Transition Probabilities of the Hydrogen Atom from Noncompact Dynamical Groups*

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(Received 30 September 1966)

In order to describe electromagnetic dipole transitions of the H atom within the framework of dynamical groups, an explicit irreducible representation of the Lie algebra $O(4.2)$ has been found on the space of bound-state wave functions. This representation remains irreducible when restricting to the subalgebra $O(4,1)$. The transformation properties of the dipole operator in $O(4,2)$ have been specified. The description becomes particularly simple by the introduction of a one-parameter family of representations of $O(4,2)$. Finally, position representations of the generators of $O(4,2)$ have been given.

I. INTRODUCTION

IN the preceding paper,[1] external interactions have been introduced into the formalism of noncompact dynamical groups describing all states of a quantum-mechanical system, and the reduced matrix elements between different rotational and vibrational levels have been calculated.

In the present paper we carry out such a discussion for the H atom. As dynamical groups of the H atom one can use the groups E_4 or $O(4,1)$.[2] These groups contain irreducible triangular representations which describe all the states of the discrete spectrum.

The following new results will be presented in this paper:

1. The larger group $O(4,2)$ is shown to be a dynamical group of the H atom by explicit construction of a matrix representation on the space of bound-state wave function. This representation remains irreducible when restricting the group to the subgroup $O(4,1)$.[3]

2. It is shown how the inclusion of the electromagnetic dipole transition operator leads to $O(4,2)$ as the dynamical group of the H atom. The transformation properties of the dipole operator in $O(4,2)$ are specified.

3. The inclusion of the dipole operator reveals a fiber bundle structure of the hydrogen wave functions in terms of a one-parameter family of representations of the dynamical group $O(4,2)$. For describing this structure the position representation has a special significance.

4. The position representation of all operators of $O(4,2)$ is given.

II. CONSTRUCTION OF $O(4,2)$

As is well known, the group of degeneracy of energy is $O(4)$ which is generated by the orbital angular momentum $\mathbf{L}$ and the Lenz vector $\mathbf{M}$ and has the invariant operator $\mathbf{L}\cdot\mathbf{M}=0$. In the $O(3)\times O(3)$ diagonalization of $O(4)$ the generators are defined as

$$\mathbf{J}=\tfrac{1}{2}(\mathbf{L}+\mathbf{M});\quad \mathbf{K}=\tfrac{1}{2}(\mathbf{L}-\mathbf{M}) \tag{1}$$

with

$$[\mathbf{J},\mathbf{K}]=0$$

and the states are labeled by $|j,k;j_3,k_3\rangle$. The states of the H atom for fixed n are given by the representations with $j=k=\frac{1}{2}(n-1)$. These states correspond to the wave functions $|n_1n_2m\rangle$ obtained from the Schrödinger equation in parabolic coordinates.[4] The quantum numbers j_3 and k_3 are given by observing that the eigenvalues of L_3 and M_3 on $|n_1n_2m\rangle$ are m and n_1-n_2, respectively. Remember that the principal quantum number constrains n_1, n_2, m by the equation

$$n=n_1+n_2+m+1. \tag{2}$$

Using (1) we now find

$$\begin{aligned} j_3&=\tfrac{1}{2}[m+(n_1-n_2)],\\ k_3&=\tfrac{1}{2}[m-(n_1-n_2)]. \end{aligned} \tag{3}$$

If we define the operators N, N_1, N_2 by $N|n_1n_2m\rangle=n|n_1n_2m\rangle$, etc., Eq. (3) gives the operator relation

$$N=N_1+N_2+L_3+1. \tag{4}$$

Let us now introduce operators $N_1^{\pm}$, $N_2^{\pm}$ which raise and lower the parabolic quantum numbers n_1, n_2, respectively, by the equations

$$\begin{aligned} N_1^{+}|n_1n_2m\rangle&=-[(n_1+1)(n_1+m+1)]^{1/2}|n_1+1,n_2,m\rangle,\\ N_1^{-}|n_1n_2m\rangle&=-[n_1(n_1+m)]^{1/2}|n_1-1,n_2,m\rangle;\\ N_2^{+}|n_1n_2m\rangle&=+[(n_2+1)(n_2+m+1)]^{1/2}|n_1,n_2+1,m\rangle,\\ N_2^{-}|n_1n_2m\rangle&=+[n_2(n_2+m)]^{1/2}|n_1,n_2^{-1},m\rangle, \end{aligned} \tag{5}$$

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* Research sponsored by the U. S. Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Grant No. AF-AFOSR-30-65.

