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Multiparticle Production in High-Energy Collisions*

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The "incoherent-droplet" model proposed earlier is solved for the general case of multiparticle production in high-energy hadron collisions. For small transverse momenta the result is equivalent to taking the invariant square matrix element to be

$$C \exp[-(n/Nk_0^2)\sum_{i=1}^{n} q_i^2],$$

where q_i is the magnitude of the transverse momentum of the ith final particle, n is the total number of particles in the final state, N is an increasing function of energy, otherwise unspecified, and k_0 is a constant. Independent of the choice of parameters in the model, it is found that in the high-energy limit two of the heaviest final particles share equally almost all of the available energy. Parameters of the model can be so chosen as to reproduce the experimental constancy of the average transverse momentum and the total cross section. The simplest choice leads to the prediction that the average multiplicity increases logarithmically with the total c.m. energy. Illustrative examples of energy and angular distributions are given.

I. INTRODUCTION

IN this paper we complete the solution of a phenomenological model proposed earlier[1] to explore the idea of "incoherence" in high-energy hadron reactions. The idea originated in an attempt to understand high-energy p-p elastic scattering at large momentum transfers, by imagining that the incident particles "see" each other as collections of "bits" that act independently and incoherently. We assume that these bits mix thoroughly during a collision and redivide into two outgoing particles. By assuming that the momentum distribution of the bits is spherically symmetric in the rest system of a particle, it follows immediately that to another fast-approaching particle the distribution appears ellipsoidal, containing a much higher proportion of longitudinal components of momentum than transverse components of momentum. Thus the model naturally leads to a strong inhibition of transverse momentum transfer, which seems to be in agreement with experiments. In order that the maximum possible transverse momentum transfer increase with energy, the number of bits should increase with incident energy. Thus we may look upon the bits as the potential number of pieces into which a hadron can be broken up. The harder we hit, the greater the number. The idea of incoherent scattering may be contrasted with that of coherent scattering, in which the incident particle sees the target as an optical potential (direct or exchange). While the latter is designed for forward or backward scattering, the former is designed for scattering near 90° in the c.m. system. In this sense the two pictures complement each other.

The model can be extended immediately to multiparticle production processes, for all we have to do is to consider a redivision of the bits into more than two outgoing particles. In this case there seems to be no obvious theoretical reason nor experimental evidence to impose *a priori* limits on the applicability of the model. We can therefore assume with good conscience that the model applies at all angles of emission of the final particles, until proven otherwise.

We solve the model explicitly for small transverse momenta of the final particles. The result is equivalent to taking the invariant squared matrix element of the production process to be

$$C \exp\left(\frac{-n}{Nk_0^2}\sum_{i=1}^{n} q_i^2\right),$$

where q_i is the magnitude of the transverse momentum of the ith final particle, n is the total number of particles in the final state, N is the total number of bits in the initial state, and k_0^2 is the mean-square transverse momentum of these bits. It turns out that N and the average multiplicity $\bar{n}$ are proportional to each other.

Because of the strong inhibition of transverse momenta, phase space becomes essentially one dimensional. In the high-energy limit this leads to a most probable energy distribution in which two of the heaviest final particles each take up almost half of the total available energy.

We do not attempt any detailed comparison with experiments in this paper, but merely make use of two pieces of experimental information concerning the constancy of the average transverse momentum $\langle\bar{q}\rangle$ and the total cross section[2]:

$$\langle\bar{q}\rangle \approx 400\ \text{MeV}/c,$$
$$\sigma_{\text{tot}}(p\text{-}p) \approx 40\ \text{mb}.$$

The first of these facts imposes a restriction only on the

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[1] K. Huang, Phys. Rev. **146**, 1075 (1966).
[2] For a review, see D. H. Perkins, in *Proceedings of the International Conference on Theoretical Aspects of Very High Energy Phenomena* (CERN, Geneva, 1961), p. 97.

quantity C in the squared matrix element. Choosing C to have the weakest possible energy dependence, we deduce from the constancy of σ_{tot} that the average multiplicity should increase logarithmically with the total c.m. energy. This choice is arbitrary, but a particularly simple one. The conclusion seems to be in disagreement with experiments, but in our opinion not ruled out by them, owing to the considerable experimental uncertainties. Some illustrative examples of energy and angular distributions are given later, but no detailed comparison with experiments will be attempted, for that requires careful analyses of the experimental data and numerical calculations in the model. We leave this task for the future.

II. THE MODEL

We adopt without change the "incoherent-droplet" model of Ref. 1, which is recapitulated here in a form suitable for multiparticle production. The reaction considered is

$$A_1+A_2\to B_1+B_2+\cdots+B_n\,, \tag{1}$$

in which all participants are hadrons, stable or unstable. The total energy is supposed to tend to infinity. The reaction must be a "true" n-body reaction, in the sense that none of the final particles are decay products of unstable particles that were emitted in an earlier stage of the reaction. We always describe the reaction in the center-of-mass system, with the following notations:

$$\begin{aligned}&W=\text{total c.m. energy},\\&M_i=\text{mass of }A_i,\quad(i=1,2),\\&m_i=\text{mass of }B_i,\quad(i=1,\cdots,n),\\&P_{i\mu}=(\mathbf{P}_i,E_i)=\text{4-momentum of }B_i,\quad(i=1,\cdots,n).\end{aligned}\tag{2}$$

The basic assumption is that A_1 and A_2 see each other as collections of "bits." Thus A_i is supposed to be composed of N_i bits, each with a definite 4-momentum p_μ, whose sum should equal the 4-momentum of A_i. There is otherwise no restriction on p_μ, which may be spacelike or timelike. We do not specify in more detail about the nature of these bits, nor do we necessarily regard them as intrinsic to an isolated hadron. As used in the model they are properties of the initial two-body colliding system only. In fact we require N_1 and N_2 to increase with W, so that $N_i\to\infty$ in the high-energy limit. The total number of bits,

$$N=N_1+N_2\,, \tag{3}$$

is a Lorentz invariant characterizing the initial state.

