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Some Relativistic Oddities in the Quantum Theory of Observation

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The orthodox quantum theory of observations is summarized and then applied to particle detection. Probabilities and state vectors are worked out for instantaneous observations, at stated times, by one counter, two counters, and three counters. The same cases are then examined in the context of a relativistic theory. It is found that, if attempts are made to detect a given particle in two space-time regions that have space-like separation, the nonrelativistic probability formulas have to be supplemented by additional conditions, and the wave function of the particle being detected becomes ambiguous and noncovariant. This ambiguity does not affect any probabilities for observations whose effect on the wave function has already been taken into account, and these are, by implication, the only observations whose probabilities could possibly be affected by the ambiguities in question. Possible consequences of these results are briefly discussed.

I. THE NONRELATIVISTIC THEORY OF OBSERVATIONS

LET us first summarize the orthodox nonrelativistic account of an instantaneous measurement of a complete set V of commuting observables, performed on a microscopic system S. We shall suppose that there is a macroscopic apparatus A which interacts very briefly and strongly with S at time t_a (being turned on only for a very short time at t_a), and that later a second apparatus A' performs the actual observation by "observing" and recording the state in which A has been left by its interaction with S. The division between A and A' is quite arbitrary; it marks the stage in the chain of cause and effect at which one stops using the Schrödinger equation to calculate fully determined state vectors, and introduces the unpredictability of the measuring process.

It is normally assumed that, before t_a, the Schrödinger state vector of the system $S+A$ is a product:

$$|\Psi_i(t)\rangle = |\mathcal{S}(t)\rangle |\mathcal{A}(t)\rangle, \tag{1}$$

where the first factor pertains to S and the second to A. The brief interaction of S and A converts this vector into a sum of products:

$$|\Psi_f(t)\rangle = \sum_v c_v |S_v(t)\rangle |A_v(t)\rangle, \tag{2}$$

where the sum covers all sets v of eigenvalues of the observables V, the c_v are c-number constants, the S_v are a set of orthonormal time-dependent states of S which, at time t_a, are eigenstates of V with the indicated eigenvalues v:

$$|S_v(t_a)\rangle = |v\rangle, \tag{3}$$

and the A_v are orthonormal time-dependent states of A. The initial state of S can be expanded in terms of the $|S_v\rangle$ at times before t_a:

$$|\mathcal{S}(t)\rangle = \sum_v b_v |S_v(t)\rangle. \tag{4}$$

In order to get the right probabilities for the various eigenvalues v, one must assume that

$$|b_v|^2 = |c_v|^2 \tag{5}$$

or

$$c_v = b_v e^{i\theta_v}.$$

Thus, the final state vector of $S+A$ is still that of a pure state; it is uniquely determined by the Schrödinger equation from the initial state. In principle, there might be interference among the various terms of the sum, which would influence a simultaneous probability distribution of variables of S and variables of A. In fact, the macroscopic nature of A would cause such interference to be unobservable.

Regardless of this question, there cannot be interference among the terms of the sum in any distribution of variables of S *alone*, for one would calculate such a distribution by summing or integrating over all the variables of A, and the orthogonality of the states of A labeled by different values of v would eliminate the interference terms in the distribution of S. Thus, after the interaction, the state of S alone (or of A alone) is a statistical mixture; the different eigenstates of V combine incoherently.

The "observation" of A by A' is the real *observation* in this account. Its outcome cannot be predicted by means of the Schrödinger equation, but only the probabilities $|c_v|^2$ of various outcomes. The state of A is a statistical mixture, and A' "finds out" which of the possible states in the mixture is the correct one. This process eliminates all the $|A_v\rangle$ except one; it reduces the state vector $|\Psi_f\rangle$ of $S+A$ to one term, and then replaces the c_v of that term by a constant of absolute value 1. This reduction is reflected in the mixture that describes S, by the elimination of all components except one $|S_v\rangle$, which is then the correct state of S after the measurement. In repetitions of the measurement on similarly prepared systems, the state of $S+A$, or the mixture describing either S or A alone, is the same on every occasion, but A' "discovers" different eigenvalues on different occasions. Possibly its action on any given occasion depends on hidden variables, but at the present stage of knowledge this action cannot be predicted.

