Reciprocal Bootstrap of the N and N^* Using the Static-Model Bethe-Salpeter Equation*

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The Bethe-Salpeter equation is written down for the static model of πN scattering. This equation satisfies unitarity exactly in the elastic region, but includes at least some inelasticity at higher energies. It is solved below the inelastic threshold by using the Noyes technique, and then making a Pagels-type approximation. It is applied to the reciprocal bootstrap problem of Chew, where N^* exchange is assumed to provide the dominant force for binding the N, and N exchange for giving rise to the N^*. Experimental values are used for the crossed-channel masses and couplings. The cutoff is adjusted to give the correct position for the direct-channel nucleon pole. This gives an output N^* mass and residues which are in rough agreement with experiment.

I. INTRODUCTION

THE simplest model for P-wave πN scattering is the static limit of the N/D method, in which the D function is approximated by a straight line. This model was used by Chew[1] to argue that the binding of the N is due primarily to N^* exchange, while that of the N^* is a result of N exchange. Since then, numerous other N/D reciprocal bootstrap calculations have been made,[2] which tend to support the notion that the Chew analysis is at least roughly valid, even though other effects may also be quite important.[3,4]

Recently, it has been proposed by several authors that the Bethe-Salpeter equation[5] may have certain advantages over N/D methods in studying strong-interaction dynamics.[6–8] It shares with the N/D equations the features that it satisfies two-body unitarity in the elastic region, and reproduces the nearby singularities correctly. It can thus be thought of as another way of unitarizing exchange graphs. On the other hand, it does include at least some inelastic effects and satisfies the Mandelstam representation. It would therefore be interesting to see how it compares with the more usual techniques.

A complete relativistic solution of the Bethe-Salpeter equation is not likely to be particularly simple.[7] This led us to the static model, where the equation simplifies considerably. Such an equation has already been considered by several authors,[8,9] who did not, however, use it to make actual calculations in bootstrap situations. In the following sections we shall apply it to the N,N^* reciprocal bootstrap.

II. THE BETHE-SALPETER EQUATION IN THE STATIC LIMIT

Consider P-wave πN scattering. A given state (I,J) is characterized by its spin J, isotopic spin I, and pion energy ω. We shall take the pion mass$=1$, so that its momentum q is given by $q^2=\omega^2-1$. The nucleon mass m is considered to be so heavy that its recoil can be neglected and the P wave can be decoupled from all other orbital angular momenta. Suppose we normalize the off-shell T matrix $T_{IJ}(\omega',\omega,E)$ so that it reduces to $e^{i\delta}\sin\delta$ when we go on-shell by setting $E=m+\omega$ and $\omega'=\omega$; here δ is the phase shift in the (I,J) state. If we exchange a baryon N^x of mass m_x, spin J', and isotopic spin I', then the Born term (Fig. 1) is[8]

$$T_B(\omega',\omega,E)=f(\omega')f(\omega)/(m_x+\omega+\omega'-E)\,, \qquad (1)$$

where $f(\omega)$ is a cutoff function containing the coupling constant and kinematic factors.

If we go on the shell, we know that Fig. 1 gives[1]

$$T_B(\omega,\,\omega,\,m+\omega)=q^3\gamma^x/(m_x-m+\omega)\,, \qquad (2)$$

where $-\gamma^x$ is the residue of the pseudopole due to N^x exchange in the on-shell amplitude

$$g_{IJ}(\omega)=e^{i\delta}(\sin\delta)/q^3\,. \qquad (3)$$

It is related to the residue of the direct-channel pole in the (I',J') state through crossing

$$\gamma_{IJ}{}^x=\alpha_{II'}\beta_{JJ'}\gamma_{I'J'}\,, \qquad (4)$$

where α and β are the crossing matrices for isotopic spins and spin, respectively. In the πN problem, α and β

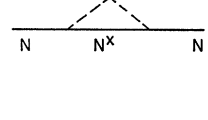

FIG. 1. The Born graph due to the exchange of the baryon N^x.