The model consists of a recipe for obtaining the differential cross section of the final state. During the collision the bits are supposed to mix freely without change in their individual 4-momenta. To produce the final state the N bits are divided into n nonempty groups, arbitrary except for the requirement that the 4-momenta of the bits in these groups add up, respectively, to $P_{1\mu},\cdots,P_{n\mu}$, which have been prescribed. The transition rate $\mathfrak{M}$ for reaction (1) is taken to be the number of ways in which such groupings can be made, divided by n^N, the total number of groupings possible. It is through the normalization factor n^N that we supply the information that there are n and only n particles in the final state.

We have ignored the effects of spin and other quantum numbers. The rationalization is that the exchange of these quantum numbers between different "parts" of the two initial hadrons is relatively easy, compared to the exchange of energy and momentum, so that it would have a relatively mild effect on the differential cross section. A refinement of the model to include these effects might be indicated if the present crude attempt works.

The counting problem involved in calculating $\mathfrak{M}$ has been solved in Ref. 1. For large N_1, N_2, the result is

$$\ln\mathfrak{M}=-\sum_{i=1}^{n-1}\lambda_i\cdot P_i+\sum_p\ln\{n^{-1}[1+\sum_{i=1}^{n-1}\exp(\lambda_i\cdot p)]\}\,, \tag{4}$$

where $\lambda_i\cdot P_i$ denotes 4-vector scalar product, $\lambda_{i\mu}$ is a 4-vector to be specified later, and $\{P_{1\mu},\cdots,P_{n-1,\mu}\}$ refers to $n-1$ of the final 4-momenta, chosen arbitrarily. The summation over p denotes a sum over all the 4-momenta p_μ of the N bits. The $n-1$ 4-vectors $\lambda_{i\mu}$ are determined by the conditions

$$P_{i\mu}=\sum_p p_\mu\exp(\lambda_i\cdot p)/(1+\sum_{j=1}^{n-1}\exp(\lambda_j\cdot p))\,,\quad(i=1,\cdots,n-1). \tag{5}$$

We have by definition

$$\sum_p \mathbf{p}=0\,,\quad \sum_p p_0=W\,, \tag{6}$$

and energy-momentum conservation is implied by (5) in the form

$$\begin{aligned}\mathbf{P}_n&\equiv-\sum_{i=1}^{n-1}\mathbf{P}_i\,,\\E_n&=W-\sum_{i=1}^{n-1}E_i\,.\end{aligned}\tag{7}$$

In the limit $N_i\to\infty$, the summations in (4) and (5) may be replaced by integrations over the 4-momentum distribution of the bits. Let the 4-momentum distribution function for A_i be denoted by $N_if_i(p)$ in the c.m. system, with

$$\int d^4p\, f_i(p)=1\,,\quad(i=1,2)\,. \tag{8}$$

We require that under a Lorentz transformation, which takes p_μ into p_μ', the function f_i goes into f_i', with $f_i(p)=f_i'(p')$. We further assume that f_i becomes spherically symmetric in the 3-momentum when it is transformed to the rest system of A_i. Let $g_i(p)$ denote this spherically symmetric function. Then

$$f_1(p)=g_1(p')\,,\qquad (9)$$
$$f_2(p)=g_2(p'')\,,$$

where $p_\mu \to p_\mu'$ is the Lorentz transformation that takes A_1 from the c.m. system to its rest system, and $p_\mu \to p_\mu''$ is the corresponding transformation for A_2. Through these Lorentz transformations the spheres of constant momentum in $g_i(p)$ are transformed into ellipsoids with major axes along the incident direction. In the limit $W\to\infty$, therefore, f_i contains overwhelmingly more longitudinal components of momentum that the transverse components. It is clear that our recipe produces final particles that rarely have large transverse momenta.

With summations replaced by integrations, (4) and (5) read

$$\ln\mathfrak{M}=-\sum_{i=1}^{n-1}\lambda_i\cdot P_i+\int d^4p\;(N_1f_1+N_2f_2)\times\ln\{n^{-1}[1+\sum_{i=1}^{n-1}\exp(\lambda_i\cdot p)]\}\,,\qquad (10)$$

$$P_{i\mu}=\int d^4p\;(N_1f_1+N_2f_2)p_\mu\exp(\lambda_i\cdot p)/(1+\sum_{j=1}^{n-1}\exp(\lambda_j\cdot p)).\qquad (11)$$

We expect N_i to depend on the nature of A_i, whether it is a proton or pion, etc. To ensure time-reversal invariance in two-body reactions, it is sufficient to assume that for the same energy hadrons having the same quantum numbers conserved by the strong interactions have the same N_i. We found in Ref. 1 that N_1, N_2 for a p-p initial state should be proportional to W. This is based on experiments below 30 BeV/c, and does not rule out a different dependence on W at higher energies. In the theoretical developments we assume no special W dependence, but require only that $N_1\to\infty$, $N_2\to\infty$, with finite N_1/N_2, as $W\to\infty$.

The differential cross section for reaction (1) is taken to be

$$d\sigma=\sigma_0\kappa^{-2n+4}\prod_{i=1}^{n}\frac{d^3P_i}{E_i}\delta^3(\sum_{i=1}^{n}\mathbf{P}_i)\delta(W-\sum_{i=1}^{n}E_i)\mathfrak{M}\,,\qquad (12)$$

where σ_0 is of the dimension of area, and κ is of the dimension of energy. They may both be functions of W, and may depend on the nature of A_1 and A_2. In fact any effects oversimplified in our treatment would be phenomenologically taken into account through σ_0 and κ. For the model to be self-consistent, however, σ_0 and κ must vary with energy much more slowly than $\mathfrak{M}$, if at all.

It is clear that $\mathfrak{M}$ serves as an invariant squared matrix element. Consistency demands that invariant phase-space elements be employed, as done in (12).