In the foregoing account the term "statistical mixture" has been used ambiguously, in both of its common meanings. Before A' enters the discussion, the post-interaction state of S (or of A) is spoken of as a statistical mixture because it is really a superposition, but with coefficients that are themselves orthogonal vectors and thus eliminate interference. But then the action of A' on the mixture is spoken of as a discovery of which, among several alternative states, is the *right* one, as if the mixture were describing a situation in which there is one and only one correct state vector for A, and the mixture description were being used simply because one did not know which state is the right one. In calculating probabilities, one gets the same answers for a given mixture, whether the mixture is used because one does not know the right state, or because the state is in fact a superposition with orthogonal coefficients.

If there had been no A', so that no observation of S took place, one would use the entire mixture describing S in calculating the probability distribution of results of a later measurement on S, getting simply a sum of the partial distributions that come from the various individual $|S_v\rangle$. If A' is there to eliminate all the components of the mixture except one, one decides that all the partial distributions were mistaken on this occasion, except the one that came from the surviving component of the mixture; the aggregate of occasions in which a given result v' has been obtained gives the partial distribution derived from $|S_{v'}\rangle$. The later distribution of variables of S, following measurements in which *all* the eigenvalues of V have been obtained with their respective probabilities $|c_v|^2$, is just what it would have been if S had simply interacted with A and there had been no A' to make observations.

In fact, on account of the complexity of the macroscopic system A, each of the $|A_v\rangle$ and $|\mathcal{A}\rangle$ should itself be regarded as a mixture rather than a pure state. But as long as all the states entering the mixture corresponding to each given v are orthogonal to all the states in the mixture corresponding to each other v, the foregoing arguments remain valid and lead to the conclusions that have been stated.

Once one has worked out this account of an ideal observation, one can simplify it for future use by moving the division between A and A' right up to the point of interaction with S, i.e., by eliminating A altogether. Then one abbreviates the narrative by speaking of a rapid "collapse" of the state of S at time t_a, from a superposition of $|S_v\rangle$ to a single $|S_v\rangle$.

II. THE NONRELATIVISTIC THEORY OF OBSERVATIONS MADE BY COUNTERS

A. A Single Counter

If one wishes to apply the foregoing analysis to counter experiments, one must realize that a counter with 100% efficiency at all energies is a device for measuring P, the projection operator into the sensitive volume of the counter: The counter tells whether $P=0$ (the particle is outside the counter) or $P=1$ (the particle is inside the counter). Since P alone is far from being a complete set of commuting observables for S, it should be supplemented by the remainder of a complete set, for which remainder we now use the symbol W with eigenvalues w. Thus the complete set is now P, W, and the initial state of S can be written

$$|\mathcal{S}(t)\rangle=\sum_{p=0,1}\sum_{w}\alpha_{wp}|S_{wp}(t)\rangle, \tag{6}$$

where p denotes an eigenvalue of P, the α_{wp} are c-number constants, and the sum extends over the complete time-dependent orthonormal set $|S_{wp}\rangle$ each of which is, at time t_a, an eigenstate of P and all the W:

$$|S_{wp}(t_a)\rangle=|wp\rangle. \tag{7}$$

The initial state of the combined system $S+A$ is as expressed in Eq. (1). The probability of a given value p' being obtained in the measurement of P is

$$\mathcal{P}(p')=\sum_{w}|\alpha_{wp'}|^2. \tag{8}$$

The probabilities of the two possible values, 0 and 1, can be expressed also as the initial expectation values of the corresponding projection operators P and $1-P$.

Since A is a device for measuring P, it must have two sets of states, quiescent and "having-counted" states, which sets can be labeled by the two eigenvalues of P (0 and 1) and can be distinguished from each other by the "observer" A'. The complexity of the macroscopic system A ensures that there will be many states in each of these sets; in addition to the label P which differentiates the two sets, there is another label n which distinguishes among the different orthonormal states $|A_{pn}(t)\rangle$ in each set. We shall assume that A' is insensitive to this second label, that it responds only to the value of p and thus collapses the state onto one p value, leaving the state of A as a sum over the second index, and the state of S as a sum over w. Despite the insensitivity of A', it is still possible that the interaction between A and S establishes a partial or total correlation between w values and values of n; conceivably a different A' might have gained more information about the state of S. We shall simplify our work by assuming that, after S and A have interacted, there is no correlation of states of A with w values, beyond that required by the correlation of w values with p values.

