$$d=-\frac{1}{m_\rho^2}\Big[\tfrac{1}{3}m-(1/3m_*^2)(m+m_*)(m_*^2+m^2-\mu^2)-(1/3m_*)(mm_*+m_*^2-\mu^2)+\frac{1}{6m_*}(m_*^2+m^2-\mu^2)$$
$$+\frac{(m_*^2-m^2-\mu^2)}{m_\rho^2-\mu^2}\{(m+m_*)(1-(1/3m_*^2)(m_*^2+m^2-\mu^2))-(1/3m_*)(mm_*+m_*^2-\mu^2)\}\Big], \quad (18)$$

$$e=\frac{1}{mm_\rho^2}\Big[\frac{(m_*^2-m^2-m_\rho^2)}{(m_\rho^2-\mu^2)}\Big\{\frac{1}{6m_*}(m+m_*)(m_*^2+m^2-\mu^2)-\tfrac{1}{3}(m_*^2+mm_*-\mu^2)\Big\}$$
$$-m^2+(1/6m_*^2)(m_*^2+m^2-m_\rho^2)(m_*^2+m^2-\mu^2-mm_*)\Big], \quad (19)$$

$$f=1/m_\rho^2. \quad (20)$$

We now use $g_{NN\pi}=\sqrt{15}$ and $|g_{N^*N\pi}|=3.12\ \text{BeV}^{-1}$.[10] The coupling constant $g_{NN\rho}^{(1)}$ is taken to be[11] equal to 4.52 and[12]

$$g_{NN\rho}^{(2)}/g_{NN\rho}^{(1)}\simeq-3.66. \quad (21)$$

From (13) and (14) we obtain

$$F_{N^*N\rho}^{(1)}=-2.84,\quad F_{N^*N\rho}^{(2)}=1.03. \quad (22)$$

These results should be compared with the $SU(6)$ values $F_{N^*N\rho}^{(1)}=-3.5$ and $F_{N^*N\rho}^{(2)}=1.5$.[13] The experimental analysis of Albright and Liu (Ref. 9) gives a large number of sets of values for $F_{N^*N\rho}^{(1)}$ and $F_{N^*N\rho}^{(2)}$ which are quite consistent with ours.

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[10] V. Gupta and V. Singh, Phys. Rev. **135**, B1442 (1964).
[11] The $g_{NN\rho}^{(1)}$ used here is equal to g_1+g_2 of A. Scotti and D. Y. Wong [Phys. Rev. **138**, B145 (1965)].
[12] This ratio is taken from the data on nucleon form factors. [See, for example, J. S. Ball and D. Y. Wong, Phys. Rev. **130**, 2112 (1963).]
[13] C. H. Albright and L. S. Liu, Phys. Rev. Letters **14**, 324 (1965); M. A. B. Bég, B. W. Lee, and A. Pais, *ibid.* **13**, 514 (1964).

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Dressed Particle States in Local Quantum Field Theory

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The approximate stationary quantum state which describes a dressed fermion elementary particle in a model theory of interacting local fields is reported. Showing no dependence on the infinite, positive, bare-mass constant, a finite, positive, dressed fermion mass value is obtainable.

IN contrast to the quantum-theoretic models studied heretofore, the essentially nonlinear field theory analyzed in this paper by means of a nonperturbative method of solution admits the possibility of a finite, positive, dressed fermion mass value, independent of the infinite, positive, bare-mass constant. There is a similarity between this work and that of Nelson,[1] with the fermion field operator treated here in a patently nonrelativistic fashion.