The identity of final particles becomes relevant when we integrate (12) over regions of phase space. In a correct theory, phase space should be reduced by a factor $1/n!$ for each group of n identical particles in the final state, while simultaneously the matrix element should acquire extra exchange terms. We shall assume that (12) is not changed by the presence of identical particles. This implicitly assumes that all exchange matrix elements have the same sign and magnitude as the direct matrix element, so that all factors $n!$ cancel. The ultimate correctness of this assumption can be established only if we are able to derive the model from a more fundamental theory.

III. GENERAL SOLUTION

We set up rectangular coordinates in the c.m. system and let the x axis lie along the incident direction. The first task is to show that as $W\to\infty$

$$\lambda_{ix}\sim O(NW)^{-1},\quad \lambda_{i0}\sim O(NW)^{-1}.\qquad (13)$$

The qualitative reason is as follows. The quantity $\exp(\lambda_{ix}p_x)$ may be regarded as the probability that a bit possessing longitudinal momentum p_x is chosen to go into the final hadron B_i. Because of the preponderance of longitudinal momenta in the distribution $N_1f_1(p)+N_2f_2(p)$, B_i would almost certainly end up with an enormous longitudinal momentum violating energy conservation, unless we correlate the choices, such that a chosen p_x is almost always canceled by a simultaneous choice of $-p_x$. This requires $\exp(\lambda_{ix}p_x)\approx1$ hence $\lambda_{ix}\approx0$. The reason for a small λ_{i0} is similar.

To prove (13) we start with (11) and change the variables of integration. In the term involving N_1f_1 we let $p_\mu\to L_1p_\mu$, and in the term involving N_2f_2 we let $p_\mu\to L_2p_\mu$, where L_1 and L_2 are, respectively, inverse Lorentz transformations to those in (9). After this is done (11) becomes a sum of two integrals containing, respectively, the functions g_1 and g_2, which are spherically symmetric in 3-momentum and are independent of W. The integrands are positive-definite. For P_{ix} and P_{i0}, the two integrals mentioned are proportional to NW, where the factor W arises from the Lorentz transformation. Since $|P_{ix}|$ and P_{i0} are bounded by W, and since by assumption $N\to\infty$, the integrands must vanish as $W\to\infty$. By inspection we see that this is possible only if (13) holds. By expanding the right side of (11) in powers of λ_{ix} and λ_{i0}, and retaining only linear terms we find that λ_{ix} and λ_{i0} are solutions to a system of $2(n-1)$ linear algebraic equations:

$$\sum_{k=1}^{n-1}\left(a_{ik}^{(+)}\frac{\lambda_{kx}}{NW}-a_{ik}^{(-)}\frac{\lambda_{k0}}{NW}\right)=\frac{P_{ix}}{W}-b_i^{(-)}, \tag{14}$$

$$\sum_{k=1}^{n-1}\left(a_{ik}^{(-)}\frac{\lambda_{kx}}{NW}-a_{ik}^{(+)}\frac{\lambda_{k0}}{NW}\right)=\frac{E_i}{W}-b_i^{(+)}, \quad (i=1,\cdots,n-1), \tag{15}$$

where $a_{ik}^{(\pm)}$, $b_i^{(\pm)}$ are finite constants independent of W:

$$a_{ik}^{(\pm)}=\frac{2}{N}\int d^4p\left(\frac{N_2}{M_2}g_2\pm\frac{N_1}{M_1}g_1\right)(p_x-p_0)^2 \times[\delta_{ik}(1+\sum_{j=1}^{n-1}\alpha_j)-\alpha_i\alpha_k]/(1+\sum_{j=1}^{n-1}\alpha_j^2), \tag{16}$$

$$b_i^{\pm}=\int d^4p\ (g_2\pm g_1)\alpha_i/(1+\sum_{j=1}^{n-1}\alpha_j^2), \tag{17}$$

$$\alpha_i\equiv\exp(\lambda_{iy}p_y+\lambda_{iz}p_z). \tag{18}$$

By (13) we have $\lambda_{ix}P_{ix}\sim O(N^{-1})$ and $\lambda_{i0}E_i\sim O(N^{-1})$. Therefore, all longitudinal and time components of $\lambda_{i\mu}$ and $P_{i\mu}$ drop out of (10), and the problem reduces to one in the transverse plane (the y-x plane). Let all vectors be decomposed into longitudinal and transverse components:

$$\mathbf{P}_i=\hat{x}P_{ix}+\mathbf{q}_i,\quad \boldsymbol{\lambda}_i=\hat{x}\lambda_{ix}+\boldsymbol{\beta}_i,\quad \mathbf{p}=\hat{x}p_x+\mathbf{k}. \tag{19}$$

Further define a transverse momentum distribution function:

$$F(k^2)=\frac{1}{N}\int_{-\infty}^{\infty}dp_x\int_{-\infty}^{\infty}dp_0\,[N_1f_1(p)+N_2f_2(p)],\quad \int d^2k\,F(k^2)=1, \tag{20}$$

where $k^2=p_x^2+p_y^2$. The W dependence in f_1, f_2 is taken away by the integration, and N_1/N, N_2/N are independent of W by assumption. Therefore $F(k^2)$ is independent of W. We can now write (10) and (11) in the forms

$$\ln\mathfrak{M}=-\sum_{i=1}^{n-1}\boldsymbol{\beta}_i\cdot\mathbf{q}_i+N\int d^2k\,F(k^2)\times\ln\{n^{-1}[1+\sum_{j=1}^{n-1}\exp(\boldsymbol{\beta}_j\cdot\mathbf{k})]\}, \tag{21}$$

$$\mathbf{q}_i=N\int d^2k\,F(k^2)\mathbf{k}\exp(\boldsymbol{\beta}_i\cdot\mathbf{k})/(1+\sum_{1=j}^{n-1}\exp(\boldsymbol{\beta}_j\cdot\mathbf{k})). \tag{22}$$

As $W\to\infty$, $|q_i|/W$ approaches a finite limit. Hence $\boldsymbol{\beta}_i$ can depend on W only through the combination N/W.