If the interaction Hamiltonian of S with A is

$$H'=\delta(t-t_a)JP, \tag{9}$$

then its effect on the state of $S+A$ is given by

$$|\Psi_f(t_a)\rangle=\exp(-iJP/\hbar)|\Psi_i(t_a)\rangle=\sigma_A|\Psi_i(t_a)\rangle. \tag{10}$$

We shall suppose that the initial state of A contains only quiescent states $|A_{0n}(t)\rangle$ with coefficients a_n, and that $|\Psi_f\rangle$ embodies no correlations except those required for the measurement of P. We are thus led to the following form for σ_A:

$$\sigma_A = 1 - P + P[\sum_{ww'} |w\rangle L_{ww'} \langle w'|] \times \sum_{mn} [|A_{1m}(t_a)\rangle M_{mn} \langle A_{0n}(t_a)| + |A_{0m}(t_a)\rangle N_{mn} \langle A_{1n}(t_a)|], \quad (11)$$

the unitarity of which is assured by that of the matrices L, M, and N. If we had wanted an σ_A that established more correlations, we would have written single matrices labeled by four indices instead of the factored pairs L, M and L, N. The unitarity of σ_A requires the presence of the N term, which can in general produce "uncounting," or transitions of A from "having-counted" to quiescent states; but with our choice of initial state, this term produces no effect. Application of σ_A to $|\Psi_i(t_a)\rangle$ gives a state of the form

$$|\Psi_f(t_a)\rangle = \sum_p (\sum_w |\alpha_{wp}|^2)^{1/2} |\Psi_p(t_a)\rangle = \sigma_A |\Psi_i(t_a)\rangle, \quad (12)$$

where

$$|\Psi_0(t_a)\rangle = \sum_{wn} a_{0n} |A_{0n}(ta)\rangle \alpha_{w0} |w0\rangle (\sum_w |\alpha_{w0}|^2)^{-1/2} = [\mathcal{P}(0)]^{-1/2}(1-P)\sigma_A |\Psi_i(t_a)\rangle,$$

and

$$|\Psi_1(t_a)\rangle = \sum_{w,w'} |w1\rangle L_{ww'} \alpha_{w'1} \times \sum_{mn} |A_{1m}(t_a)\rangle M_{mn} a_n (\sum_w |\alpha_{w1}|^2)^{-1/2} = [\mathcal{P}(1)]^{-1/2} P \sigma_A |\Psi_i(t_a)\rangle. \quad (13)$$

These are the two normalized states into which $|\Psi_f\rangle$ can be collapsed by the action of A', i.e., by the action of a projection operator divided by the square root of a probability—a nonlinear operation. Our assumption of minimum correlation of states of A with those of S in $|\Psi_f\rangle$ results in each of these $|\Psi_p\rangle$ being a pure state of S. If σ_A had introduced more correlations, the state of S in each of the two $|\Psi_p\rangle$ would have been a mixture in which the interference of different eigenstates of W would have been reduced or absent.

B. Two Counters

Now we shall suppose that the counter A acts at time t_a as stated above, and that a second counter B with projection operator Q acts similarly at time t_b. The two counter volumes are assumed not to overlap:

$$PQ = 0. \quad (14)$$

In any case, P and Q commute, so they can be taken as two of a set of commuting observables; we shall let Y with eigenvalues y be the remainder of the set. The operator σ_B will resemble σ_A, *mutatis mutandis*; the two counters A and B will be assumed similar in the sense that the matrices L, M, and N are the same for the two.

Now let us consider the case $t_b > t_a$, or A is turned on and off first, and B is turned on and off later. The initial state of the system $S+A+B$ is a product of three factors. After the operation of σ_A, the state is that shown in Eq. (12), multiplied by the state of B. When the effect of A' is taken into account, the state is one of the states (13), multiplied by the state of B. This state is then propagated to the time t_b by the propagator $U(t_b - t_a)$, which is, for noninteracting systems, a product of $K(t_b - t_a)$, which is the propagator for S, and $T(t_b - t_a)$—itself a product—the propagator for $A+B$. This state is then the initial state for the observation at t_b, and is thus acted on by the operator σ_B and then collapsed onto one value of Q.

