The large $|\mathbf{q}|$ behavior of $\varphi_1(k-q)|_{q_0=\pm\omega,\mathbf{k}=0}$ is clearly controlled by the constant term on the right-hand side of Eq. (7), $[2M_1(f_0)-F_0(f_0)]$. However, because of a cancellation between positive- and negative-frequency terms, the large $|\mathbf{q}|$ behavior of $J(q,k_0)$ is *not* controlled by the above constant term, but instead is controlled by the C term of Eq. (7). Thus,

$$J(\mathbf{q},k_0) \xrightarrow[|\mathbf{q}|\to\infty]{} -2C[\ln(|\mathbf{q}|/m)]^2/(2k_0|\mathbf{q}|)^2, \quad (9)$$

which depends[3] *on* k_0; therefore the subtractions indicated in the computation of $\mathfrak{M}(k;\varphi_1)$ from $M(k;\varphi_1)$ [see Eq. (2)] cannot produce a finite result for $\mathfrak{M}(k;\varphi_1)$. This result when coupled with Eq. (5) completes our proof that the power-series-expansion solution of Eq. (1) fails in fifth order of the coupling constant.

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[3] It should be noted that the inclusion of higher-order terms in Eq. (7) cannot alter Eq. (9).

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Off-Shell Unitarity for Two Spin-$\frac{1}{2}$ Particles

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The implications of off-shell unitarity for the system of two spin-$\frac{1}{2}$ particles are examined. It is found that unitarity and time-reversal invariance imply a parametrization of the half-off-shell scattering amplitude which is identical to a formula previously found from a potential model. The amplitude is expressed in terms of the on-shell phase shifts and additional real quasi-phase-parameters.

IN recent papers[1] a parametric representation was given for the half-off-energy-shell element of the proton-proton scattering matrix. This representation was derived from potential theory, and describes each partial wave in terms of the on-shell phase shifts and mixing parameters, together with additional real numbers called quasi-phase parameters. The purpose of this paper is to show that the parametrization found is a consequence of off-shell unitarity and time-reversal invariance for a system of two spin-$\frac{1}{2}$ particles. We also give a simple derivation of a factorization theorem discussed recently by Kowalski[2] for the full-off-shell amplitude.

The general transition matrix T is defined in terms of the kinetic-energy operator K and the potential V by

$$T(E)=V[1+(E+i\epsilon-K-V)^{-1}V]. \quad (1)$$

The matrix elements of $T(E)$ between initial and final plane wave states $\varphi_{\mathbf{P}_i}$ and $\varphi_{\mathbf{P}_f}$ are related to the center-of-mass (c.m.) M matrix by

$$\langle\varphi_{\mathbf{P}_f}|T(E)|\varphi_{\mathbf{P}_i}\rangle=-(4\pi^2\mu)^{-1}M_\kappa(\mathbf{k}',\mathbf{k})\delta^3(\mathbf{P}_f-\mathbf{P}_i). \quad (2)$$

Here $\mathbf{P}_i$ ($\mathbf{P}_f$) is the initial (final) momentum, E is the total energy, and $\mathbf{k}$ ($\mathbf{k}'$) and $\kappa^2/2\mu$ are these quantities in the c.m. system; μ is the reduced mass. M_κ is a 4×4 matrix if the interacting particles have spin $\frac{1}{2}$. Only the on-energy-shell amplitudes, for which $\kappa=k=k'$, are measured by elastic-scattering experiments. Double-scattering processes involve half-off-shell elements for which either $\kappa=k\neq k'$ or $\kappa=k'\neq k$. It is well known that these can be calculated from a potential model by integration over the potential. Full-off-shell amplitudes appear in higher-order processes, and cannot be directly calculated from a potential.