Our purely local model theory is based on the Lorentz-invariant Lagrangian density

$$\mathcal{L}=-\tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi+\psi^\dagger(i\sigma^\mu\partial_\mu-m_0)\psi+\lambda\phi^2\psi^\dagger\psi, \quad (1)$$

with ϕ a real scalar field, ψ a two-component complex Weyl spinor field, m_0 a positive bare-mass constant, and λ a positive coupling constant. From (1) we obtain the associated Hamiltonian operator

$$H=\int\Big\{-\frac{1}{2}\frac{\delta^2}{\delta\phi(\mathbf{x})^2}+\tfrac{1}{2}|\nabla\phi(\mathbf{x})|^2-i\psi^\dagger(\mathbf{x})\sigma\cdot\nabla\psi(\mathbf{x})$$
$$+[m_0-\lambda\phi(\mathbf{x})^2]\psi^\dagger(\mathbf{x})\psi(\mathbf{x})\Big\}d^3x \quad (2)$$

in which the boson field operator $\phi(\mathbf{x})$ is represented in diagonalized form and the fermion field operator $\psi(\mathbf{x})$ satisfies the anticommutation relations $\{\psi(\mathbf{x}),\psi(\mathbf{x}')\}=0$, $\{\psi(\mathbf{x}),\psi^\dagger(\mathbf{x}')\}=\delta(\mathbf{x}-\mathbf{x}')$.

[1] E. Nelson, J. Math. Phys. **5**, 1190 (1964).

The nonrelativistic approximation to the vacuum state, following from the specific ordering in (2), is

$$|\text{vac}\rangle=\left\{\exp\left[-\frac{1}{2}\int\int\phi(\mathbf{x})f_{\text{vac}}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})d^3xd^3y\right]\right\}\times\Omega_{\text{vac}}, \quad (3)$$

where Ω_{vac} denotes the nonrelativistic fermion vacuum, $\psi(x)\Omega_{\text{vac}}\equiv 0$ with $\Omega_{\text{vac}}{}^\dagger\Omega_{\text{vac}}=1$, and $f_{\text{vac}}(\mathbf{x},\mathbf{y})=(-\nabla_{\mathbf{x}}{}^2)^{1/2}\times\delta(\mathbf{x}-\mathbf{y})$ is the real symmetric distribution inverse to

$$g_{\text{vac}}(\mathbf{x},\mathbf{y})=\frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\frac{d^3k}{|\mathbf{k}|}=\lim_{K\to\infty}(1-\cos K|\mathbf{x}-\mathbf{y}|)/2\pi^2|\mathbf{x}-\mathbf{y}|^2, \quad (4)$$

the wave-number integration being restricted to $|\mathbf{k}|\leqslant K$, a cutoff constant. Normalized with respect to an appropriate displacement-invariant measure for the inner product functional integration over all fields ϕ, the vacuum state (3) is such that the associated *energy functionality*[2]

$$E_{\text{vac}}\equiv\langle\text{vac}|H|\text{vac}\rangle/\langle\text{vac}|\text{vac}\rangle=\int\{\tfrac{1}{4}f_{\text{vac}}(\mathbf{x},\mathbf{x})+\tfrac{1}{4}\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}g_{\text{vac}}(\mathbf{x},\mathbf{y})|_{\mathbf{y}=\mathbf{x}}\}d^3x \quad (5)$$

satisfies the Rayleigh-Ritz equation $\delta E_{\text{vac}}/\delta f_{\text{vac}}=0$, an integral equation which yields the expression (4). Finally, the vacuum state energy can be obtained by evaluating (5),

$$E_{\text{vac}}=\lim_{K\to\infty}\int\frac{K^4}{16\pi^2}d^3x.$$

Let us seek a physical one-fermion stationary state, a simultaneous eigenstate of the Hamiltonian (2) and the fermion number operator $\int\psi^\dagger(\mathbf{x})\psi(\mathbf{x})d^3x$ (eigenvalues E_{one} and unity, respectively) with the approximate form

$$|\text{one}\rangle=N^{-1}\left[\exp\left(-\frac{1}{2}\int\int[\phi(\mathbf{x})-\phi_1(\mathbf{x})]\times f(\mathbf{x},\mathbf{y})[\phi(\mathbf{y})-\phi_1(\mathbf{y})]d^3xd^3y\right)\right]\times\int\psi^\dagger(\mathbf{x})\psi_1(\mathbf{x})d^3x\Omega_{\text{vac}}, \quad (6)$$