[1] A. O. Barut, preceding paper, Phys. Rev. **156**, 1538 (1967).

[2] The first group proposed as the dynamical group of the H atom was E_4 [A. O. Barut, Phys. Rev. **135**, B839 (1964)]. Later it was found that $O(4,1)$ has also a representation containing all the bound states of the H atom [A. O. Barut, P. Budini, and C. Fonsdal, Proc. Roy. Soc. (London) **A291**, 106 (1966); Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17**, 148 (1965); R. H. Pratt and T. F. Jordan, Phys. Rev. **148**, 1276 (1966); R. Musto, *ibid.* **148**, 1274 (1966); M. Bander and C. Itzykson, Rev. Mod. Phys. **38**, 330 (1966); **38**, 346 (1966)]. The relevant representations of E_4 and $O(4,1)$ are related to each other by group contraction and their $O(4)$ content is the same.

[3] I. A. Malkin and V. I. Man'ko, JETP Pis'ma v Redaktsiyn **2**, 230 (1966) [English transl.: JETP Letters **2**, 146 (1966)] have noticed the use of $O(4,2)$ for the H spectrum in Fock coordinates in analogy to the Klein-Gordon equation with zero mass. They have not discussed the problem of transition probabilities.

[4] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Academic Press Inc., New York, 1957).

then we obtain immediately

$$[N_i^-,N_j^+]=\delta_{ij}[N-(-1)^i M_3],\quad [N_i^+,N_j^+]=0,\quad i,j=1,2. \tag{6}$$

As we can see from Eqs. (3) and (4), N_i do not belong to a pure tensor of the rotation group generated by **L**. However, the combinations

$$A_3^\pm\equiv\tfrac{1}{2}(N_1^\pm-N_2^\pm),\quad B^\pm\equiv\tfrac{1}{2}(N_1^\pm+N_2^\pm) \tag{7}$$

are the third component of a vector and a scalar, respectively, as is shown in Appendix A.

Using Eq. (6) one finds for A_3, B the commutation rules:

$$\begin{aligned}
&[A_3^\mp,A_3^\pm]=\pm\tfrac{1}{2}N,\\
&[A_3^\mp,B^\pm]=\pm\tfrac{1}{2}M_3,\\
&[A_3^\mp,B^\mp]=0,\\
&[B^-,B^+]=\tfrac{1}{2}N,\\
&[N,A_3^\pm]=\pm A_3^\pm,\\
&[N,B^\pm]=\pm B^\pm,\\
&[M_3,A_3^\pm]=\pm B^\pm,\\
&[M_3,B^\pm]=\pm A_3^\pm.
\end{aligned} \tag{8}$$

The other components of $\mathbf{A}^\pm$ satisfy the commutation rules (see Appendix A)

$$\begin{aligned}
&[A_i^\mp,A_j^\pm]=-\tfrac{1}{2}iL_{ij}\pm\tfrac{1}{2}N\delta_{ij},\\
&[A_i^\mp,A_j^\pm]=0,\\
&[A_i^\pm,M_j]=\mp B^\pm\delta_{ij}.
\end{aligned} \tag{9}$$

In the remaining terms of Eq. (8) one can replace the index 3 by i, as follows from the vector character of **A** and **M**.