By rotational covariance a solution to (22) must have the form

$$\boldsymbol{\beta}_i=\sum_{j=1}^{n-1}D_{ij}(\mathbf{q}_1,\cdots,\mathbf{q}_{n-1})\mathbf{q}_j, \tag{23}$$

where D_{ij} is a scalar function. The set of $n-1$ equations (22) is invariant under a simultaneous permutation of $\{\mathbf{q}_1,\cdots,\mathbf{q}_{n-1}\}$ and $\{\boldsymbol{\beta}_1,\cdots,\boldsymbol{\beta}_{n-1}\}$. Let P be a permutation that takes $\{1,\cdots,n-1\}$ into $\{P1,\cdots,P(n-1)\}$. Then

$$D_{Pi,Pj}(\mathbf{q}_1,\cdots,\mathbf{q}_{n-1})=D_{ij}(\mathbf{q}_{P1},\cdots,\mathbf{q}_{P(n-1)}). \tag{24}$$

Therefore, among the $(n-1)^2$ functions D_{ij}, only two are independent, e.g., D_{11} and D_{12}. Choosing $i=j=1$, and choosing P to be any permutation that leaves 1 fixed, we see that D_{11} is symmetric in $\{\mathbf{q}_2,\cdots,\mathbf{q}_n\}$. Choosing $i=1$, $j=2$, and choosing P to be any permutation that leaves 1 and 2 fixed, we see that D_{12} is symmetric in $\{\mathbf{q}_3,\cdots,\mathbf{q}_n\}$.

Further developments of the solution will be carried out for special cases.

IV. TWO-BODY REACTIONS

The case $n=2$, corresponding to $A_1+A_2\to B_1+B_2$, is particularly simple. Let the reaction take place in the x-y plane, and let P be the incident momentum, θ the scattering angle. When W is much larger than the particle masses, P may be replaced by $W/2$. The transverse momentum transfer is

$$|\mathbf{q}|=P\sin\theta\approx\tfrac{1}{2}W\sin\theta. \tag{25}$$

Let

$$G(k_y)=\int_{-\infty}^{\infty}dk_z\,F(k_y^2+k_z^2),\quad \int_{-\infty}^{\infty}dk\,G(k)=1. \tag{26}$$

$G(k)$ is an even function independent of W. Now (21) and (22) can be reduced to

$$\ln\mathfrak{M}=-\tfrac{1}{2}\beta W\sin\theta+\frac{4N}{\beta}\int_0^{\infty}du\,G\left(\frac{2u}{\beta}\right)\ln\cosh u, \tag{27}$$

$$\beta^2\sin\theta=\frac{8N}{W}\int_0^{\infty}du\,G\left(\frac{2u}{\beta}\right)u\tanh u. \tag{28}$$

The parameter β is to be eliminated from (27) with the help of (28). By definition $G(k)\geq 0$, hence $\beta\geq 0$. We can rewrite (27) in the form

$$\ln\mathfrak{M}=-Wh(\sin\theta), \tag{29}$$

$$h(\sin\theta)=\beta\sin\theta\left\{1-\int_0^{\infty}du\,G(2u/\beta)\ln\cosh u\Big/\int_0^{\infty}du\,G(2u/\beta)u\tanh u\right\}. \tag{30}$$

The differential cross section is given by

$$d\sigma/d\Omega=\tfrac{1}{2}\sigma_0 e^{-Wh(\sin\theta)}. \tag{31}$$

The function $h(\sin\theta)$ is positive-definite. It can depend on W only through the combination N/W, and depends on θ only through $\sin\theta$. Thus at fixed W the differential cross section is symmetrical about $\theta=\pi/2$. The model, however, is not valid near $\theta=0$, π, where we know physically that coherent scattering dominates. At fixed θ, $\mathfrak{M}$ generally decreases with W. It decreases exponentially with W if and only if $N\propto W$.

In Ref. 1 (where N was written as $N/2$) detailed calculations were made assuming $G(k)$ to be a Gaussian distribution. Comparison with p-p scattering at 13–30 BeV/c at large angles led to the conclusion that $N\propto W$. Defining the constants

$$N_0\equiv MN/W, \tag{32}$$

$$k_0^2\equiv 4\int_0^\infty dk\, k^2G(k)$$
$$=N^{-1}\int d^4p\,(N_1f_1+N_2f_2)(p_y^2+p_z^2), \tag{33}$$

where M is the proton mass, we found that a fit to experiments can be achieved by choosing

$$(k_0/M)^2\approx 1/3N_0, \tag{34}$$

$$\sigma_0\approx 60\text{ mb/sr}. \tag{35}$$

N_0 was not precisely determined, but it should be of the order of 10. In later applications we arbitrarily choose $N_0=6$.

V. MULTIPARTICLE PRODUCTION

1. Differential Cross Section

We now consider $n>2$, and assume that our model applies to regions of large as well as small transverse momenta.

The region of large transverse momenta, being much more difficult mathematically, and much less likely to be reached in experiments, will not be considered in this paper. If all transverse momenta are small, then all $\boldsymbol{\beta}_i$ in (22) will be small, and we can approximate (22) by the linear equation

$$\mathbf{q}_i=\frac{Nk_0^2}{2n}\left(\boldsymbol{\beta}_i-\frac{1}{n}\sum_{j=1}^{n-1}\boldsymbol{\beta}_j\right), \tag{36}$$

where k_0 is the same constant defined in (33). Solving for $\boldsymbol{\beta}_i$, we obtain

$$\boldsymbol{\beta}_i=\frac{2n}{Nk_0^2}(\mathbf{q}_i-\mathbf{q}_n), \tag{37}$$

where

$$\mathbf{q}_n\equiv-\sum_{j=1}^{n}\mathbf{q}_j.$$

Treating (21) in the same approximation yields

$$\ln\mathfrak{M}=-\frac{n}{Nk_0^2}\sum_{j=1}^{n}|\mathbf{q}_j|^2. \tag{38}$$

The validity of the approximation is restricted by the condition $k_0|\boldsymbol{\beta}_i|\ll 1$, or

$$(2n/Nk_0)|\mathbf{q}_i-\mathbf{q}_j|\ll 1, \tag{39}$$

for all pairs i, j of final particles. Since (38) indicates an extremely rapid decrease of the differential cross section for large transverse momenta, the approximation is adequate for most calculations, except when we specifically constraint the transverse momenta to be large, which we shall not do.