where $\phi_1(\mathbf{x})$, $\psi_1(\mathbf{x})$ (c-number functions) and $f(\mathbf{x},\mathbf{y})=f(\mathbf{y},\mathbf{x})$ (a c-number distribution) are to be determined by the Rayleigh–Ritz procedure and N is a normalization constant. The *energy functionality* associated with (2) and (6) is defined and evaluated by functional integration[2] as

$$E_{\text{one}}\equiv\langle\text{one}|H|\text{one}\rangle/\langle\text{one}|\text{one}\rangle=\int\{\tfrac{1}{4}f(\mathbf{x},\mathbf{x})+\tfrac{1}{4}\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}g(\mathbf{x},\mathbf{y})|_{\mathbf{y}=\mathbf{x}}+\tfrac{1}{2}|\nabla\phi_1|^2-i\psi_1{}^\dagger\boldsymbol{\sigma}\cdot\nabla\psi_1+[m_0-\lambda\phi_1{}^2-\tfrac{1}{2}\lambda g(\mathbf{x},\mathbf{x})]\psi_1{}^\dagger\psi_1\}d^3x \quad (7)$$

in which $g(\mathbf{x},\mathbf{y})$ is the real symmetric distribution inverse to $f(\mathbf{x},\mathbf{y})$,

$$\int f(\mathbf{x},\mathbf{z})g(\mathbf{z},\mathbf{y})d^3z\equiv\delta(\mathbf{x}-\mathbf{y}). \quad (8)$$

From (7) we derive the Rayleigh-Ritz equations

$$\frac{\delta E_{\text{one}}}{\delta\phi_1}=-\nabla^2\phi_1-2\lambda\psi_1{}^\dagger\psi_1\phi_1=0, \quad (9)$$

$$\frac{\delta E_{\text{one}}}{\delta\psi_1{}^\dagger}=-i\boldsymbol{\sigma}\cdot\nabla\psi_1+[m_0-\lambda\phi_1{}^2-\tfrac{1}{2}\lambda g(\mathbf{x},\mathbf{x})]\psi_1=0, \quad (10)$$

$$\frac{\delta E_{\text{one}}}{\delta f}=\tfrac{1}{4}\delta(\mathbf{x}-\mathbf{y})+\frac{1}{4}\int g(\mathbf{x},\mathbf{z})[\nabla_{\mathbf{z}}{}^2+2\lambda\psi_1{}^\dagger(\mathbf{z})\psi_1(\mathbf{z})]\times g(\mathbf{z},\mathbf{y})d^3z=0. \quad (11)$$

It follows immediately from (11) that the singular character of $g(\mathbf{x},\mathbf{y})$ as $\mathbf{y}\to\mathbf{x}$ is identical to the singular character of $g_{\text{vac}}(\mathbf{x},\mathbf{y})$ as $\mathbf{y}\to\mathbf{x}$, $g(\mathbf{x},\mathbf{x})=g_{\text{vac}}(\mathbf{x},\mathbf{x})=\lim_{K\to\infty}(K^2/4\pi^2)$ by (4). Hence, in order to have a meaningful c-number equation (10), we put

$$m_0\equiv\tfrac{1}{2}\lambda g(\mathbf{x},\mathbf{x})=\lim_{K\to\infty}(\lambda K^2/8\pi^2). \quad (12)$$

