Critique of an Application of the N-Quantum Approximation*

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The renormalization prescription given by Greenberg in his application of the N-quantum approximation to the Hurst-Thirring field is shown to be divergent.

A METHOD for obtaining approximate solutions to the Heisenberg field operators in a theory characterized by a specific Lagrangian has been recently proposed by Greenberg.[1] The method was made explicit by applying the general ideas in first order of the approximation to the model with $\mathcal{L}_I(x)=gA(x)^3$. The explicit procedure used results in a quadratic integral equation for the vertex function, and it was "shown" that the power-series-expansion solution yields finite results in all orders of the coupling constant. The purpose of this article is to point out an oversight in the proof of finiteness which, when included, leads to the conclusion that the power-series-expansion fails in fifth order of the coupling constant.

The integral equation in question is

$$(m^2-k^2)\varphi(k)=\Phi(k;f_0+\varphi)+\mathfrak{M}(k;f_0+\varphi)+(2g)^{-1}[\mathfrak{M}(k;f_0+\varphi)-F_0(f_0+\varphi)]\times(m^2-k^2)\varphi(k)\,, \quad (1)$$

where

$$f_0(k)\equiv 2g(m^2-k^2)^{-1}\,,$$

and the functionals Φ, $\mathfrak{M}$, and F_0 are defined by

$$\Phi(k;\chi)=F(k;\chi)-F_0(\chi)\,,$$

$$F(k_1+k_2;\chi)=(2\pi)^{-3}g\int\{\chi(k_1+q)\chi(k_2-q)+\chi(k_1+k_2-q)[\chi(k_1-q)+\chi(k_2-q)]\}\times\delta(q^2-m^2)\,|_{k_1{}^2=k_2{}^2=m^2}d^4q\,,$$

$$F_0(\chi)=F(k;\chi)\,|_{k^2=m^2};$$

$$\mathfrak{M}(k;\chi)=2(m^2-k^2)^{-1}[M(k;\chi)-M_0(\chi)-M_1(\chi)(k^2-m^2)]\,,$$

$$M(k;\chi)=(2\pi)^{-3}g^2\int\chi(k-q)\delta(q^2-m^2)d^4q\,,$$

$$M_0(\chi)=M(k;\chi)\,|_{k^2=m^2}\,,$$

$$M_1(\chi)=\left.\frac{dM(k;\chi)}{dk^2}\right|_{k^2=m^2}. \quad (2)$$

In the above equations it is to be understood that φ satisfies the retarded boundary condition

$$\varphi(k)=\lim_{\epsilon\to 0}\varphi(\mathbf{k},k_0+i\epsilon)\,.$$

Expanding φ as

$$\varphi(k)=\sum_{n=1}^{\infty}\varphi_n(k)\,, \quad (3)$$

with φ_n of order g^{2n+1}, we have[2]

$$(m^2-k^2)\varphi_1(k)=\Phi(k;f_0)+\mathfrak{M}(k;f_0) \quad (4)$$

and

$$(m^2-k^2)\varphi_2(k)=[\Phi(k;f_0+\varphi_1)-\Phi(k;f_0)-\Phi(k;\varphi_1)]+\mathfrak{M}(k;\varphi_1)(2g)^{-1}[\mathfrak{M}(k;f_0)-F_0(f_0)]\times(m^2-k^2)\varphi_1(k)\,. \quad (5)$$

It was pointed out in Ref. 1 that $F(k;f_0)$ varies as $(k^2)^{-1}[\ln(k^2/m^2)]^2$ for large $|k^2|$, and therefore the asymptotic form of $\Phi(k;f_0)$ is

$$\Phi(k;f_0)\sim -F_0(f_0)+C[\ln(k^2/m^2)]^2/(k^2)\,. \quad (6)$$

It is a straightforward matter to show that $M(k;f_0)-M_0(f_0)$ varies as $\ln(k^2/m^2)$ for large $|k^2|$ so that the asymptotic form of $\varphi_1(k)$ is given by

$$(m^2-k^2)\varphi_1(k)\sim[2M_1(f_0)-F_0(f_0)]+C[\ln(k^2/m^2)]^2/(k^2)\,. \quad (7)$$

In Ref. 1 it is claimed that the asymptotic form of $\varphi_1(k)$ guarantees that the Φ and $\mathfrak{M}$ quantities involved in $\varphi_2(k)$ [see Eq. (5)] are finite. This statement is true with respect to Φ, but, as will be shown below, it is false with respect to $\mathfrak{M}$.

We shall write the Lorentz-invariant quantity $M(k;\varphi_1)$ in the reference frame with $\mathbf{k}=0$ as

$$M(k;\varphi_1)\,|_{\mathbf{k}=0}=\int\frac{d^3q}{2\omega}J(\mathbf{q},k_0)\,,$$

where

$$J(\mathbf{q},k_0)=[\varphi_1(k-q)\,|_{q_0=\omega}+\varphi_1(k-q)\,|_{q_0=-q}]_{\mathbf{k}=0}\,,$$

and

$$\omega=(|\mathbf{q}|^2+m^2)^{1/2}\,. \quad (8)$$

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[1] O. W. Greenberg, Phys. Rev. **139**, B1038 (1956).

[2] The $\mathfrak{M}$ term in Eq. (4) was omitted in Ref. 1, but its presence there does not materially affect the subsequent discussion.

