$ be real (for real Δ^2) we obtain a simple relation between the $SU(3)$ amplitudes for the two channels: $A_i{}^s=A_i{}^u$ for $i=27$, $8s$, $8a$, 10, $\overline{10}$, and 1 and $A_{8s8a}{}^s=A_{8a8s}{}^u$ and $A_{8a8s}{}^s=A_{8s8a}{}^u$.

[7] S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40A**, 1171 (1965).

[8] M. Ademollo and R. Gatto, Phys. Rev. Letters **13**, 264 (1964).

[9] This is due to the presence of only one divergence in the continuum expression in (4).

[10] Note that the s and u amplitudes are evaluated at ν and $-\nu$, respectively.

For the special case in which the scalar densities belong to the same octet as the $D^{(\alpha)}$'s,[3] we have[11] in addition that $A_{8s8a}{}^{j}=A_{8a8s}{}^{j}$, $j=s$ and u.

The form-factor sum rules can be obtained most directly by taking all possible matrix elements of the commutator $[Q^{K0},J^{K0}]=\frac{1}{2}J^{\pi 0}-\frac{1}{2}\sqrt{3}J^{\eta}$ together with the matrix element of one other commutator; for example, the matrix element of $[Q^{K0},J^{\eta}]=\frac{1}{2}\sqrt{3}J^{K0}$ between the states $|\Sigma^0\rangle$ and $|\Xi^0\rangle$. By eliminating the resulting continuum contributions, as described above, we obtain the five sum rules

$$\sqrt{3}G_{\Sigma^0\Sigma^0}{}^{\eta}-\sqrt{3}G_{NN}{}^{\eta}-G_{NN}{}^{\pi 0}+4G_{N\Sigma^0}{}^{K0}-G_{\Sigma^+\Sigma^+}{}^{\pi 0}=0\,, \quad (6)$$

$$\sqrt{3}G_{\Sigma^0\Sigma^0}{}^{\eta}-\sqrt{3}G_{\Xi^0\Xi^0}{}^{\eta}-G_{\Xi^0\Xi^0}{}^{\pi 0}+4G_{\Sigma^0\Xi^0}{}^{K0}+G_{\Sigma^+\Sigma^+}{}^{\pi 0}=0\,, \quad (7)$$

$$3\sqrt{3}G_{\Sigma^0\Sigma^0}{}^{\eta}-3\sqrt{3}G_{\Lambda\Lambda}{}^{\eta}+4G_{NN}{}^{\pi 0}+4G_{\Xi^0\Xi^0}{}^{\pi 0}+8G_{N\Sigma^0}{}^{K0}+8G_{\Sigma^0\Xi^0}{}^{K0}+6\sqrt{3}G_{\Sigma^0\Lambda}{}^{\pi}=0\,, \quad (8)$$

$$\sqrt{3}G_{N\Lambda}{}^{K0}-\sqrt{3}G_{\Sigma^0\Lambda}{}^{\pi 0}-2G_{NN}{}^{\pi 0}+G_{N\Sigma^0}{}^{K0}-2G_{\Sigma^0\Xi^0}{}^{K0}-G_{\Sigma^+\Sigma^+}{}^{\pi 0}=0\,, \quad (9)$$

$$\sqrt{3}G_{\Lambda\Xi^0}{}^{K0}-\sqrt{3}G_{\Sigma^0\Lambda}{}^{\pi 0}-2G_{\Xi^0\Xi^0}{}^{\pi 0}-2G_{N\Sigma^0}{}^{K0}+G_{\Sigma^0\Xi^0}{}^{K0}+G_{\Sigma^+\Sigma^+}{}^{\pi 0}=0\,. \quad (10)$$

In the special case in which the scalar densities belong to the same octet as the $D^{(\alpha)}$'s we obtain *one additional* sum rule:

$$S_{\Lambda\Xi^0}{}^{K0}-S_{N\Lambda}{}^{K0}-\sqrt{3}S_{N\Sigma^0}{}^{K0}+\sqrt{3}S_{\Sigma^0\Xi^0}{}^{K0}=0\,, \quad (11)$$

where $S_{B_1B_2}{}^{\beta}(\Delta^2)$ denotes the scalar form factor *only*.

4. APPLICATION

The most immediate application of our analysis is to strong-coupling problems. With the single assumption that the meson source density satisfies the commutator (1)[12] we can rederive the Muraskin-Glashow[2] sum rules for strong-interaction coupling constants since relations (6) to (10) are identical with these sum rules. Since the derivation of Ref. 2 was based on group theory our result is closely analogous to the current algebra derivations[13] of the Gell-Mann–Okubo mass formula.