The unitarity condition expressed in terms of M is[2]

$$\tfrac{1}{2}i[M_\kappa{}^\dagger(\mathbf{k}',\mathbf{k})-M_\kappa(\mathbf{k},\mathbf{k}')]$$
$$=(\kappa/4\pi)\int d\Omega\, M_\kappa{}^\dagger(\mathbf{k}',\boldsymbol{\kappa})M_\kappa(\boldsymbol{\kappa},\mathbf{k}), \quad (3)$$

where Ω describes the direction of $\boldsymbol{\kappa}$.

Consider first the case of the singlet states. The singlet element of M can be expanded in the form

$$M_\kappa(\mathbf{k}',\mathbf{k})=\frac{1}{i\kappa}\sum_l\left(\frac{2l+1}{2}\right)\alpha_\kappa{}^l(k',k)P_l(\hat{k}'\cdot\hat{k}), \quad (4)$$

where P_l is a Legendre polynomial. Equation (3) then becomes

$$\alpha_\kappa{}^*(k',k)+\alpha_\kappa(k,k')=-\alpha_\kappa{}^*(k',\kappa)\alpha_\kappa(\kappa,k). \quad (5)$$

For simplicity we suppress the index l. If $\kappa=k=k'$, this equation implies that

$$\alpha_\kappa(\kappa,\kappa)=2ie^{i\delta^s(\kappa)}\sin\delta^s(\kappa) \quad (6)$$

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[1] M. I. Sobel, Phys. Rev. **138**, B1517 (1965); A. H. Cromer and M. I. Sobel, *ibid.* **152**, 1351 (1966).

[2] K. L. Kowalski, Phys. Rev. **144**, 1239 (1966); C. Lovelace, in *Lectures at the 1963 Edinburgh Summer School*, edited by R. G. Moorhouse (Oliver and Boyd, London, 1964).

for some real phase shift δ^s, which is the standard condition of elastic unitarity. For $\kappa=k\neq k'$ we have

$$\alpha_\kappa^*(k',\kappa)+\alpha_\kappa(\kappa,k')=-\alpha_\kappa^*(k',\kappa)\alpha_\kappa(\kappa,\kappa)\,, \tag{7}$$

which is the half off-shell unitarity condition. If time-reversal invariance is imposed, we have $\alpha_\kappa(\kappa,k')=\alpha_\kappa(k',\kappa)$, so that the left side of Eq. (7) becomes $2\,\mathrm{Re}\alpha_\kappa(k',\kappa)$. Thus the equation implies that the phase of $\alpha_\kappa^*(k',\kappa)$ is opposite to that of $\alpha_\kappa(\kappa,\kappa)$, and

$$\alpha_\kappa(k',\kappa)=2ie^{i\delta^s(\kappa)}\Delta_\kappa^s(k',\kappa)\,. \tag{8}$$

Here Δ^s is a real number called the singlet quasi-phase-parameter.

For the triplet states,[3] the M matrix is expanded in terms of spherical harmonics of the angle between $\mathbf{k}$ and $\mathbf{k}'$. We consider a particular total angular momentum j. The states for which the orbital angular momentum l is equal to j do not mix with the states $l=j\pm1$. Therefore the argument used for the singlet state applies and the scattering amplitude takes the form $2i\exp[i\delta(\kappa)]\Delta_\kappa(k',\kappa)$, where δ is the standard triplet $l=j$ phase shift.

For the mixed states $l=j\pm1$, α becomes a 2×2 matrix with rows (columns) corresponding to initial (final) states $l=j-1$ and $l=j+1$, respectively,

$$\alpha_\kappa(k',\kappa)=\begin{pmatrix}\alpha_\kappa^{11}(k',\kappa) & \alpha_\kappa^{12}(k',\kappa)\\ \alpha_\kappa^{21}(k',\kappa) & \alpha_\kappa^{22}(k',\kappa)\end{pmatrix}. \tag{9}$$