If we now make the following identifications:

$$\begin{aligned}
&L_{l4}=M_l, && i=1,\cdots,3\\
&L_{\mu5}=\begin{pmatrix}\mathbf{A}^++\mathbf{A}^-\\ -i(B^+-B^-)\end{pmatrix}, && \mu=1,\cdots,4\\
&L_{\mu6}=\begin{pmatrix}i(\mathbf{A}^+-\mathbf{A}^-)\\ B^++B^-\end{pmatrix},\\
&L_{56}=N,
\end{aligned} \tag{10}$$

we can find by straightforward calculation that the 15-generators $L_{\alpha\beta}$; $1\leqq\alpha\leqq\beta\leqq6$ satisfy the commutation rules of $O(4,2)$ with the metric: $g_{\mu\mu}=+1$, $g_{55}=g_{66}=-1$. The explicit irreducible representation of $O(4,2)$ is obtained by using the definitions (10), (7) and inserting the matrices for $N_1^\pm$, $N_2^\pm$ given in (5). This representation remains irreducible when one restricts $O(4,2)$ to the subgroup $O(4,1)$ by dropping $L_{\mu6}$, L_{56}. To see this just observe that $\mathbf{A}^+$ in $L_{\mu5}$ can raise n arbitrarily high by repeated application.

III. THE ELECTROMAGNETIC DIPOLE OPERATOR

It is sufficient to consider the third component of the dipole operator; the others are then obtained by rotation. In the position representation the wave functions $|n_1n_2m\rangle$ are[4]

$$u_{n_1n_2m}=e^{\pm im\varphi}N_{n_1n_2m}e^{-(\xi+\eta)/2n}\left(\frac{\xi\eta}{n^2}\right)^{m/2}\times L_{n_1+m}{}^m\left(\frac{\xi}{n}\right)L_{n_2+m}{}^m\left(\frac{\eta}{n}\right), \tag{11}$$

where

$$\xi=r+z,\quad \eta=r-z,\ \phi \tag{12}$$

are the parabolic coordinates and $L_{n+m}{}^m(z)$ are the Laguerre polynomials. We have to find the effect of $z=\frac{1}{2}(\xi-\eta)$ on $u_{n_1n_2m}$. For this we simply use the recursion relation[5]

$$zL_{n_1+m}{}^m(z)=-\frac{n_1+1}{n_1+m+1}L_{n_1+m+1}{}^m(z)+(2n_1+m+1)L_{n_1+m}{}^m(z)-(n_1+m)^2L_{n_1+m-1}{}^m(z). \tag{13}$$

Therefore we obtain

$$\begin{aligned}
(\xi/n)u_{n_1n_2m}
&=-\frac{n_1+1}{n_1+m+1}\frac{N_{n_1n_2m}}{N_{n_1+1,n_2,m}}D_{(n+1)/n}u_{n_1+1,n_2,m}\\
&\quad+(2n_1+m+1)u_{n_1n_2m}\\
&\quad-(n_1+m)^2\frac{N_{n_1n_2m}}{N_{n_1-1,n_2,m}}D_{(n-1)/n}u_{n_1-1,n_2,m},
\end{aligned} \tag{14}$$

where D_a is defined as the dilatation operator by a, i.e.,

$$D_af(x)=f(ax). \tag{15}$$

A position representation to D_a is $D_a=a^{xd/dx}$. An equation similar to (14) holds for $(\eta/n)u_{n_1n_2m}$. Now, with the normalization constant

$$N_{n_1n_2m}=\frac{(-1)^{n_2+\frac{1}{2}(|m|-m)}}{\pi^{1/2}n^2}\left[\frac{n_1!n_2!}{(n_1+m)!^3(n_2+m)!^3}\right]^{1/2} \tag{16}$$

and remembering the definitions (5) for $N_1^\pm$, $N_2^\pm$, we

[5] E. Jahnke, F. Emde, and F. Lösch, *Tables of Higher Functions* (McGraw-Hill Book Company, Inc., New York, 1960). Observe, however, that our Laguerre polynomials are the ones used by Bethe and Salpeter (Ref. 4) which are related to the ones in in Jahnke-Emde by $L_{n+m}{}^m=(-1)^{n+m}(m+n)!L_n{}^{(m)}$.