The large $|\mathbf{q}|$ behavior of $\varphi_1(k-q)|_{q_0=\pm\omega,\mathbf{k}=0}$ is clearly controlled by the constant term on the right-hand side of Eq. (7), $[2M_1(f_0)-F_0(f_0)]$. However, because of a cancellation between positive- and negative-frequency terms, the large $|\mathbf{q}|$ behavior of $J(q,k_0)$ is *not* controlled by the above constant term, but instead is controlled by the C term of Eq. (7). Thus,

$$J(\mathbf{q},k_0) \xrightarrow[|\mathbf{q}|\to\infty]{} -2C[\ln(|\mathbf{q}|/m)]^2/(2k_0|\mathbf{q}|)^2, \quad (9)$$

which depends[3] *on* k_0; therefore the subtractions indicated in the computation of $\mathfrak{M}(k;\varphi_1)$ from $M(k;\varphi_1)$ [see Eq. (2)] cannot produce a finite result for $\mathfrak{M}(k;\varphi_1)$. This result when coupled with Eq. (5) completes our proof that the power-series-expansion solution of Eq. (1) fails in fifth order of the coupling constant.

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[3] It should be noted that the inclusion of higher-order terms in Eq. (7) cannot alter Eq. (9).

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Off-Shell Unitarity for Two Spin-$\frac{1}{2}$ Particles

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The implications of off-shell unitarity for the system of two spin-$\frac{1}{2}$ particles are examined. It is found that unitarity and time-reversal invariance imply a parametrization of the half-off-shell scattering amplitude which is identical to a formula previously found from a potential model. The amplitude is expressed in terms of the on-shell phase shifts and additional real quasi-phase-parameters.

IN recent papers[1] a parametric representation was given for the half-off-energy-shell element of the proton-proton scattering matrix. This representation was derived from potential theory, and describes each partial wave in terms of the on-shell phase shifts and mixing parameters, together with additional real numbers called quasi-phase parameters. The purpose of this paper is to show that the parametrization found is a consequence of off-shell unitarity and time-reversal invariance for a system of two spin-$\frac{1}{2}$ particles. We also give a simple derivation of a factorization theorem discussed recently by Kowalski[2] for the full-off-shell amplitude.

The general transition matrix T is defined in terms of the kinetic-energy operator K and the potential V by

$$T(E)=V[1+(E+i\epsilon-K-V)^{-1}V]. \quad (1)$$

The matrix elements of $T(E)$ between initial and final plane wave states $\varphi_{\mathbf{P}_i}$ and $\varphi_{\mathbf{P}_f}$ are related to the center-of-mass (c.m.) M matrix by

$$\langle\varphi_{\mathbf{P}_f}|T(E)|\varphi_{\mathbf{P}_i}\rangle=-(4\pi^2\mu)^{-1}M_\kappa(\mathbf{k}',\mathbf{k})\delta^3(\mathbf{P}_f-\mathbf{P}_i). \quad (2)$$

Here $\mathbf{P}_i$ ($\mathbf{P}_f$) is the initial (final) momentum, E is the total energy, and $\mathbf{k}$ ($\mathbf{k}'$) and $\kappa^2/2\mu$ are these quantities in the c.m. system; μ is the reduced mass. M_κ is a 4×4 matrix if the interacting particles have spin $\frac{1}{2}$. Only the on-energy-shell amplitudes, for which $\kappa=k=k'$, are measured by elastic-scattering experiments. Double-scattering processes involve half-off-shell elements for which either $\kappa=k\neq k'$ or $\kappa=k'\neq k$. It is well known that these can be calculated from a potential model by integration over the potential. Full-off-shell amplitudes appear in higher-order processes, and cannot be directly calculated from a potential.

The unitarity condition expressed in terms of M is[2]

$$\tfrac{1}{2}i[M_\kappa^\dagger(\mathbf{k}',\mathbf{k})-M_\kappa(\mathbf{k},\mathbf{k}')] = (\kappa/4\pi)\int d\Omega\, M_\kappa^\dagger(\mathbf{k}',\boldsymbol{\kappa})M_\kappa(\boldsymbol{\kappa},\mathbf{k}), \quad (3)$$

where Ω describes the direction of $\boldsymbol{\kappa}$.

Consider first the case of the singlet states. The singlet element of M can be expanded in the form

$$M_\kappa(\mathbf{k}',\mathbf{k})=\frac{1}{i\kappa}\sum_l\left(\frac{2l+1}{2}\right)\alpha_\kappa{}^l(k',k)P_l(\hat{k}'\cdot\hat{k}), \quad (4)$$

where P_l is a Legendre polynomial. Equation (3) then becomes

$$\alpha_\kappa{}^*(k',k)+\alpha_\kappa(k,k')=-\alpha_\kappa{}^*(k',\kappa)\alpha_\kappa(\kappa,k). \quad (5)$$

For simplicity we suppress the index l. If $\kappa=k=k'$, this equation implies that

$$\alpha_\kappa(\kappa,\kappa)=2ie^{i\delta^s(\kappa)}\sin\delta^s(\kappa) \quad (6)$$

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[1] M. I. Sobel, Phys. Rev. **138**, B1517 (1965); A. H. Cromer and M. I. Sobel, *ibid.* **152**, 1351 (1966).

[2] K. L. Kowalski, Phys. Rev. **144**, 1239 (1966); C. Lovelace, in *Lectures at the 1963 Edinburgh Summer School*, edited by R. G. Moorhouse (Oliver and Boyd, London, 1964).