Then the singularity-free spherically symmetric "particle-like" solution to the coupled c-number equations[3] (9) and (10) is obtainable in closed form and given exactly by[4]

$$\phi_1=\pm(3a/\lambda)^{1/2}(|\mathbf{x}|^2+a^2)^{-1/2}, \quad (13)$$

$$\psi_1=(3/2\lambda)^{1/2}a(|\mathbf{x}|^2+a^2)^{-3/2}(i\sigma\cdot\mathbf{x}+a)u, \quad (14)$$

with a denoting a free positive constant of integration, a so-called "homology constant" stemming from the scale (dilatation) invariance of Eqs. (9) and (10) with the condition (12); in (14) u denotes a constant Weyl spinor of unit length, $u^\dagger u=1$. The final Rayleigh-Ritz equation (11) with (14) is

$$-\int g(\mathbf{x},\mathbf{z})[\nabla_{\mathbf{z}}{}^2+3a^2(|\mathbf{z}|^2+a^2)^{-2}]g(\mathbf{z},\mathbf{y})d^3z=\delta(\mathbf{x}-\mathbf{y}), \quad (15)$$

[2] G. Rosen, Phys. Rev. Letters **16**, 704 (1966). In symbolic notation, the appropriate displacement-invariant measure is

$$\mathfrak{D}(\phi)=\prod_{\mathbf{x}}\left(\frac{d^3x}{\pi}\right)^{1/2}\delta[(-\nabla_{\mathbf{x}}{}^2)^{1/4}\phi(\mathbf{x})],$$

and the squared-norm of (3) takes the form

$$\langle\text{vac}|\text{vac}\rangle=\int\left\{\exp\left[-\int[(-\nabla_{\mathbf{x}}{}^2)^{1/4}\phi(\mathbf{x})]^2d^3x\right]\right\}\mathfrak{D}(\phi)\Omega_{\text{vac}}{}^\dagger\Omega_{\text{vac}}=1.$$

For a discussion of this measure $\mathfrak{D}(\phi)$, see: K. O. Friedrichs *et al.*, New York University Institute of Mathematical Sciences Report, 1957 (unpublished), p. II-9.

[3] Note that these are just the classical field equations derived from (1) with $\phi=\phi_1(\mathbf{x})$ and $\psi=e^{-im_0t}\psi_1(\mathbf{x})$.

[4] G. Rosen, J. Math. Phys. **7**, 2066 (1966).

or equivalently by (8),

$$-[\nabla_{\mathbf{x}}^2+3a^2(|\mathbf{x}|^2+a^2)^{-2}]\delta(\mathbf{x}-\mathbf{y}) = \int f(\mathbf{x},\mathbf{z})f(\mathbf{z},\mathbf{y})d^3z. \quad (16)$$

Introducing the Fourier transforms

$$f(\mathbf{x},\mathbf{y})=f_{\text{vac}}(\mathbf{x},\mathbf{y})+\frac{1}{(2\pi)^3}\int \alpha(\mathbf{k},\mathbf{k}')e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{y}}d^3kd^3k', \quad (17)$$

$$g(\mathbf{x},\mathbf{y})=g_{\text{vac}}(\mathbf{x},\mathbf{y})+\frac{1}{(2\pi)^3}\int \frac{\beta(\mathbf{k},\mathbf{k}')}{|\mathbf{k}||\mathbf{k}'|} \times e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{y}}d^3kd^3k', \quad (18)$$

we see that the equations obtained from (16) and (8)

$$(|\mathbf{k}|+|\mathbf{k}'|)\alpha(\mathbf{k},\mathbf{k}')+\frac{3a}{8\pi}e^{-a|\mathbf{k}-\mathbf{k}'|} = -\int \alpha(\mathbf{k},\mathbf{k}'')\alpha(\mathbf{k}'',\mathbf{k}')d^3k'', \quad (19)$$

$$\alpha(\mathbf{k},\mathbf{k}')+\beta(\mathbf{k},\mathbf{k}')=-\int \alpha(\mathbf{k},\mathbf{k}'')\beta(\mathbf{k}'',\mathbf{k}')\frac{d^3k''}{|\mathbf{k}''|} \quad (20)$$

can be solved by an obvious iteration procedure, the first approximation

$$\alpha(\mathbf{k},\mathbf{k}')\cong-\beta(\mathbf{k},\mathbf{k}')\cong-\frac{3a}{8\pi}\frac{e^{-a|\mathbf{k}-\mathbf{k}'|}}{(|\mathbf{k}|+|\mathbf{k}'|)} \quad (21)$$