recognize

$$\xi u_{n_1n_2m}=\left[D_{N/(N-1)}N_1^+\frac{(N+1)^2}{N}+M_3+N+D_{N/(N+1)}N_1^-\frac{(N-1)^2}{N}\right]u_{n_1n_2m},$$
$$\eta u_{n_1n_2m}=\left[D_{N/(N-1)}N_2^+\frac{(N+1)^2}{N}-M_3+N+D_{N/(N+1)}N_2^-\frac{(N-1)^2}{N}\right]u_{n_1n_2m}. \tag{17}$$

From this follows by addition and subtraction:

$$r=\left[D_{N/(N-1)}B^+\frac{(N+1)^2}{N}+N+D_{N/(N+1)}B^-\frac{(N-1)^2}{N}\right],$$
$$x_j=\left[D_{N/(N-1)}A_j^+\frac{(N+1)^2}{N}+M_j+D_{N/(N+1)}A_j^-\frac{(N-1)^2}{N}\right]. \tag{18}$$

We see that the dipole operator contains the operators $A_j^{\pm}$, $B^{\pm}$ which are elements of the complex extension of the Lie algebra of $O(4,2)$[6] but do not belong to $O(4,1)$. In this sense $O(4,2)$ turns out to be the more natural dynamical group.

To see the physical meaning of the dilatation operator, observe that $A_j^{\pm}$, $B^{\pm}$ allow only jumps of the principal quantum number by one. The dilatation operator is the one which causes transitions in n over the whole spectrum. It is also the more complicated operator to calculate. Its representations will be given and discussed in the next section.

We also note that the operators B^+, B^-, N occurring in the *magnitude* of the dipole operators generate an algebra isomorphic to $SU(1,1)\sim O(2,1)$.

IV. THE FIBER SPACE OF THE HYDROGEN WAVE FUNCTION

In view of later applications we consider in this discussion the Hilbert space of wave functions defined with diagonal L^2 and L_3. If we define

$$S_{nl}(r)=e^{-r}r^lF(-n_r,\,2l+2,\,2r)\,, \tag{19}$$

where $n_r=n-l-1$ is the radial quantum number and $F(x,y,z)$ the hypergeometric function,[4] then the radial wave functions of the H atom are

$$R_{nl}(r)=N_{nl}S_{nl}(r/n)\,, \tag{20}$$

where N_{nl} is a normalization constant.

Now consider the normalized family of wave functions

$$\psi_{nlm}{}^{\tau}(\mathbf{r})=N_{nl}{}^{\tau}S_{nl}(r/\tau)Y_{lm}(\varphi,\theta) \tag{21}$$

[6] The complex extension of the Lie algebra of $O(4,2)$ is D_3.

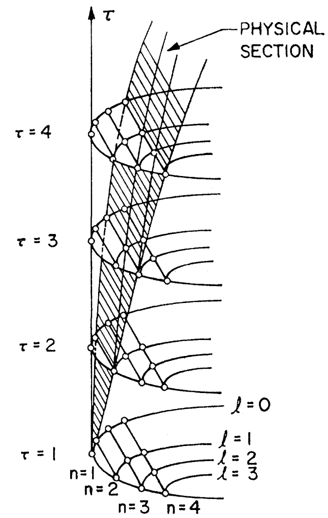

FIG. 1. The bundle of wave functions $\psi_{nl\tau}(\mathbf{r})$. Each fiber contains ψ's with fixed n,l and varying τ; each cross section represents a fixed τ. The physical wave functions lie on the oblique-lined section.

for all τ. The normalization constant $N_{nl}{}^{\tau}$ is

$$N_{nl}{}^{\tau}=(n/\tau)^{3/2}N_{nl}=(n/\tau)^{3/2}\times\frac{2^{l+1}}{n^2(2l+1)!}\left[\frac{(n+l)!}{(n-l-1)!}\right]^{1/2}. \tag{22}$$

We can represent this family as a bundle of functions, each cross section of which is covered with $\psi_{nlm}{}^{\tau}(\mathbf{r})$'s for fixed τ, τ increasing along each fiber (see Fig. 1). The physical wave functions (20) are given by the steep section with $\tau=n$ for $\psi_{nlm}{}^{\tau}(\mathbf{r})$.