The probabilities of the two values of P are given in Eq. (8), and the corresponding states that can follow the first observation are given in (13). Thus the possible initial states $|\Psi_{ip}\rangle$ for the second observation are

$$|\Psi_{i0}(t_b)\rangle = K(t_b - t_a) \sum_{y,q} |y0q\rangle \alpha_{y0q} \times \sum_n |A_{0n}(t_b)\rangle a_n (\sum_{yq} |\alpha_{y0q}|^2)^{-1/2},$$

$$|\Psi_{i1}(t_b)\rangle = K(t_b - t_a) \sum_{yy'} |y10\rangle L_{yy'} \alpha_{y'10} \times \sum_{mn} |A_{1m}(t_b)\rangle M_{mn} a_n (\sum_y |\alpha_{y10}|^2)^{-1/2}. \quad (15)$$

In the second of these expressions there is no sum over the q values appearing in $|ypq\rangle$ because $p=1$ implies $q=0$. These states can be written as

$$|\Psi_{i0}(t_b)\rangle = [\mathcal{P}(0)]^{-1/2} U(t_b - t_a)(1-P)\sigma_A |\Psi_i(t_a)\rangle,$$
$$|\Psi_{i1}(t_b)\rangle = [\mathcal{P}(1)]^{-1/2} U(t_b - t_a) P \sigma_A |\Psi_i(t_a)\rangle. \quad (16)$$

From these one can calculate the probabilities $\mathcal{P}_p(q)$ of the two values of Q, *given* that P has a certain value p:

$$\mathcal{P}_p(0) = \langle \Psi_{ip}(t_b) | 1-Q | \Psi_{ip}(t_b)\rangle,$$
$$\mathcal{P}_p(1) = \langle \Psi_{ip}(t_b) | Q | \Psi_{ip}(t_b)\rangle = 1 - \mathcal{P}_p(0). \quad (17)$$

The two $\mathcal{P}_p(1)$ are thus

$$\mathcal{P}_0(1) = \sum_{y,q,y',q'} \bar{\alpha}_{y0q} \langle y0q | K^\dagger(t_b - t_a) Q K(t_b - t_a) \times |y'0q'\rangle \alpha_{y'0q'} (\sum_{yq} |\alpha_{y0q}|^2)^{-1},$$

$$\mathcal{P}_1(1) = \sum_{y,y',y'',y'''} \bar{\alpha}_{y10} \bar{L}_{yy'} \langle y'10 | K^\dagger(t_b - t_a) Q \times K(t_b - t_a) | y''10\rangle L_{y''y'''} \alpha_{y'''10} (\sum_y |\alpha_{y10}|^2)^{-1}, \quad (18)$$

and the $\mathcal{P}_p(0)$ are the complementary probabilities. The final factors are the reciprocals of the probabilities of Eq. (8), expressed in forms appropriate to the present case. Thus, multiplying by these probabilities, we can obtain the *a priori* probabilities $\mathcal{P}(p,q)$ of given pairs p, q

$$
\begin{aligned}
\mathcal{P}(0,1)=&\sum_{y,q,y',q'}\bar{\alpha}_{y0q}\langle y0q|K^\dagger(t_b-t_a)\\
&\qquad\times QK(t_b-t_a)|y'0q'\rangle\alpha_{y'0q'}\\
=&\sum_y|\langle y01|K(t_b-t_a)\sigma_A(1-P)|\Psi_i(t_a)\rangle|^2,\\
\mathcal{P}(1,1)=&\sum_{yy'y''y'''}\bar{\alpha}_{y10}\bar{L}_{yy'}\langle y'10|K^\dagger(t_b-t_a)\\
&\qquad\times QK(t_b-t_a)|y''10\rangle L_{y''y'''}\alpha_{y'''10}\\
=&\sum_y|\langle y01|K(t_b-t_a)\sigma_AP|\Psi_i(t_a)\rangle|^2,\\
\mathcal{P}(0,0)=&\sum_{yq}|\alpha_{y0q}|^2-\mathcal{P}(0,1)=\mathcal{P}(0)-\mathcal{P}(0,1)\\
=&\sum_{yp}|\langle yp0|K(t_b-t_a)\sigma_A(1-P)|\Psi_i(t_a)\rangle|^2,\\
\mathcal{P}(1,0)=&\sum_y|\alpha_{y10}|^2-\mathcal{P}(1,1)=\mathcal{P}(1)-\mathcal{P}(1,1)\\
=&\sum_{yp}|\langle yp0|K(t_b-t_a)\sigma_AP|\Psi_i(t_a)\rangle|^2.
\end{aligned}
\tag{19}
$$