Let us introduce cylindrical coordinates about the x axis, and let x_i, q_i, ϕ_i denote, respectively, the x component, transverse component, and azimuthal angle of $\mathbf{P}_i$:

$$\mathbf{P}_i=(x_i,q_i,\phi_i), \tag{40}$$
$$E_i=(x_i^2+q_i^2+m_i^2)^{1/2}.$$

The differential cross section for the production of n particles is

$$d\sigma_n=\frac{dx_1\cdots dx_n}{E_1\cdots E_n}d^2q_1\cdots d^2q_n$$
$$\times\delta\Big(\sum_{i=1}^{n}x_i\Big)\delta^2\Big(\sum_{i=1}^{n}\mathbf{q}_i\Big)\delta\Big(W-\sum_{i=1}^{n}E_i\Big)\sigma_0\kappa^{-2n+4}\mathfrak{M}, \tag{41}$$

where $\mathfrak{M}$ is given by (38), $\mathbf{q}_i$ is the two-dimensional vector (q_i,ϕ_i), and $d^2q=qdqd\phi$.

2. Integrated Cross Sections

Because of the strong suppression of transverse momenta by $\mathfrak{M}$, particles prefer to be emitted within small forward and backward cones. Since $\mathfrak{M}$ depends only on the transverse momenta, the longitudinal momentum distribution is determined mainly by the phase-space volume in a one-dimensional relativistic phase space, defined by

$$\Gamma_n(W^2)=\int_{-\infty}^{\infty}\frac{dx_1\cdots dx_n}{E_1\cdots E_n}\delta\Big(\sum_{i=1}^{n}x_i\Big)\delta\Big(W-\sum_{i=1}^{n}E_i\Big), \tag{42}$$

where

$$E_i=(x_i^2+\Delta_i^2)^{1/2}, \tag{43}$$
$$\Delta_i=(q_i^2+m_i^2)^{1/2}.$$

Clearly $\Gamma_n(W^2)=0$ for $W<U_n$, where

$$U_n=\sum_{i=1}^{n}\Delta_i \tag{44}$$

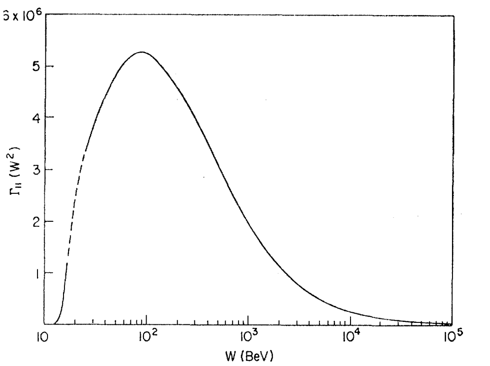

FIG. 1. Total longitudinal phase space volume $\Gamma_{11}(W^2)$ for 11 final particles at total c.m. energy W, with $(q_i^2+m_i^2)^{1/2}=1$ BeV/c, where q_i and m_i are, respectively, the transverse momentum and mass of the ith particle. Solid portions of the curve represent the threshold and asymptotic approximations (52) and (55).

is the effective threshold energy. The formula (42) represents the total phase-space volume for n particles of respective masses Δ_i moving along the x axis, with total energy-momentum $(W,0)$. The expression, however, is invariant under a Lorentz transformation along the x axis. Therefore, $\Gamma_n(W^2)$ is also the phase-space volume for total energy-momentum (W',P') satisfying $W'^2-P'^2=W^2$. Using this fact we easily establish the recursion formula

$$\Gamma_n(W^2)=\int_{-\infty}^{\infty}\frac{dx_n}{(x_n^2+\Delta_n^2)^{1/2}}\Gamma_{n-1}[(W-E_n)^2-x_n^2]. \quad (45)$$

Changing the variable of integration to the energy $E_n=(x_n^2+q_n^2+m_n^2)^{1/2}$, and remembering that $\Gamma_{n-1}(W^2)=0$ for $W\leq U_{n-1}$, we have

$$\Gamma_n(W^2)=2\int_{\Delta_n}^{a_n}\frac{dE_n}{(E_n^2-\Delta_n^2)^{1/2}}\Gamma_{n-1}(W^2-2WE_n+\Delta_n^2), \quad (46)$$

where

$$a_n=\frac{1}{2W}[W^2+\Delta_n^2-(\sum_{i=1}^{n-1}\Delta_i)^2]. \quad (47)$$

For $n=2$, 3, $\Gamma_n(W^2)$ can be calculated exactly:

$$\Gamma_2(W^2)=4[W^2-(\Delta_1+\Delta_2)^2]^{-1/2}[W^2-(\Delta_1-\Delta_2)^2]^{-1/2}, \quad (48)$$

$$\Gamma_3(W^2)=16[(E-\Delta_3)^2-(\Delta_1-\Delta_2)^2]^{-1/2}\times[(E+\Delta_3)^2-(\Delta_1+\Delta_2)^2]^{-1/2}K(k), \quad (49)$$

where

$$K(k)=\int_0^{\pi/2}d\phi\,(1-k^2\sin^2\phi)^{-1/2}, \quad (50)$$

$$k^2=\frac{[(W-\Delta_3)^2-(\Delta_1+\Delta_2)^2][(W+\Delta_3)^2-(\Delta_1-\Delta_2)^2]}{[(W-\Delta_3)^2-(\Delta_1-\Delta_2)^2][(W+\Delta_3)^2-(\Delta_1+\Delta_2)^2]}. \quad (51)$$