being asymptotic to the exact solution for both $a|\mathbf{k}|$, $a|\mathbf{k}'|\gg1$.[5] The energy of the physical one-fermion stationary state can be expressed by evaluating (7) with (12), (13), (14), (17), and (18):

$$E_{\text{one}}=E_{\text{vac}}+\frac{1}{4}\int[\alpha(\mathbf{k},\mathbf{k})+\beta(\mathbf{k},\mathbf{k})]d^3k+\frac{9\pi^2}{8\lambda}, \quad (22)$$

and so the dressed fermion mass value is

$$m\equiv E_{\text{one}}-E_{\text{vac}}=\frac{9\pi^2}{8\lambda}-\frac{1}{4}\int \alpha(\mathbf{k},\mathbf{k}')\beta(\mathbf{k}',\mathbf{k})\frac{d^3k'}{|\mathbf{k}'|}d^3k \quad (23)$$

in view of (20). Making use of the approximation (21), we finally obtain

$$m=\frac{9\pi^2}{8\lambda}+\left(\frac{3}{16}\right)^2\lim_{K\to\infty} a^{-1}\ln(aK) \quad (24)$$

for $(aK)\gg1$. Since the first approximation (21) is asymptotic to the exact solution of Eqs. (19) and (20), Eq. (24) is an exact consequence of the Rayleigh-Ritz approximation theory.

Now all of the equations in the theory remain perfectly regular in the limit $a\to\infty$, that is, as the homology constant a increases without bound.[6] In view of the fact that the homology constant is a free parameter in the theory, the result (24) can be made finite by requiring a to manifest a suitable dependence on the cutoff constant K so that $a\to\infty$ as $K\to\infty$, say by putting $a\equiv\lambda^2K$. With such an $a=a(K)$, the finite, positive, dressed fermion mass value $m=9\pi^2/8\lambda$ is obtained without ambiguity, independent of the infinite, positive, bare-mass constant (12).[7]

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[5] Formally more direct but difficult to justify with mathematical rigor, an alternative method for solving (15) or (16) can be based on the fact that the effective Schrödinger potential $-3a^2(|\mathbf{x}|^2+a^2)^{-2}$ is a smooth singularity-free function which admits no "bound states," the Schrödinger operator $-[\nabla_{\mathbf{x}}^2+3a^2(|\mathbf{x}|^2+a^2)^{-2}]$ having no eigenfunction with a negative eigenvalue in the space of bounded C^1 piecewise C^2 functions of $\mathbf{x}$, the nodeless eigenfunction $(|\mathbf{x}|^2+a^2)^{-1/2}$ possessing the eigenvalue zero. Thus we have the WKB approximations

$$f(\mathbf{x},\mathbf{y})\cong\frac{1}{(2\pi)^3}\int_{|\mathbf{k}|\geqslant\mu(\mathbf{x},\mathbf{y})}[|\mathbf{k}|^2-\mu(\mathbf{x},\mathbf{y})^2]^{1/2}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}d^3k,$$

$$\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}g(\mathbf{x},\mathbf{y})\cong\frac{1}{(2\pi)^3}\int_{|\mathbf{k}|\geqslant\mu(\mathbf{x},\mathbf{y})}|\mathbf{k}|^2[|\mathbf{k}|^2-\mu(\mathbf{x},\mathbf{y})^2]^{-1/2} \times e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}d^3k,$$

where $\mu(\mathbf{x},\mathbf{y})=\mu(\mathbf{y},\mathbf{x})$ is a certain non-negative real function such that $\mu(\mathbf{x},\mathbf{x})^2=3a^2(|\mathbf{x}|^2+a^2)^{-2}$. Substituting these WKB approximations into (7), we obtain