Here, in taking squares of absolute values, we imply that the states of the counters are multiplied to give unity.

The four possible states $|\Psi_{pq}(t_b)\rangle$ that can be present immediately after t_b can be expressed as

$$
\begin{aligned}
|\Psi_{00}(t_b)\rangle=&[\mathcal{P}(0,0)]^{-1/2}(1-Q)\sigma_BU(t_b-t_a)(1-P)\sigma_A\\
&\qquad\times|\Psi_i(t_a)\rangle,\\
|\Psi_{10}(t_b)\rangle=&[\mathcal{P}(0,1)]^{-1/2}(1-Q)\sigma_BU(t_b-t_a)P\sigma_A\\
&\qquad\times|\Psi_i(t_a)\rangle,\\
|\Psi_{01}(t_b)\rangle=&[\mathcal{P}(1,0)]^{-1/2}Q\sigma_BU(t_b-t_a)(1-P)\sigma_A\\
&\qquad\times|\Psi_i(t_a)\rangle,\\
|\Psi_{11}(t_b)\rangle=&[\mathcal{P}(1,1)]^{-1/2}Q\sigma_BU(t_b-t_a)P\sigma_A|\Psi_i(t_a)\rangle.
\end{aligned}
\tag{20}
$$

These states can be written out as in Eq. (15), but we shall not display them in that form.

The foregoing expressions apply to the case $t_b>t_a$. The case $t_b=t_a$ can be analyzed as a single measurement of the two commuting observables P and Q, with the Y constituting the rest of the commuting set. Alternatively, it can be treated as a limiting case of the successive observations already discussed. If $t_b=t_a$, the propagators U, T, K all reduce to the identity. The probabilities of Eq. (19) then reduce to

$$
\begin{aligned}
\mathcal{P}(0,1)&=\sum_y|\alpha_{y01}|^2,\\
\mathcal{P}(1,1)&=0,\\
\mathcal{P}(1,0)&=\sum_y|\alpha_{y10}|^2,\\
\mathcal{P}(0,0)&=\sum_y|\alpha_{y00}|^2,
\end{aligned}
\tag{21}
$$

which are clearly the same expressions that would be obtained in an analysis of simultaneous observations. The final states (20) turn out to be

$$
\begin{aligned}
|\Psi_{00}(t_b)\rangle&=[\mathcal{P}(0,0)]^{-1/2}(1-Q)(1-P)\sigma_B\sigma_A|\Psi_i(t_a)\rangle,\\
|\Psi_{01}(t_b)\rangle&=[\mathcal{P}(0,1)]^{-1/2}(1-Q)P\sigma_B\sigma_A|\Psi_i(t_a)\rangle,\\
|\Psi_{10}(t_b)\rangle&=[\mathcal{P}(1,0)]^{-1/2}Q(1-P)\sigma_B\sigma_A|\Psi_i(t_a)\rangle,\\
|\Psi_{11}(t_b)\rangle&=0,
\end{aligned}
\tag{22}
$$

where $t_b=t_a$, and the commutativity of the projection operators with σ_A and σ_B has been used. The last of these vectors is indeterminate when taken as the limiting form of a normalized state, but clearly it should vanish; i.e., it should not occur in the final state. Thus these states are just the ones that would be obtained in the analysis of a single observation of P and Q, with the effect of the interaction of S with A and B being given by the operator

$$\sigma_{AB}=\sigma_B\sigma_A=\sigma_A\sigma_B. \tag{23}$$

So it appears that all probabilities and final states for simultaneous observations are the same, whether the observations are treated as a single one, or as a limiting case of successive observations. The intermediate states (13), (15), and (16) do not appear, since their duration has become zero.