Similarly, the assumption that the matrix elements of the pseudoscalar quark density are proportional to the corresponding matrix elements of the meson source current, $\langle B_1|\bar{q}\gamma_5\lambda^{(\alpha)}q|B_2\rangle=C\langle B_1|J_\pi{}^{(\alpha)}|B_2\rangle$, where C is independent of α, will also lead to the Muraskin-Glashow sum rules. It is interesting to note that Moffat[14] has found that this assumption leads to the Johnson-Treiman relations in the high-energy limit.

These coupling-constant sum rules can also be obtained from the analogous commutator to (1) for the axial-vector current $A_\mu{}^{(\beta)}$:

$$[Q^{(\alpha)},A_\mu{}^{(\beta)}(x)]=-\sqrt{3}\begin{pmatrix}8 & 8 & 8_a\\ \alpha & \beta & \gamma\end{pmatrix}A_\mu{}^{(\gamma)}(x)\,. \quad (12)$$

Taking the divergence, we obtain

$$[\dot{Q}^{(\alpha)},A_0{}^{(\beta)}(x)]+[Q^{(\alpha)},D_5{}^{(\beta)}(x)]=-\sqrt{3}\begin{pmatrix}8 & 8 & 8_a\\ \alpha & \beta & \gamma\end{pmatrix}D_5{}^{(\gamma)}(x)\,, \quad (13)$$

where

$$D_5{}^{(\beta)}(x)=\partial_\mu A_\mu{}^{(\beta)}(x)\,.$$

Matrix elements of the first commutator on the left-hand side of (13) have the same $SU(3)$ properties as the continuum contribution to the matrix elements of the second commutator. It can therefore be eliminated at the same time as the continuum and so will not affect the sum rules. Thus, following the same analysis as above and applying PCAC we obtain the five sum rules (6) to (10) but with each coupling constant $G_{B_1B_2}{}^{\alpha}$ multiplied by the factor $C_\alpha/m_\alpha{}^2$, where C_α and m_α are the PCAC proportionality constant and mass, respectively, for the αth meson. Note that our sum rules are taken at zero momentum transfer to avoid Δ^2-dependent factors. If we now assume the universality principle[15] $C_K/m_K{}^2=C_\pi/m_\pi{}^2=C_\eta/m_\eta{}^2$, we once again obtain the Muraskin-Glashow sum rules. If, however, we do not make this assumption we have two extra parameters $C_Km_\pi{}^2/C_\pi m_K{}^2$ and $C_\eta m_\pi{}^2/C_\pi m_\eta{}^2$, say, to be eliminated. This correction and the first commutator in (13) make up the contributions due to the mesons not belonging to an 8-vector. After eliminating these two parameters we are left with three nonlinear sum rules which will thus exhibit fewer restrictions placed on the coupling constants than are obtained in the group-theoretical derivation.

Other methods of obtaining coupling-constant sum rules have also been explored recently.[4] These investigations resulted in sum rules exhibiting more symmetry than experiment would lead us to expect. In the case of Bose and Hara's work we suspect that the added symmetry results from neglecting the first-order corrections which would come from taking matrix elements of the *symmetry-preserving* part of the Hamiltonian between *physical* states. One can in fact show, using (12), that their method is inconsistent with PCAC in that it is equivalent to neglecting important many-particle intermediate-state contributions to the matrix elements of some of the commutators. Consider, for example, the matrix element

$$\langle\Xi^0|[Q^{(K0)},A_\mu{}^{(K-)}]|P\rangle=0\,. \quad (14)$$

[11] K. Tanaka, Phys. Rev. **135**, B1886 (1964).

[12] Similar assumptions are implicitly made by authors working on the Lie algebra of strong coupling, cf. C. J. Goebel, Phys. Rev. Letters **16**, 1130 (1966).

[13] See Refs. 7 and 1, for example. It is interesting to note that the method of M. Boiti and C. Rebbi [Nuovo Cimento **43A**, 475 (1966)] for calculating corrections to the mass formula could conceivably be applied to these sum rules as well, should sufficiently accurate data provide justification for such an extension.

[14] J. W. Moffat, Phys. Letters **23**, 148 (1966).

[15] See W. Krolikowski, Trieste Report IC/66/52 (unpublished) and B. Renner, Cambridge Report, 1966 (unpublished), for a discussion of this point.

