Calculation of Transition Probabilities from Noncompact Dynamical Groups*

A. O. BARUT
Department of Physics, University of Colorado, Boulder, Colorado
(Received 30 September 1966)

External electromagnetic interactions have been introduced into the formalism of noncompact dynamical groups describing *all* states of a quantum-mechanical system. Complete reduced matrix elements can be evaluated if the hadron levels are approximated as rotational and "vibrational" levels. The relation of the method to the S matrix and the algebra of currents is discussed.

I. INTRODUCTION

THE purpose of this paper is to show explicitly how external interactions are introduced into the noncompact groups describing *all* the states of a quantum-mechanical system. We compare the results with the S-matrix elements of the vertex functions and with the postulates of the algebra of currents.

The hypothesis[1] that all states of a quantum-mechanical system belong to a single irreducible representation of a dynamical noncompact group G which contains the group of generacy of energy, G_E, as a subgroup has been verified for all the well-known quantum-mechanical systems.[2–9] The hypothesis has been extended to strongly interacting particles[1,4,7,8,10] with the hope that one might predict the higher levels of hadron resonances.

In the formalism of noncompact groups, physical particles are assigned to fixed multiplets of the maximal compact subgroups, but the operators corresponding to transition probabilities under external interactions are in general complicated, that is, they do not transform simply as the noncompact generators of the group. On the other hand, if one starts with the physical currents, they obey simple commutation relations of an algebra [say, $SU(3)\times SU(3)$, or $SU(6)$ if one includes the scalar and pseudoscalar currents as well], but then the physical particles are *not* assigned to fixed multiplets of this algebra of currents, but to its mixed representations. Thus these two complementary approaches point to a larger algebraic structure containing both the noncompact group and the group of currents and that these two groups should be kept separately.

In this paper we treat some simple cases where the algebraic structure contains fixed multiplets and the transition operators transform like noncompact generators, so that complete reduced matrix elements can be evaluated. Inasmuch as the spin dependence of levels of hadrons can be easily approximated as rotational levels, for which there is good experimental evidence, and the isospin dependence as the analog of vibrational levels, these simple cases have also direct physical applications for strongly interacting particles.

II. TRANSITIONS BETWEEN ROTATIONAL AND VIBRATIONAL LEVELS

Whereas in ordinary quantum mechanics a system is characterized by a phenomenological potential or Hamiltonian, it can now also be characterized completely by the dynamical group G with the specified physical interpretation of its generators and its invariant operators. This result would be only an elegant geometrical description if we did not know how to deal with the external interactions of the system, which is really the essence of dynamics. We want to show that the description of the interactions fits quite naturally into the formalism of noncompact groups.

Consider first ordinary atomic systems under external electromagnetic interactions. The system without external fields is characterized by a specific representation[11] of the group G, corresponding to a Hamiltonian H_0. We shall say that the system has the geometry of the group G. The states of this representation are the possible states of the physical system. In order to actually produce these states, external interactions are necessary. Thus transition probability amplitudes or form factors of the transitions must be identified with the matrix elements of definite operators between the states of G.

The problem is to specify the transformation property of the external interactions and to compute reduced

* Supported in part by the U. S. Air Force Office of Scientific Research under Grant No. AF-AFOSR-30-65.

[1] A. O. Barut, Phys. Rev. **135**, B839 (1964); *Proceedings of the First Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman, San Francisco, 1965), p. 81.

[2] A. O. Barut and A. Böhm, Phys. Rev. **139**, B1107 (1965) (rotational levels).

[3] A. O. Barut, Phys. Rev. **139**, B1107 (1965) (N-dimensional anisotropic oscillator).

[4] Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17**, 148 (1965) (strong coupling model and models of hadron levels); Y. Dothan and Y. Ne'eman, California Institute of Technology Report No. 68-41, (unpublished).

[5] T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965) (strong-coupling theory).

[6] A. O. Barut, P. Budini, and C. Fronsdal, Proc. Roy. Soc. (London) **A291**, 106 (1966) (H atom, Bethe-Salpeter equation).

[7] A. O. Barut, in *Proceedings of the Seminar on High Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965), p. 679.

[8] C. Fronsdal, in Ref. 7, p. 665.

[9] N. Mukunda, L. O'Raifertaigh, and E. C. G. Sudarshan, Phys. Rev. Letters **15**, 1041 (1965).

[10] A. O. Barut, in *Proceedings of the Conference on Noncompact Groups in Particle Physics*, edited by Y. Chow (W. A. Benjamin, Inc., New York, 1966), p. 1.

[11] The specific representation is picked up by additional group-extension operators like parity (Refs. 8 and 10).

matrix elements as well. It will turn out that, in general, the external interactions lie in a larger group $\mathcal{G}$, whose relevant irreducible representation is also irreducible for G.