The threshold behavior of $\Gamma_n(W^2)$ for general n can be obtained by induction, using (45) and (48):

$$\Gamma_n(W^2)\xrightarrow[W\to U_n]{}(2\pi)^{(n-1)/2}(W-U_n)^{(n-3)/2}\Big/\Gamma\left(\frac{n-1}{2}\right)[\sum_{i=1}^{n}\Delta_i]^{1/2}[\prod_{i=1}^{n}\Delta_i]^{1/2}. \quad (52)$$

The asymptotic behavior of $\Gamma_n(W^2)$ for general n can be obtained as follows. The substitution $x_i=\frac{1}{2}[u_i-(\Delta_i^2/u_i)]$ reduces (42) to

$$\Gamma_n(W^2)=2\int_0^{\infty}\frac{du_1\cdots du_n}{E_1\cdots E_n}\delta(W-\sum_{i=1}^{n}u_i)\delta\left(W-\sum_{i=1}^{n}\frac{\Delta_i^2}{u_i}\right). \quad (53)$$

The range of each u_i is restricted by the δ functions to lie between Δ_i^2/W and W. Therefore, we can also write

$$\Gamma_n(W^2)=2\int_{\Delta_1^2/W}^{W}\frac{du_1}{u_1}\cdots\int_{\Delta_n^2/W}^{W}\frac{du_n}{u_n}\times\delta(W-\sum_{i=1}^{n}u_i)\delta\left(W-\sum_{i=1}^{n}\frac{\Delta_i^2}{u_i}\right). \quad (54)$$

as $W\to\infty$ for fixed n, u_i can be neglected in the argument of the first δ function unless it is of order W, and it can be neglected in the argument of the second δ function unless it is of order W^{-1}. To calculate the leading asymptotic term we consider three possibilities for each u_i: (a) it is retained in the first δ function only; (b) it is retained in the second δ function only; (c) it is retained in neither δ function. The leading asymptotic term is obtained by retaining only one u_i in each δ function. The result is

$$\Gamma_n(W^2)\xrightarrow[W\to\infty]{}\frac{2n(n-1)}{W^2}(\ln W^2)^{n-2}\left[1+O\left(\frac{1}{\ln W^2}\right)\right]. \quad (55)$$

We note that this is independent of transverse momenta and masses. The scale for W^2 in $\ln W^2$ is not significant until we calculate $\Gamma_n(W^2)$ to order W^{-2}.

The general feature of $\Gamma_n(W^2)$ for $n\geqslant 4$ is then as follows. It is zero at threshold, but rises rapidly, passes through a maximum, then decreases at large W essentially like W^{-2}. The maximum lies in the asymptotic region if n is sufficiently larger. According to (54), it is at $W=\exp\frac{1}{2}(n-2)$. The unfamiliar feature that the phase-space volume decreases with energy is due to the one-dimensionality of the problem, plus the use of relativistic phase-space elements dx_i/E_i. For illustration Fig. 1 shows $\Gamma_{11}(W^2)$ for $\Delta_i=1$ BeV, $(i=1,\cdots,n)$.

Our method of deriving (54) also reveals the most probable distribution of longitudinal momentum in the limit $W\to\infty$, to wit, two of the final particles have, respectively, $u_i\approx W$, and $u_j\approx\Delta_i^2/W$, while the rest of the particles have neither of these properties. This means that two of the particles have longitudinal momenta$\approx\pm W/2$, while the rest of the particles have longitudinal momenta small compared to W but large compared to $1/W$. Thus for W sufficiently large, the two particles mentioned travel in opposite directions along the x axis, and equally share almost all of the available energy. Closer examination shows that these two particles are the heaviest particles among the final particles. The criterion for W to be large is $\ln W^2 \gg n$, where W is measured in units of some typical particle mass.

The asymptotic cross section for production of n particles is obtained by integrating (41) over all phase space:

$$\sigma_n(W)=(n-1)\sigma_0\left(\frac{\kappa_0}{W}\right)^2\left(\frac{\pi N k_0^2}{n\kappa_0^2}\right)^{n-1}, \tag{56}$$

where

$$\kappa_0=\kappa/(2\ln W). \tag{57}$$

The average multiplicity $\bar{n}$ is the value of n that maximizes σ_n:

$$\bar{n}=\frac{\pi}{e}\left(\frac{k_0}{\kappa_0}\right)^2 N. \tag{58}$$

Near $n=\bar{n}$, $\sigma_n(W)$ may be approximated by

$$\sigma_n(W)\approx\sigma_{\bar{n}}(W)\exp[-(n-\bar{n})^2/2\bar{n}], \tag{59}$$

where

$$\sigma_{\bar{n}}(W)=\sigma_0(\bar{n}-1)(\kappa_0/W)^2 e^{\bar{n}-1}. \tag{60}$$

We see that fluctuations about $n=\bar{n}$ are rather large for $\bar{n}<100$, so that $\bar{n}$ has only qualitative significance for these cases. To estimate the total cross section we integrate (59) over n-$\bar{n}$ from $-\infty$ to $+\infty$, obtaining

$$\sigma_{\text{tot}}(W)\approx(2\pi\bar{n})^{1/2}\sigma_{\bar{n}}(W)$$
$$=\sigma_0\left(\frac{\kappa_0}{W}\right)^2(2\pi\bar{n})^{1/2}(\bar{n}-1)e^{\bar{n}-1}. \tag{61}$$

3. One-Particle Distributions

The distribution of transverse momentum for the ith particle can be obtained by integrating (41) over all coordinates except q_i. The result is independent of i:

$$T_n(q)=2c_n q e^{-c_n q^2},$$
$$c_n=\left(\frac{n}{n-1}\right)\frac{n}{Nk_0^2}, \tag{62}$$
$$\int_0^\infty dq\, T_n(q)=1.$$