Observe that each cross section contains orthogonal wave functions only for fixed n. For different n and equal l, m, only the states in the physical section are orthogonal and

$$d_{nln'l}{}^{\tau\tau'}=\int_0^{\infty}R_{nl}{}^{\tau}(r)R_{n'l}{}^{\tau'}(r)r^2dr \tag{23}$$

is the scalar product of these different functions. Inserting (20) and (19) we obtain

$$d_{nln'l}{}^{\tau\tau'}=\frac{N_{nl}{}^{\tau}N_{n'l}{}^{\tau'}}{\tau^l\tau'^l}\int e^{-(1/\tau+1/\tau')r}r^{2l}\times F(-n_r,\,2l+2,\,2r/\tau)\times F(-n_r{}'2l+2,\,2r/\tau')r^2dr. \tag{24}$$

Gordon[7] has calculated the integral

$$J_{\rho}{}^{(\sigma,\tau)}(n_r,k;\,n'r;\,k')=\int_0^{\infty}e^{-(k+k')\xi/2}\xi^{\rho+\sigma}F(-n_r,\,\rho+1,\,k\xi)\times F(-n_r{}';\,\rho+1-\tau,\,k'\xi)d\xi \tag{25}$$

[7] W. Gordon, Ann. Physik 2, 1031 (1929).

in terms of which we find

$$d_{nln'l}{}^{\tau\tau'}=\frac{N_{nl}{}^{\tau}N_{n'l}{}^{\tau'}}{\tau^{l}\tau'^{l}}J_{2l+1}{}^{(1,0)}(n_r,2/\tau;n_r'2/\tau'). \quad (26)$$

This $d_{nln'l}{}^{\tau\tau'}$ now allows us to represent the dilatation operator D_a on the physical section. Obviously

$$D_a\psi_{nlm}(r)\equiv\psi_{nlm}(ar)$$
$$=a^{-3/2}\sum_{n'} d_{nln'l}{}^{(n/a)n'}\psi_{n'lm}(r). \quad (27)$$

Geometrically, D_a translates ψ a piece along a fiber and this new function can be expanded again along the physical section.

On each of the fixed τ cross sections we can define a representation of $O(4,2)$ by the same procedure as used in Sec. II; just take $|n_1n_2m;\tau\rangle$ as state vectors for fixed τ (see Appendix II). In these representations the dipole operator x_i causes only transitions between neighboring states, since it is essentially an element of the Lie algebra of $O(4,2)$;

$$r=\tau\left[B^+\left(\frac{N+1}{N}\right)^{1/2}+N+B^-\left(\frac{N-1}{N}\right)^{1/2}\right],$$
$$x_l=\tau\left[A_i^+\left(\frac{N+1}{N}\right)^{1/2}+M_i+A_i^-\left(\frac{N-1}{N}\right)^{1/2}\right]. \quad (28)$$

The transition (27) to the physical section lifts the selection rules in n and causes quite a complication of the calculation. It is important to note that the calculation of $d_{nln'l}{}^{\tau\tau'}$ needs a specific position representation of the state vectors. This is so since D_a is a dilatation in the physical space. While for the description of observed spectra one can pursue a purely group-theoretical approach, for the calculation of transition probabilities, one had to include at the present time-space properties into the consideration.

V. THE POSITION REPRESENTATION OF $O(4,2)$

This representation is well known for the angular momentum vector $\mathbf{L}$ and the Lenz vector $\mathbf{M}$, namely,

$$\mathbf{L}=\mathbf{r}\times\mathbf{p},$$
$$\mathbf{M}=\tfrac{1}{2}(\mathbf{p}\times\mathbf{L}-\mathbf{L}\times\mathbf{p})-\mathbf{r}/r. \quad (29)$$