There are three practically important electromagnetic transitions, namely, those between rotational, vibrational, and Coulombic levels. The rotational levels of a rigid body have the symmetry of the group O(4,1), with one vanishing invariant operator, or E_4 (Euclidian group in four dimensions).[12] The group of degeneracy of the completely symmetric rigid body is O(4) in $O(3)\times O(3)$ diagonalization, where one $O(3)$ refers to $\mathbf{J}_s$, angular momentum with respect to space-fixed axes, and the other $O(3)$ to $\mathbf{J}_b$, angular momentum with respect to body-fixed axes. The vanishing of one of the invariant operators means here $\mathbf{J}_s^2=\mathbf{J}_b^2=j(j+1)$. The eigenvalues of the J_{s3} and J_{b3} are denoted by μ and m, respectively. The dipole transition operator is a vector $\mathbf{d}$ with three commuting components. Such a vector can be constructed in E_4 or even in the larger group $O(4,2)$, the conformal group. It is remarkable that the conformal group has an irreducible representation (again with one invariant operator being zero) which remains also irreducible for the group $O(4,1)$.[13] This particular representation consists precisely of all the states of the rotator. The bigger group contains in addition the transition operators. Now, for this simple case, since $\mathbf{d}$ is a vector with respect to space-fixed axis, only one of the $O(3)$ is important—that is, transition elements are independent of m but they depend on j and μ. Thus it is sufficient to consider the group E_3 (Euclidian group in three dimensions) consisting of $O(3)$ and the three translations $\mathbf{d}$. The *two invariants* are $\mathbf{d}^2$ and $\mathbf{d}\cdot\mathbf{J}=d\Lambda$ (Λ is the component of the angular momentum along the dipole moment axis); $\mathbf{d}^2$ is essentially the fundamental coupling constant e^2. Using the known representation of E_3,[14] we obtain immediately not only the nine well-known nonvanishing matrix elements[15]

$$\begin{aligned}&\langle j,\mu|\,d_3\,|j,\mu'\rangle=d_j{}^j\mu\delta_{\mu\mu'},\\ &\langle j,\mu|\,d_+\,|j+1,\mu-1\rangle\\ &\qquad=d_j{}^{j+1}[(j-\mu+2)(j-\mu+1)]^{1/2},\ \text{etc.}\end{aligned}\tag{1}$$

but also the reduced matrix elements[16]

$$d_j{}^j=\frac{d\Lambda}{j(j+1)},$$

$$d_{j-1}{}^j=d_j{}^{j-1}=\frac{d}{j}[(j-\Lambda)(j+\Lambda)/(2j-1)(2j+1)]^{1/2}.\tag{2}$$

Here Λ is the lowest value of j that occurs in the infinite-dimensional unitary representation of E_3. Thus, the dynamical group of the rotational levels *including* the electromagnetic dipole transitions can be taken to be E_4 [the contracted $O(4,1)$]. The coupling constant is specified by the invariants of the group which fit the representation.

In the case of the three-dimensional oscillator the dynamical group in the absence of interactions is[3] $SU(3,1)$ with the group of degeneracy $SU(3)$ [with one vanishing invariant operator] in which $O(3)$ is diagonalized (not SU_2). The representations of $SU(3)$ that occur are of the type $(n,0)$ with dimensions $\frac{1}{2}(n+1)(n+2)$. The electromagnetic dipole operator $\mathbf{d}$ can be constructed in the group $SU(3,1)$ as in the case of conformal group; it is a vector with respect to the diagonalized $O(3)$ subgroup. In the convenient spinor form of the representation with the basis $\xi_1{}^a\xi_2{}^b\xi_3{}^c$, $a+b+c+=n$, $\mathbf{d}$ transforms like ξ^j.[17] Thus, one obtains for the m dependence of the matrix elements $\langle nlm|\,d\,|n'l'm'\rangle$ the equations (1), plus, from the fact that the magnitude of $\mathbf{d}$ transform like the generators of the noncompact subgroup $SU(1,1)$, the reduced matrix elements

$$d_{n-1}{}^n=d_n{}^{n-1}=(1/\alpha)(\tfrac{1}{2}n)^{1/2}.\tag{3}$$

The case of the H atom is slightly pathological in that an additional dilatation operation occurs which causes transitions between any n and n'. Nevertheless, it is shown in a separate paper[18] that in the presence of the electromagnetic interactions the dynamical group of the hydrogen atom, $O(4,1)$, is extended to $O(4,2)$ or $O(5,1)$, the conformal type groups,[1] which also contain both the cases of discrete and continuous spectrum as different little groups. Thus, in all the practically important electromagnetic transitions the conformal group is the dynamical group which includes external interactions and provides a unified framework of electromagnetic interactions.[19]

The above calculations incorporate the usual approximations of atomic physics. Firstly, we have multipole transitions as well, in which case the external interaction

[12] Both O(4,1) and E_4 have a representation which has the same "content" as the states of the rigid rotator or the H atom. In fact, the first group suggested for the dynamical group of the H atom was E_4 (Ref. 1). E_4 is the contracted group from O(4,1).