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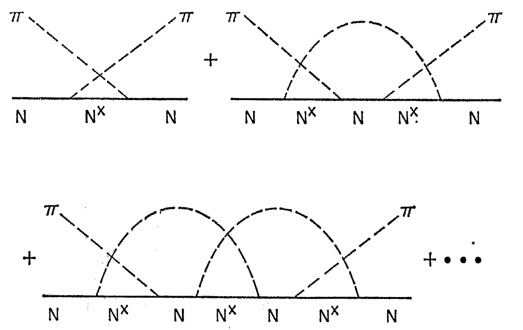

FIG. 2. The ladder graphs.

are the same

$$\alpha=\beta=\begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}. \tag{5}$$

Now the extrapolation of Eq. (2) off the shell is always a somewhat ambiguous procedure. We therefore took the simplest possible extrapolation, which is simply to set

$$f(\omega)=(q^3\gamma^x)^{1/2}\theta(\Lambda-\omega) \tag{6}$$

in Eq. (1). The θ function introduces a straight cutoff at $\omega=\Lambda$, and is thus a rough way of incorporating high-energy effects.

The Bethe-Salpeter equation is equivalent to summing the ladder graphs of Fig. 2 and has the form[8]

$$T(\omega',\omega,E)=f(\omega')f(\omega)/(m_x+\omega+\omega'-E) -\frac{1}{\pi}\int_1^\infty d\omega''\frac{f(\omega')f(\omega'')}{m_x+\omega'+\omega''-E}\frac{T(\omega'',\omega,E)}{E-\omega''-m}, \tag{7}$$

where we have suppressed all I, J subscripts. We have also neglected any self-energy and vertex modifications. Moreover, our Born term consists only of N^x exchange, although it would be straightforward to include the exchange of additional particles. Actually, Eq. (7) is too general for our purposes, since we shall ultimately be interested only in the on-shell amplitude. We therefore set $E=m+\omega$ from now on. Then for $\omega<\Lambda$ and $\omega'<\Lambda$, Eq. (7) becomes

$$\phi(\omega',\omega)=\frac{\gamma^x}{\Delta+\omega'}+\frac{1}{\pi}\int_1^\Lambda d\omega''\frac{q''^3\gamma^x}{\Delta+\omega'+\omega''-\omega}\frac{\phi(\omega'',\omega)}{\omega''-\omega}, \tag{8}$$

where $\Delta=m_x-m$ and

$$\phi(\omega',\omega)=T(\omega',\omega,m+\omega)/(q'q)^{3/2}, \tag{9}$$

so that $\phi(\omega,\omega)=g(\omega)$.

Equation (6) is a singular integral equation, which has to be solved for each value of ω. It would be convenient to reduce it to a form which is at least nonsingular in certain regions. This can be done by following a procedure first suggested by Noyes.[10] We write

$$\phi(\omega',\omega)=f(\omega',\omega)g(\omega), \tag{10}$$

which means that $f(\omega,\omega)=1$. From Eqs. (8) and (10) we obtain for the amplitude

$$g(\omega)=[\gamma^x/(\Delta+\omega)]d^{-1}(\omega), \tag{11}$$

with

$$d(\omega)=1-\frac{1}{\pi}\int_1^\Lambda d\omega''\frac{q''^3}{\omega''-\omega}\frac{\gamma^x}{\Delta+\omega''}f(\omega'',\omega), \tag{12}$$

where $f(\omega',\omega)$ is obtained by solving the integral equation

$$f(\omega',\omega)=\frac{\Delta+\omega}{\Delta+\omega'}+\frac{\gamma^x}{\pi}\int_1^\Lambda d\omega''\frac{q''^3}{\omega''-\omega} \times\left[\frac{1}{\Delta+\omega''+\omega'-\omega}-\frac{\Delta+\omega}{\Delta+\omega'}\frac{1}{\Delta+\omega''}\right]f(\omega'',\omega). \tag{13}$$

This equation is nonsingular in the elastic region. It becomes singular for $\omega>\Delta_x+2$, the point at which the Bethe-Salpeter equation begins to pick up inelastic contributions.