To find the other generators $L_{\mu u}$ we just have to find $N_1^\pm$, $N_2^\pm$ in the position representation. This is readily done starting from the definition (5)

$$N_1^+|n_1n_2m\rangle=-[(n_1+1)(n_1+m+1)]^{1/2}|n_1+1,n_2,m\rangle,$$
$$N_1^-|n_1n_2m\rangle=-[n_1(n_1+m)]^{1/2}|n_1-1,n_2m\rangle \quad (30)$$

and using the recursion formula[5]

$$L_{n_1+1+m}{}^m(\xi)=\frac{n_1+m+1}{n_1+1}\left(\xi\frac{\partial}{\partial\xi}+n_1+m+1-\xi\right)L_{n_1+m}{}^m(\xi),$$

$$L_{n_1-1+m}{}^m(\xi)=(n_1+m)^{-2}\left(\xi\frac{\partial}{\partial\xi}-n_1\right)L_{n_1+m}{}^m(\xi). \quad (31)$$

One finds

$$N_1^+=-D_{N/(N+1)}\left(\xi\frac{\partial}{\partial\xi}-\xi+\frac{\xi}{2N}+\tfrac{1}{2}L_3+N_1+1\right)\left(\frac{N}{N+1}\right)^2,$$
$$N_1^-=-D_{N/(N-1)}\left(\xi\frac{\partial}{\partial\xi}+\frac{\xi}{2N}-\tfrac{1}{2}L_3-N_1\right)\left(\frac{N}{N-1}\right)^2, \quad (32)$$

where the index $N/(N\pm1)$ of D acts to the right of the parenthesis. From this it follows that

$$B^+=-\tfrac{1}{2}D_{N/(N+1)}\left(\mathbf{r}\frac{\partial}{\partial\mathbf{r}}-2r+r\frac{1}{N}+N+1\right)\left(\frac{N}{N+1}\right)^2,$$
$$\mathbf{A}^+=-\tfrac{1}{2}D_{N/(N+1)}\left(\frac{\partial}{\partial\mathbf{r}}+\mathbf{r}\partial_r-2\mathbf{r}+\mathbf{r}\frac{1}{N}+\mathbf{M}\right)\left(\frac{N}{N+1}\right)^2,$$
$$B^-=-\tfrac{1}{2}D_{N/(N-1)}\left(\mathbf{r}\frac{\partial}{\partial\mathbf{r}}+r\frac{1}{N}-N+1\right)\left(\frac{N}{N-1}\right)^2, \quad (33)$$
$$\mathbf{A}^-=-\tfrac{1}{2}D_{N/(N-1)}\left(r\frac{\partial}{\partial\mathbf{r}}+\mathbf{r}\partial_r+\mathbf{r}\frac{1}{N}-\mathbf{M}\right)\left(\frac{N}{N-1}\right)^2.$$

Inserting these equations into (10), one obtains the remaining generators $L_{\mu5}$, $L_{\mu6}$, and L_{56}.

VI. CONCLUSIONS

The present discussion suggests that many other potential problems might have a similar structure of the transition operator. It seems that by means of existing simple recursion relations a dynamical group can always be constructed which combined with a dilatation operator gives not only a representation of the spectra but also transition amplitudes. For the description of the dilatation operator, a position representation of the dynamical group has been used. It is not clear yet how the dilatation operator can be found within the framework of a purely group-theoretical approach to dynamics.[8]

APPENDIX A

(a) We first show that $B^\pm$ is a scalar. For this one only has to verify that

$$[L_\pm,B^+]=0,\quad [L_\pm,B^-]=0. \quad (A1)$$

[8] *Note added in proof*. This last problem is solved in a forthcoming paper [A. O. Barut and H. Kleinert (to be published)].

But

$$L_\pm|n_1n_2m\rangle=(J_\pm+K_\pm)|n_1n_2m\rangle$$
$$=+[(j\pm j_3)(j\pm{}_3+1)]^{1/2}|n_1,\, n_2\mp1,\, m\pm1\rangle$$
$$+[(j\mp k_3)(j\pm k_3+1)]^{1/2}|n_1\mp1,\, n_2,\, m\pm1\rangle. \quad \text{(A2)}$$