[13] This fact has been noticed in Fock's coordinates by I. A. Malkin and V. I. Man'ko, JETP Pis'mav Redaktsiyn **2**, 230 (1966) [English transl.: JETP Letters, **2**, 146 (1966)]. These authors, however, do not discuss the external electromagnetic interactions.

[14] W. Pauli, *Continuous Groups in Quantum Mechanics*, (Ergebnisse der exakten Naturwissenschaften, 1965), Vol. 37. In this representation the O_3 subgroup is diagonalized.

[15] There is no need to write all nine of these well-known expressions; see, for example. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1958), p. 93.

[16] Reference 15, p. 295; H. Honl and F. London, Z. Physik **33**, 803 (1925).

[17] In the representation with creation and annihilation operators, the three generators $(a_i+a_i{}^*+b_i+b_i{}^*)$ transform like a vector with respect to the angular momentum $\mathbf{J}=i(\mathbf{a}\times\mathbf{a}^*+\mathbf{b}\times\mathbf{b}^*)$. The U_3 is generated by $\mathbf{J}$ and $M_{ij}=b_i{}^*b_k+b_k{}^*b_i-(a_i{}^*a_k+a_k{}^*a_i)$; $\mathbf{J}$, M_{ij}, $\mathbf{a}+\mathbf{a}^*+\mathbf{b}+\mathbf{b}^*$, and $\mathbf{a}-\mathbf{a}^*+\mathbf{b}-\mathbf{b}^*$, form the conformal-type algebra (Ref. 1, Sec. II). In our case $\mathbf{b}=0$ and $\mathbf{a}^*$ increase the quantum number n by one.

[18] A. O. Barut and H. Kleinert, following paper, Phys. Rev. **156**, 1541 (1967).

[19] In the above simple cases of rotation and oscillation, only a subgroup of the conformal type groups is relevant for dipole interactions.

operator is expected to transform like a group element $e^{\alpha d}$ in $\mathcal{G}$ thus have a *definite direction* in the group space. Secondly, we have to take recoil effects into account. This means that the states of the H atom, for example, must be labeled in addition to the quantum numbers $n, l, m, \cdots$, by the energy-momentum k. Then we will have to evaluate reduced matrix elements of the form $\langle k,j|d|k',j'\rangle$. Only if the spin-orbit effects can be neglected can the energy-momentum effects be taken into effect in a simple way by the space-time translation group. Otherwise, due to mass differences between the states, the Hilbert space of states forms a reducible representation of the Poincaré group with the various parts connected in the rest frame by the noncompact group G.[10] Alternatively, one can take the direction of the interaction in group space to be momentum-dependent in a definite way. This procedure is reasonable because only via external interactions does the system acquire energy-momentum, and it constitutes a new way of treating internal and space-time quantum numbers together. [*Note added in proof.* This program on the further development of the theory has been explicitly carried out in a series of papers now in the process of publication.]

III. APPLICATION TO HADRONS

The application of this approach to fundamental particle interactions is immediate. It is, in fact, the generalization to noncompact groups of what is being done for compact groups. We start from a representation of the noncompact group G giving all the baryon states, for example, $|N=1,J,I,YJ_3I_3\rangle$. Those strong, weak, and electromagnetic interactions which cause transitions among these states, for examples,

$$N^* \xrightarrow{(\pi)} N(\text{strong}), \quad \Sigma^0 \xrightarrow{(\gamma)} \Lambda(\text{electromagnetic}),$$

$$n \xrightarrow{(e\nu)} p(\text{weak}), \quad \Omega^- \xrightarrow{(K^-)} \Lambda(\text{weak}),$$

are characterized by definite operators in G (or in the corresponding $\mathcal{G}$, see above) whose matrix elements are the S-matrix elements of the reactions in the corresponding spin states. Again, in general the energy-momentum dependence of the states must also be specified to obtain form factors. Note that the existence of the external interactions is conceptually of fundamental importance to the use of groups in particle physics. For example, the $n \to p$ transition operation would violate the superselection rule on charge if one would perform this group operation of SU_2 without the external weak interaction (via the lepton pair $e\nu$), although quantitatively the weak interactions are negligible in the considerations of strong-interaction symmetries. Thus only the existence of weak and electromagnetic interactions makes the group operations of strong interactions physically meaningful.

The above approach and the approach via the algebra of currents attack the problem of particle interactions from opposite directions, and should therefore complement each other. In the latter case, one postulates commutation relations between current operators whose products are essentially the final S-matrix operators and associates the observed particles to an approximate zero-momentum representation of these commutation relations.[20] In the former case, one starts from the single-particle states as given by a group representation and introduces external interactions thereafter which are associated with the transition operators of the group. In both cases the transformation properties of the various physical interactions have to be specified in terms of currents or in terms of a group element.

Finally, because the choice of the dynamical group is not entirely unique,[21] we formulate the following requirement: The group and the physical states should be so chosen that the transition operators of external interactions transform like the group generators.

[20] R. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966).

[21] Because every Hilbert space is isomorphic to every other, one could in principle transform the representation space of one group to that of another. But then the physical quantities may take such a complicated form as to make the new group entirely useless.