From Eq. (11) we see that a resonance or bound state will occur at $\omega=\omega_R$ if

$$\mathrm{Re}\,d(\omega_R)=0. \tag{14}$$

The residue of the corresponding pole in $g(\omega)$ will then be

$$\gamma=-[\gamma^x/(\Delta+\omega_R)]/[\mathrm{Re}\,d'(\omega_R)]. \tag{15}$$

In the $(\frac{1}{2},\frac{1}{2})$ state, Chew's reciprocal bootstrap[1] suggests that N^* exchange should be sufficient to produce the nucleon as a bound state. Equations (14) and (15) will then give its mass and residue. Since this mass must be the same as that of the external nucleon, we must have $\omega_R=0$ for consistency. We therefore vary the cutoff Λ until this condition is satisfied. We can then use the same value of Λ in computing ω_R and γ for the N^* in the $(\frac{3}{2},\frac{3}{2})$ state with N exchange. To keep the calculations as simple as possible, experimental values were taken for the exchanged mass and residue in both cases. Thus $\Delta=2.15$ for N^* exchange, and $\Delta=0$ for N exchange. For $\gamma_{3/2,3/2}$ we use the Chew-Low prediction[11] $\gamma_{3/2,3/2}=\frac{1}{2}\gamma_{1/2,1/2}$, which agrees with experiment, while $\gamma_{1/2,1/2}=3f^2=0.24$, where f is the pseudovector πN coupling constant.

III. THE PAGELS APPROXIMATION

Equation (13) can be solved quite simply by using an approximation similar to the one proposed for the N/D equations by Pagels.[12] We shall continue to restrict ourselves to the single-particle-exchange one-channel

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case, although it would be straightforward to extend the approximation to more complicated potentials in multichannel problems. To see how it arises and what its limitations are, we first note that for $\omega<\Delta+2$, $f(\omega'',\omega)$ is nonsingular for $\omega''>1$; physically this means that there is no inelasticity for $\omega<\Delta+2$, as can be seen from Eqs. (11) and (12).[13] From Eq. (13) it then follows that the only singularities of $f(\omega',\omega)$ in the variable ω' are for $\omega'<\omega'_{\max}=\max\{(\omega-\Delta-1),-\Delta\}$. In fact, we can always write the function f as

$$f(\omega',\omega)=\frac{1}{\pi}\int_{\omega'_{\min}}^{\omega'_{\max}}dy\frac{h(y,\omega)}{y-\omega'}, \tag{16}$$

where $\omega'_{\min}=\min\{(\omega-\Delta-\Lambda),-\Delta\}$. The weight function $h(y,\omega)$ is not necessarily smoothly varying; it may include delta functions in y, for instance. Substituting Eq. (16) into Eq. (13), we obtain

$$f(\omega',\omega)=\frac{\Delta+\omega}{\Delta+\omega'}-\frac{\gamma^x}{\pi^2}\int_{\omega'_{\min}}^{\omega'_{\max}}dyK(y,\omega',\omega)h(y,\omega), \tag{17}$$

where

$$K(y,\omega',\omega)=\int_1^\Lambda d\omega''\frac{q''^3}{(\omega''-y)(\omega''-\omega)}\times\left[\frac{1}{\omega''+\Delta+\omega'-\omega}-\frac{\Delta+\omega}{\Delta+\omega'}\frac{1}{\omega''+\Delta}\right] \tag{18}$$

$$=\frac{I(y)}{y-\omega}\left[\frac{1}{y+\Delta+\omega'-\omega}-\frac{\Delta+\omega}{\Delta+\omega'}\frac{1}{y+\Delta}\right]-\frac{I(\omega-\Delta-\omega')}{(\omega-\Delta-\omega'-y)(\Delta+\omega')}-\frac{\Delta+\omega}{\Delta+\omega'}\frac{I(-\Delta)}{(\Delta+y)(\Delta+\omega)}, \tag{19}$$

with

$$I(x)=\int_1^\Lambda d\omega''\frac{q''^3}{\omega''-x}. \tag{20}$$

If we take the discontinuity of $f(\omega',\omega)$ in Eq. (17) with respect to the ω' variable, we obtain an integral equation for $h(y,\omega)$. In principle, we could always solve this equation instead of Eq. (13), and then find $f(\omega',\omega)$ from Eq. (16). As long as $\omega<\Delta+2$, however, we see from Eq. (19) that we only need $I(x)$ for $x<1$, i.e., in a nonsingular region. Thus we can always make the approximation

$$I(x)\simeq\sum_{i=1}^n\frac{c_i}{a_i-x}, \tag{21}$$

with $1<a_i<\Lambda$. The constants c_i and a_i are adjusted so that Eq. (21) is a good approximation for $\omega'_{\min}<x<\omega'_{\max}$, the only region where $I(x)$ is needed for solving our integral equation and obtaining the amplitude.