Using the assumption of Ref. 4 that the symmetry-breaking Hamiltonian is $\lambda S^{(8)}$, where $S^{(\alpha)}=\bar{q}\lambda^{(\alpha)}q$, we find $\dot{Q}^{(\alpha)}=i\lambda[S^{(8)},Q^{(\alpha)}]\propto -i\lambda S^{(\alpha)}$ so that in this case, the first commutator in (13) is identically zero. We therefore obtain

$$\langle\Xi^0|[Q^{(\bar{K}^0)},D_5^{(K^-)}]|P\rangle=0. \quad (15)$$

If we take only the single-particle intermediate states into account and use PCAC, we obtain the Bose and Hara sum rule

$$\sqrt{3}g_{\Lambda P}{}^{K^-}-g_{\Sigma^0 P}{}^{K^-}+\sqrt{2}g_{\Xi^0\Sigma^+}{}^{K^-}=0\,, \quad (16)$$

where $g_{B_1B_2}{}^{\alpha}$ is the $B_1B_2\ \pi_\alpha$ strong-coupling constant. Thus, these sum rules are obtained only at the expense of ignoring the contributions from *all* other intermediate states, including those belonging to the decuplet. It does not appear that such an assumption can be justified in any way.

In the case of River's quark-model method one has octet dominance built in by the nature of the model and hence one would expect the added symmetry from the beginning. An assumption of dominance of the octet intermediate states in our analysis is sufficient to produce the sum rules predicted by Rivers.

Scalar densities have been used in nonleptonic decay calculations by Riazuddin and Mahanthappa[16] and by Gaillard.[17] Theoretical predictions for S-wave decays are made which are in good agreement with experiment and which are based on taking the symmetry-limit values of the matrix elements of the scalar quark density. However, it is conceivable that $SU(3)$ breaking will provide contributions sufficiently significant to destroy the $\Delta I=\frac{1}{2}$ rules and the Lee-Sugawara triangle. Our method provides a useful tool for investigating these contributions and the assumptions necessary to preserve the sum rules. We should note that this is true for the current x current Hamiltonian[18] also.

Further applications of our analysis can be found within the framework of the $U(12)$ algebra. For example, an investigation of the renormalization of the leptonic decay coupling constants in broken $SU(3)$ can be easily carried out. Such a study would be the current-algebra analog and extension of the work done by Kawarabayashi and Wada.[19] Also, with the aid of a hypothesis of "partial conservation of tensor current"[20] we could use the commutators involving tensor currents to investigate VPP and VBB coupling-constant sum rules. Another interesting point is the type of sum rules obtained when, instead of eliminating the continuum contribution, we attempt to saturate the individual matrix elements with just decuplet (and octet) intermediate states. The results of these studies will be reported in a separate paper.

5. DISCUSSION

We have reported a general current-algebra method for obtaining broken-symmetry sum rules to first order in the breaking. Our analysis has demonstrated that the sum rules obtained are just those that were previously derived group-theoretically, provided that similar physical assumptions are made. The current-algebra method has several advantages over group theory, however: (1) It has greater applicability in that the same basic sum rules are produced for the form factors of any current or density; (2) it leaves much more room for dynamical assumptions to be added as we demonstrated with our coupling-constant example; (3) the effects of the symmetry-breaking Hamiltonian and any subsidiary dynamical assumptions become more apparent; (4) higher-order corrections can conceivably be calculated[13] should experimental data make such calculations meaningful.

In order to obtain sum rules which differ from those derived group-theoretically one must take advantage of the latter three points as was evidenced in our example involving the divergence of the axial-vector current.

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[16] Riazuddin and K. T. Mahanthappa, Phys. Rev. **147**, 972 (1966).
[17] M. K. Gaillard, Phys. Letters **20**, 553 (1966).
[18] See, for example, H. Sugawara, Phys. Rev. Letters **15**, 870 (1965); M. Suzuki, *ibid.* **15**, 986 (1965).
[19] K. Kawarabayashi and W. W. Wada, Phys. Rev. **137**, B1002 (1965).
[20] S. Fubini, G. Segrè, and J. D. Walecka, Ann. Phys. (N. Y.) **39**, 381 (1966). We should also note at this point that R. Rockmore [Phys. Rev. **153**, 1490 (1967)] has applied the Bose and Hara technique to obtain VVP and VPP coupling-constant sum rules.