$$E_{\text{one}}=\frac{1}{4}\int\left[\frac{1}{(2\pi)^3}\int_{|\mathbf{k}|\geqslant\mu(\mathbf{x},\mathbf{x})}\{[|\mathbf{k}|^2-\mu(\mathbf{x},\mathbf{x})^2]^{1/2} +|\mathbf{k}|^2[|\mathbf{k}|^2-\mu(\mathbf{x},\mathbf{x})^2]^{-1/2}\}d^3k\right]d^3x+\frac{9\pi^2}{8\lambda}$$
$$=\lim_{K\to\infty}\int\left[\frac{K^4}{16\pi^2}+\frac{\mu(\mathbf{x},\mathbf{x})^4}{32\pi^2}\ln(2K/\mu(\mathbf{x},\mathbf{x}))\right]d^3x+\frac{9\pi^2}{8\lambda}$$
$$=E_{\text{vac}}+\left(\frac{3}{16}\right)^2\lim_{K\to\infty}a^{-1}\ln(aK)+\frac{9\pi^2}{8\lambda}$$

which agrees with (24).

[6] Moreover, a straightforward integration calculation with (6) and (14) shows that the expectation value of the total angular momentum

$$\mathbf{s}=\langle\text{one}|\int\left[\frac{1}{2}\left\{\mathbf{r}\times\nabla\phi, i\frac{\delta}{\delta\phi}\right\}+\psi^\dagger(-i\mathbf{r}\times\nabla+\tfrac{1}{2}\boldsymbol{\sigma})\psi\right] \times d^3r|\text{one}\rangle/\langle\text{one}|\text{one}\rangle$$
$$=\int\psi_1^\dagger(-i\mathbf{r}\times\nabla+\tfrac{1}{2}\boldsymbol{\sigma})\psi_1d^3r\Big/\int\psi_1^\dagger\psi_1d^3r=\tfrac{1}{2}u^\dagger\boldsymbol{\sigma}u,$$

entirely independent of the value assigned to a, is generally consistent for a spin-$\frac{1}{2}$ particle.

[7] Of course the strong divergence of the normalization constant N in (6) is a hard-set feature of the theory. With the appropriate displacement-invariant measure (Ref. 2) and the relation obtained from (14) by ordinary integration

$$\int\psi_1^\dagger(\mathbf{x})\psi_1(\mathbf{x})d^3x=\frac{3\pi^2a}{2\lambda},$$

the normalization condition $\langle\text{one}|\text{one}\rangle=1$ produces

$$|N|^2=(3\pi^2a/2\lambda)\mathfrak{F},$$

where

$$\mathfrak{F}=\int\left\{\exp\left[-\int\int\omega(\mathbf{x})h(\mathbf{x},\mathbf{y})\omega(\mathbf{y})d^3xd^3y\right]\right\} \times\prod_{\mathbf{x}}\{\exp[-\omega(\mathbf{x})^2d^3x]\}\left(\frac{d^3x}{\pi}\right)^{1/2}\delta\omega(\mathbf{x}),$$

$$h(\mathbf{x},\mathbf{y})\equiv(-\nabla_{\mathbf{x}}^2)^{-1/4}(-\nabla_{\mathbf{y}}^2)^{-1/4}[f(\mathbf{x},\mathbf{y})-f_{\text{vac}}(\mathbf{x},\mathbf{y})].$$

Now from the approximate solution (21) with (17) it follows that all iterated kernels constructed from $h(\mathbf{x},\mathbf{y})$ [i.e., $\int h(\mathbf{x},\mathbf{z})h(\mathbf{z},\mathbf{y})d^3z$, etc.] are regular as $\mathbf{y}\to\mathbf{x}$, and thus the dominant (divergent) terms in $\mathfrak{F}$ are simply the powers of $\int h(\mathbf{x},\mathbf{x})d^3x$ as $K\to\infty$; hence we have

$$\mathfrak{F}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{2}\int h(\mathbf{x},\mathbf{x})d^3x\right)^n =\lim_{K\to\infty}e^{3aK/8}.$$