The average transverse momentum is

$$\langle q\rangle=\int_0^\infty dq\, qT_n(q)=\tfrac{1}{2}k_0\left(\frac{\pi N}{n}\frac{n-1}{n}\right)^{1/2}. \tag{63}$$

When averaged over many events at the same energy, we have

$$\langle\bar{q}\rangle=\tfrac{1}{2}k_0(\pi N/\bar{n})^{1/2}=\tfrac{1}{2}e^{1/2}\kappa_0, \tag{64}$$

where we have assumed $\bar{n}\gg1$, and where (58) has been used. The criterion (39) for the validity of the basic approximation requires $(2\bar{n}/Nk_0)\langle\bar{q}\rangle\ll1$, or

$$k_0/\kappa_0\ll1. \tag{65}$$

The longitudinal momentum distribution for the ith particle, when $q_1^2, \cdots, q_n^2$ are fixed, may be read from (45):

$$L_n(x_i;q_1,\cdots,q_n)=\frac{\Gamma_{n-1}[W^2-2W(x_i^2+\Delta_i^2)^{1/2}+\Delta_i^2]}{(x_i^2+\Delta_i^2)^{1/2}\Gamma_n(W^2)},$$
$$\int_{-\infty}^\infty dx_i\, L_n=1. \tag{66}$$

Note that Γ_{n-1} vanishes for

$$(x_i^2+\Delta_i^2)^{1/2}\geqslant(2W)^{-1}[W^2+\Delta_i^2-(\sum_{j\neq i}\Delta_j)^2].$$

Thus $L_n(x_i)$ depends on masses and on $q_1^2, \cdots, q_n^2$. Over most of the range of x_i, however, we may replace Γ_{n-1} by its asymptotic form, and $(x_i^2+\Delta_i^2)^{1/2}$ by $|x_i|$, and these dependences drop out. We shall approximate $\Gamma_n(W^2)$ by its asymptotic form $\Gamma_n^{(\infty)}(W^2)$. Then L_n depends only on quantities associated with the same final particle:

$$L_n(x;q)\approx\frac{\Gamma_{n-1}^{(\infty)}[W^2-2W(x^2+q^2+m^2)^{1/2}+q^2+m^2]}{\Gamma_n^{(\infty)}(W^2)(x^2+m^2+q^2)^{1/2}}, \tag{67}$$

where m is the mass of the final particle being considered, and

$$\Gamma_n^{(\infty)}(W^2)=\frac{2n(n-1)}{W^2}\left(\ln\frac{W^2}{m^2}\right)^{n-2}, \tag{68}$$

where, for convenience we have arbitrarily supplied a scale factor m^2 in the logarithm. The range of x in (66) is taken to be

$$0\leqslant|x|\leqslant W/2. \tag{69}$$

In actuality the upper limit should be smaller than indicated, by a quantity of the order of particle masses, and near that limit (67) is inaccurate; but our approximations are good enough for qualitative purposes.

Using (67) and (62) we arrive at the approximate energy distribution $I_n(E)$ and angular distribution $A_n(\theta)$ for a final particle of mass m:

$$I_n(E)\approx2E\int_0^{(E^2-m^2)^{1/2}}dq\frac{T_n(q)L_n((E^2-q^2-m^2)^{1/2};q)}{(E^2-q^2-m^2)^{1/2}}, \tag{70}$$

$$A_n(\theta)\approx\int_0^{(W^2/4-m^2)^{1/2}}dy\, yT_n(y\sin\theta)L_n(y\cos\theta;y\sin\theta). \tag{71}$$

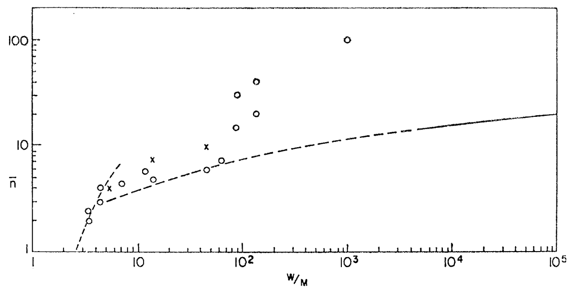

FIG. 2. Average multiplicity as a function of total c.m. energy W, based on the choices of parameters of Sec. 6, which are somewhere arbitrary. The dashed curve on the left is based on experimental fits of p-p elastic scattering at machine energies. The dashed curve on the right is an asymptotic curve. The experimental points o are taken from Ref. 2, and the points X are from L. Hansen and W. Fretter, Phys. Rev. **118**, 812 (1960).

These are not exactly normalized, for the approximation leading to them does not preserve normalization. As unnormalized distributions, they are accurate except near the ends of the ranges of E and θ, where they have only qualitative significance.

VI. ILLUSTRATIVE EXAMPLES

For illustration we choose a tentative set of parameters for the model and work out some predictions for pp-initiated reactions.

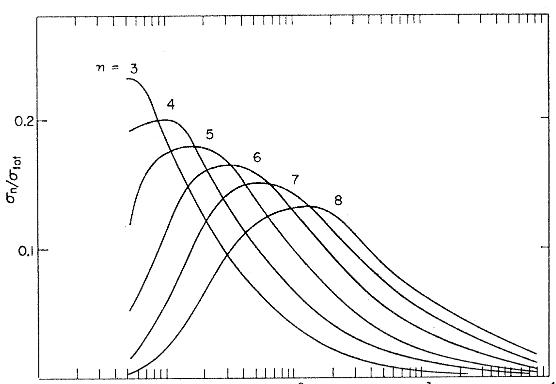

FIG. 3. Illustrative cross sections for a reaction leading to n final particles.