Hence, one finds

$$L_+|n_1n_2m\rangle=+[n_2(n_1+m+1)]^{1/2}|n_1,\, n_2-1,\, m+1\rangle$$
$$+[n_1(n_2+m+1)]^{1/2}|n_1-1,\, n_2,\, m+1\rangle,$$
$$L_-|n_1n_2m\rangle=+[(n_1+m)(n_2+1)]^{1/2}|n_1,\, n_2+1,\, m-1\rangle$$
$$+[(n_2+m)(n_1+1)]^{1/2}|n_i+1,\, n_2,\, m-1\rangle. \quad \text{(A3)}$$

On the other hand,

$$B^+|n_1n_2m\rangle=-[(n_1+1)(n_1+m+1)]^{1/2}|n_1+1,\, n_2,\, m\rangle$$
$$+[(n_2+1)(n_2+m+1)]^{1/2}|n_1,\, n_2+1,\, m\rangle. \quad \text{(A4)}$$

From this one can easily find that Eq. (A1) is true.

(b) Define the other components of $A^\pm$ by

$$A_+{}^\pm=A_1{}^\pm+iA_2{}^\pm, \quad A_-{}^\pm=A_1{}^\pm-iA_2{}^\pm$$

and

$$A_+{}^+|n_1,n_2,m\rangle=+2[(n_1+m+1)(n_2+m+1)]^{1/2}$$
$$\times|n_1,n_2,\, m+1\rangle, \quad \text{(A5)}$$
$$A_+{}^-|n_1,n_2,m\rangle=-2[n_1\cdot n_2]^{1/2}|n_1-1,\, n_2-1,\, m+1\rangle,$$
$$(A_+{}^+)^\dagger=A_-{}^-, (A_+{}^-)^\dagger=A_-{}^+.$$

Then one finds, using Eqs. (A3) and (A4),

$$[L_+,A_3{}^+]=-A_+{}^+,$$
$$[L_-,A_3{}^+]=+A_-{}^+. \quad \text{(A6)}$$

This verifies that $\mathbf{A}^\pm$ is a vector.

(c) To prove the commutation rules (9) we only have to show that

$$[A_-{}^\mp,A_3]=2L_-,$$
$$[A_-{}^\mp,A_3{}^\mp]=0,$$
$$[A_-{}^\pm,M_3]=0. \quad \text{(A7)}$$

The remaining relations follow then from the vector properties of $A_i{}^\pm$ and M_i and Eqs. (8).

APPENDIX B

(a) The functions of the fiber bundle in hyperbolic coordinates are

$$u_{n_1n_2m}{}^\tau=e^{\pm im\varphi}N_{n_1n_2m}{}^\tau e^{-(\xi+\eta)/2\tau}\left(\frac{\xi\eta}{\tau^2}\right)^{m'2}$$
$$\times L_{n_1+m}{}^m\left(\frac{\xi}{\tau}\right)L_{n_2+m}{}^m\left(\frac{\eta}{\tau}\right), \quad \text{(A8)}$$

where

$$N_{n_1n_2m}{}^\tau=(n/\tau)^{3/2}N_{n_1n_2m}. \quad \text{(A9)}$$

It now follows from Eq. (13) that

$$(\xi/\tau)u_{n_1n_2m}{}^\tau=-\frac{n_1+1}{n_1+m+1}\frac{N_{n_1n_2m}{}^\tau}{N_{n_1+1,n_2,m}{}^\tau}u_{n_1+1,n_2,m}$$
$$+(2n_1+m+1)u_{n_1n_2m}$$
$$-(n_1+m)^2\frac{N_{n_1n_2m}{}^\tau}{N_{n_1-1,n_2,m}{}^\tau}u_{n_1-1,n_2,m}.$$

Defining $N_1{}^\pm, N_2{}^\pm$ as in Eq. (5) and M, N in the obvious way, we find

$$r=\tau\left[B^+\left(\frac{N+1}{N}\right)^{1/2}+N+B^-\left(\frac{N-1}{N}\right)^{1/2}\right],$$
$$x_i=\tau\left[A_i{}^+\left(\frac{N+1}{N}\right)^{1/2}+M_i+A_i{}^-\left(\frac{N-1}{N}\right)^{1/2}\right]. \quad \text{(A10)}$$

A dilatation operator is not needed here, and the x_i consist essentially of elements of the Lie algebra $O(4,2)$.