[13] Since the inelasticity probably sets in very slowly above its threshold, it should be reasonable in practice to raise the limit $\Delta+2$ to a much higher value in the discussion which follows.

Now Eq. (21) is equivalent to setting

$$q''^3=\sum_{i=1}^n c_i\delta(\omega''-a_i) \tag{22}$$

in Eq. (20), and hence in Eqs. (18) and (13). The latter equation thus becomes in this approximation

$$f(\omega',\omega)=\frac{\Delta+\omega}{\Delta+\omega'}+\frac{\gamma^x}{\pi}\sum_{i=1}^n\frac{c_i}{a_i-\omega}\times\left[\frac{1}{\Delta+a_i+\omega'-\omega}-\frac{\Delta+\omega}{\Delta+\omega'}\frac{1}{\Delta+a_i}\right]f(a_i,\omega). \tag{23}$$

If we set $\omega'=a_i$ for $i=1,\cdots,n$, we have n linear equations which we can solve for the $f(a_i,\omega)$. These can then be substituted back into Eq. (23) to give $f(\omega',\omega)$ for any ω'.

Once we have $f(\omega',\omega)$, we can find $d(\omega)$ from Eq. (12). Here again, however, if we substitute Eq. (16) into Eq. (12) we find that we can express the integral in terms of the function $I(x)$. But this time $I(x)$ is needed for $x=\omega$ as well as for $\omega'_{\min}<x<\omega'_{\max}$. Thus we can only make the approximation (21) if $\omega<1$, since otherwise we would be making a pole approximation for $I(x)$ in a region where it is singular. Since our approximation is equivalent to Eq. (22), we obtain from Eq. (13)

$$d(\omega)=1-\frac{1}{\pi}\sum_{i=1}^n\frac{c_i}{a_i-\omega}\frac{\gamma^x}{\Delta+a_i}f(a_i,\omega). \tag{24}$$

We shall use Eq. (24) in the $(\frac{1}{2},\frac{1}{2})$ state since we are only interested in the region near $\omega=0$ there.

In the $(\frac{3}{2},\frac{3}{2})$ state, however, we need $d(\omega)$ for $\omega>1$. It turns out that we can still simplify Eq. (12) provided that we rewrite it as

$$d(\omega)=1-\frac{\gamma^x}{\pi}\frac{I(\omega)}{\Delta+\omega}-\frac{\gamma^x}{\pi}\times\int_1^\Lambda d\omega''\frac{q''^3}{\omega''-\omega}\left[\frac{f(\omega'',\omega)}{\Delta+\omega''}-\frac{1}{\Delta+\omega}\right]. \tag{25}$$

Since $f(\omega,\omega)=1$, the last term is now nonsingular for $\omega<\Delta+2$. If we again use Eq. (16), we find that this term can be expressed in terms of $I(x)$, with x needed only in the region $\omega'_{\min}<x<\omega'_{\max}$, just as it was for $f(\omega',\omega)$. Thus we can make the approximation (22) to obtain

$$d(\omega)=1-\frac{\gamma^x}{\pi}\frac{I(\omega)}{\Delta+\omega}-\frac{\gamma^x}{\pi}\sum_{i=1}^n\frac{c_i}{a_i-\omega}\left[\frac{f(a_i,\omega)}{\Delta+a_i}-\frac{1}{\Delta+\omega}\right]. \tag{26}$$

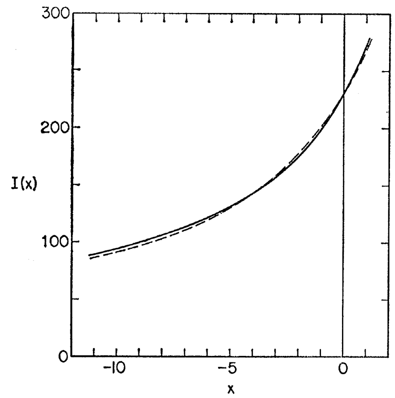

FIG. 3. Plot of $I(x)$ as given by Eq. (27) for $\Lambda=8.9$ (full line), compared with the pole approximation (16) when $n=1$, $c_1=1518$ and $a_1=6.67$ (dashed line).