Taking $\langle\bar{q}\rangle = 400\text{MeV}/c$ for all energies we find from (64) that

$$\kappa_0 = 500 \text{ MeV}/c. \tag{72}$$

With this, the condition (39) for the validity of the basic approximation becomes

$$N_0 \gg 1. \tag{73}$$

For a pp initial state we take the asymptotic total cross section to be 40 mb. Then (61) imposes a relation between σ_0 and $\bar{n}$. The simplest assumption is to take σ_0 to be a constant, the same constant as (35). Then $\bar{n}$ is given by

$$\bar{n} + \ln[(\bar{n}-1)(2\pi\bar{n})^{1/2}] \xrightarrow[W\to\infty]{} 2\ln(W/\kappa_0) + 1 + \ln(\sigma_{\text{tot}}/\sigma_0), \tag{74}$$

or

$$\bar{n} + \ln[(\bar{n}-1)\bar{n}^{1/2}] \xrightarrow[W\to\infty]{} 2\ln(W/M) + 1, \tag{75}$$

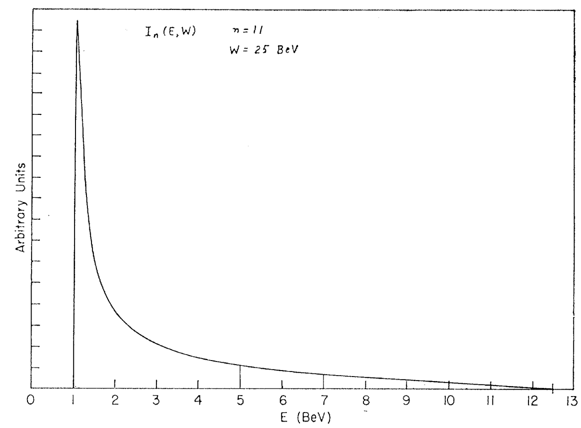

FIG. 4. Illustrative energy distribution for 11 final particles of 1 BeV mass, at c.m. energy $W = 25$ BeV. The curve for $W = 100$ BeV is not significantly different when properly scaled. The two vertical lines at $E = 5, 7$ marked off two areas each approximately equal to 1/11 of the total area under the curve.

where M is the proton mass. Thus $\bar{n}$ increases approximately logarithmically with W for very large W, and so does N, according to (58).

At lower energies ($W \lesssim 10$ BeV), $\bar{n}$ should increase linearly with W, as indicated by (58) and (32). Since $\bar{n}$ should extrapolate to zero at $W = 2M$, we take in this

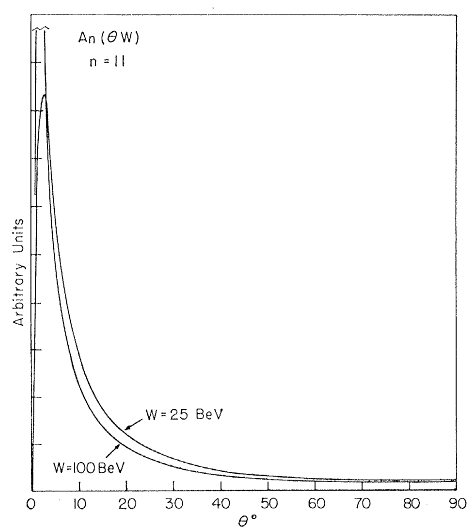

FIG. 5. Illustrative c.m. angular distribution for 11 final particles of 1-BeV mass, at c.m. energies $W = 25$ and 100 BeV. The distribution is symmetrical about 90°. As $W \to \infty$, the forward peak tends to a δ function.

region

$$\bar{n} \approx \frac{\pi}{e}\left(\frac{k_0}{\kappa_0}\right)^2 \frac{N_0}{M}(W-2M) = \frac{3}{2}\left(\frac{W}{M}-2\right). \qquad (76)$$

To determine N we arbitrarily take $N_0=6$. Thus for moderate energies

$$N \approx 6W/M\,, \qquad (77)$$

and asymptotically

$$N \approx 5.2\bar{n}. \qquad (78)$$

The two limiting behaviors (76) and (75) are shown in Fig. 2, together with some experimental points taken from Ref. 2. The apparent disagreement between the experimental data and the asymptotic curve does not necessarily mean that our assumption of a constant σ_0 must be abandoned, for the experimental values of $\bar{n}$ are subject to large uncertainties. Apart from the fact that fluctuations about $\bar{n}$ are large, uncertainties may arise because (a) a good fraction of the final particles may be decay products of unstable particles produced in the reaction; (b) in reactions with complex nuclei as targets, more than one nucleon in the target may be effective in producing particles; (c) a jet observed in cosmic-ray experiments may in fact be a superposition of an original and a number of secondary jets. These effects tend to increase the apparent $\bar{n}$.

The cross section $\sigma_n(W)$ for various n are shown in Fig. 3. It is seen that at given energies many values of n have comparable cross sections, so that the average multiplicity, though given a precise mathematical definition, has only qualitative physical meaning.

The approximate energy distribution according to (70) is plotted in Fig. 4 for 11 final particles of mass 1 BeV, at $W=25$ BeV. The curve for $W=100$ BeV, suitably scaled, is indistinguishable from this except near the ends of the range of E, where it is slightly different. If we divide the area under the curve in Fig. 4 into 11 disjoint vertical strip of equal area, then on the average each strip is "occupied" by one particle. The two strips of highest energies are marked off in Fig. 4. If we take the centers of these strips to correspond respectively to the most probable energies of the two most energetic particles we obtain an "inelasticity parameter" of 40%. This parameter decreases with W roughly like $(\ln W)^{-1}$.

The approximate c.m. angular distribution according to (71) is plotted in Fig. 5 for 11 final particles of mass 1 BeV, at $W=25$, 100 BeV. The curve is symmetrical about 90°. The general features are that between 90° and 30° the distribution is nearly isotropic. Between 30° and 10° the distribution favors small angles, and between 10° and 0° there is a very sharp peak, which tends to a δ function as $W \to \infty$. The sharp peak reflects the fact that two final particles, traveling in opposite longitudinal directions, equally share a good fraction of the available energy. Apart from the sharpening of this peak, the distribution is insensitive to W.

ACKNOWLEDGMENT

I am indebted to Saul Rappaport for computing the curves in Figs. 4 and 5.