On the energy shell, time-reversal invariance implies $\alpha^{12}=\alpha^{21}$, but in the general case the matrix is not symmetric. Time-reversal invariance implies

$$\alpha_\kappa(k',k)=\alpha_\kappa^T(k,k')\,. \tag{10}$$

Thus, instead of Eq. (7), we have

$$\alpha_\kappa^\dagger(k',\kappa)+\alpha_\kappa^T(k',\kappa)=-\alpha_\kappa^\dagger(k',\kappa)\alpha_\kappa(\kappa,\kappa)\,. \tag{11}$$

The on-shell expression for α, if we use the Blatt-Biedenharn parametrization, is[4]

$$\alpha_\kappa(\kappa,\kappa)=2i\begin{pmatrix}\cos^2\epsilon\, e^{i\delta_-}\sin\delta_-+\sin^2\epsilon\, e^{i\delta_+}\sin\delta_+ & \sin\epsilon\cos\epsilon(e^{i\delta_-}\sin\delta_--e^{i\delta_+}\sin\delta_+)\\ \sin\epsilon\cos\epsilon(e^{i\delta_-}\sin\delta_--e^{i\delta_+}\sin\delta_+) & \sin^2\epsilon\, e^{i\delta_-}\sin\delta_-+\cos^2\epsilon\, e^{i\delta_+}\sin\delta_+\end{pmatrix}. \tag{12}$$

Here ϵ and $\delta_\pm$ are the mixing parameters and the $l=j\pm1$ phase shifts, evaluated at energy $\kappa^2/2\mu$. If we put Eqs. (9) and (12) into Eq. (11) we find four equations, which imply that the quantities

$$\tfrac{1}{2}ie^{i\delta_-}(\alpha^{11*}+\alpha^{21*}\tan\epsilon)\,,\quad \tfrac{1}{2}ie^{i\delta_+}(\alpha^{11*}-\alpha^{21*}\cot\epsilon)\,,\quad \tfrac{1}{2}ie^{i\delta_-}(\alpha^{22*}+\alpha^{12*}\cot\epsilon)\,,\quad \tfrac{1}{2}ie^{i\delta_+}(\alpha^{22*}-\alpha^{12*}\tan\epsilon)\,,$$

are real. Here the α's are $\alpha_\kappa(k',\kappa)$. If we define these four quantities as Δ_-^-, Δ_+^-, Δ_-^+, and Δ_+^+, respectively, and then solve for the α's, we find

$$\alpha_\kappa(k',\kappa)=2i\begin{pmatrix}\cos^2\epsilon\, e^{i\delta_-}\Delta_-^-+\sin^2\epsilon\, e^{i\delta_+}\Delta_+^- & \sin\epsilon\cos\epsilon\,(e^{i\delta_-}\Delta_-^+-e^{i\delta_+}\Delta_+^+)\\ \sin\epsilon\cos\epsilon\,(e^{i\delta_-}\Delta_-^--e^{i\delta_+}\Delta_+^-) & \cos^2\epsilon\, e^{i\delta_+}\Delta_+^++\sin^2\epsilon\, e^{i\delta_-}\Delta_-^+\end{pmatrix}. \tag{13}$$

Equation (13), together with the forms for the unmixed states, describes the parameterization previously found from a potential model. Thus any attempt to fit the off-shell scattering matrix to experimental data by searching for real quasi-phase-parameters Δ will automatically satisfy unitarity.

Finally, we note that Eq. (5) implies in a simple way the factorization property for the full-off-shell amplitude which has been proved in some recent papers.[2,5] If $\alpha_\kappa(k',k)\equiv2i\chi_\kappa(k',k)$, then, using Eq. (5), we have

$$\mathrm{Im}\chi_\kappa(k',k)=\Delta_\kappa(k',\kappa)\Delta_\kappa(\kappa,k) \tag{14}$$

for the singlet case. So the imaginary part can be calculated in terms of quasi-phase-parameters, and hence, from a potential model. Unitarity supplies no information about $\mathrm{Re}\chi$.

[3] I am indebted to Dr. L. Heller for a discussion on this point.
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