We shall use Eq. (26) in calculating the properties of the N^*.

In practice, we just took $n=1$ in Eq. (16). The constants c_1 and a_1 were adjusted to reproduce the exact values of $I(x)$ at $x=0$ and $x=-4$, as given by the integral (20). For $\Lambda^2\gg 1$, this integral reduces to

$$I(x)=\tfrac{1}{3}\Lambda^3+\tfrac{1}{2}x[\Lambda^2-\ln 2\Lambda] +(x^2-1)[\Lambda+x\ln 2\Lambda-J(x)], \quad (27)$$

where

$$J(x)=(x^2-1)^{1/2}\ln\{(\Lambda[(1-1/2\Lambda^2)\times(x^2-1)^{1/2}+x]-1)/(x-\Lambda)\}, \quad \text{if } 0<x<1$$
$$=(1-x^2)^{1/2}\tan^{-1}\{\Lambda(1-1/2\Lambda^2)\times(1-x^2)^{1/2}/(1-x\Lambda)\}, \quad \text{otherwise.}$$

With $\Lambda=8.9$, for instance, we obtain $c_1=1518$, and $a_1=6.67$. In Fig. 3, our pole form is compared with Eq. (27), and is seen to be quite a good approximation in the region where it is needed. This value of Λ is actually the one which gives an output nucleon at $\omega=0$, with N^* exchange in the crossed channel. As mentioned in Sec. II, experimental values were taken for the N^* parameters, which give $\Delta=2.15$ and $\gamma^x=0.2133$.

If we evaluate the residue of our output nucleon pole, we obtain $\gamma_{1/2,1/2}=0.20$. The corresponding experimental value is $\gamma_{1/2,1/2}=0.24$, as we have seen. We next exchange the nucleon in the crossed channel. Taking experimental values for its mass and coupling, we have $\Delta=0$ and $\gamma^x=0.1067$ in the $(\frac{3}{2},\frac{3}{2})$ state. With the same cutoff $\Lambda=8.9$ as in the $(\frac{1}{2},\frac{1}{2})$ state, this leads to an output N^*, with $\omega_R=1.5$ and $\gamma_{3/2,3/2}=0.10$. The experimental values are $\omega_R=2.15$ and $\gamma_{3/2,3/2}=\frac{1}{2}\gamma_{1/2,1/2}=0.12$.

IV. CONCLUSION

The above results suggest that the Bethe-Salpeter equation works at least as well as the N/D method in making bootstrap calculations. Interestingly enough, it seems to lead to output residues which are smaller than the experimental values. This is the reverse of what one obtains in most N/D calculations.[14]

In calculating the $(\frac{3}{2},\frac{3}{2})$ state, we took the same cutoff $\Lambda=8.9$ as in the $(\frac{1}{2},\frac{1}{2})$ state.[15] Actually, this procedure may not be as arbitrary as it probably appears at first sight. A cutoff of this type is a rough way of parametrizing short-range forces. Now it has been suggested that higher symmetries should manifest themselves at higher energies and momentum transfers, i.e., in interactions involving small distances. Therefore, it might be reasonable to assume that parameters which represent such interactions (for instance, cutoffs) should obey whatever symmetry is appropriate to the problem—the symmetry breaking would then come from long-range interactions.[16]

Several symmetry schemes have been proposed in which the N and N^* would be degenerate except for symmetry breaking. In particular, this is true for $SU(6)$,[17] and the strong-coupling group,[18] both of which have had some success in strong-interaction physics. Thus it should be reasonable to assume the same cutoff in the $(\frac{1}{2},\frac{1}{2})$ and $(\frac{3}{2},\frac{3}{2})$ states, within which the N and N^* occur as bound systems. Of course, we should, at the same time, include whatever other channels are coupled to the πN channel through the symmetry we are assuming. Since these have higher thresholds, however, their effect is probably not crucial when the symmetry is broken. It should therefore not be unreasonable to neglect them, at least in first approximation.

ACKNOWLEDGMENT

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[14] See, for example, J. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. **137**, B1242 (1965).

[15] It is found, however, that $\gamma_{3/2,3/2}$ is not sensitive to variations in Λ. With $\Lambda=8$, for instance, we again obtain $\gamma_{3/2,3/2}\simeq 0.10$, even though $\omega_R\simeq 2$.

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