C. Three Counters

Our last examples will pertain to three similar counters A, B, and C, instantaneously turned on at the respective times t_a, t_b, and t_c. The projection operator of C will be R, and the three operators P, Q, and R will be assumed not to overlap. P, Q, R, and Z will be taken as a complete set of commuting observables for S, with eigenvectors $|zpqr\rangle$.

We shall deal first with the case of three successive observations, $t_a<t_b<t_c$. Now the initial states $|\Psi_i\rangle$ at t_c are the states (20), multiplied by the propagator for $S+A+B+C$, $U(t_c-t_b)=T(t_c-t_b)K(t_c-t_b)$:

$$
\begin{aligned}
|\Psi_{i00}(t_c)\rangle=&[\mathcal{P}(0,0)]^{-1/2}U(t_c-t_b)(1-Q)\sigma_BU(t_b-t_a)\\
&\qquad\times(1-P)\sigma_A|\Psi_i(t_a)\rangle,\\
|\Psi_{i01}(t_c)\rangle=&[\mathcal{P}(0,1)]^{-1/2}U(t_c-t_b)(1-Q)\sigma_BU(t_b-t_a)\\
&\qquad\times P\sigma_A|\Psi_i(t_a)\rangle,\\
|\Psi_{i10}(t_c)\rangle=&[\mathcal{P}(1,0)]^{-1/2}U(t_c-t_b)Q\sigma_BU(t_b-t_a)\\
&\qquad\times(1-P)\sigma_A|\Psi_i(t_a)\rangle,\\
|\Psi_{i11}(t_c)\rangle=&[\mathcal{P}(1,1)]^{-1/2}U(t_c-t_b)Q\sigma_BU(t_b-t_a)\\
&\qquad\times P\sigma_A|\Psi_i(t_a)\rangle.
\end{aligned}
\tag{24}
$$

The probabilities of the two eigenvalues r of R, *given* that certain p and q values have been observed, are expectation values of R or $1-R$, taken over the states (24). As before, we can convert these to *a priori* probabilities $\mathcal{P}(p,q,r)$ by multiplying them by the appropriate

$\mathcal{P}(p,q)$. The $\mathcal{P}(p,q,r)$ turn out to be

$$\mathcal{P}(0,0,0)=\sum_{zpq} |\langle zpq0|K(t_c-t_b)\sigma_B(1-Q) \times K(t_b-t_a)\sigma_A(1-P)|\Psi_i(t_a)\rangle|^2,$$

$$\mathcal{P}(0,0,1)=\mathcal{P}(0,0)-\mathcal{P}(0,0,0) =\sum_{z} |\langle z001|K(t_c-t_b)\sigma_B(1-Q) \times K(t_b-t_a)\sigma_A(1-P)|\Psi_i(t_a)\rangle|^2,$$

$$\mathcal{P}(0,1,0)=\sum_{zpq} |\langle zpq0|K(t_c-t_b)\sigma_B QK(t_b-t_a) \times\sigma_A(1-P)|\Psi_i(t_a)\rangle|^2,$$

$$\mathcal{P}(0,1,1)=\mathcal{P}(0,1)-\mathcal{P}(0,1,0) =\sum_{z} |\langle z001|K(t_c-t_b)\sigma_B QK(t_b-t_a) \times\sigma_A(1-P)|\Psi_i(t_a)\rangle|^2, \quad (25)$$

$$\mathcal{P}(1,0,0)=\sum_{zpq} |\langle zpq0|K(t_c-t_b)\sigma_B(1-Q) \times K(t_b-t_a)\sigma_A P|\Psi_i(t_a)\rangle|^2,$$

$$\mathcal{P}(1,0,1)=\mathcal{P}(1,0)-\mathcal{P}(1,0,0) =\sum_{z} |\langle z001|K(t_c-t_b)\sigma_B(1-Q) \times K(t_b-t_a)\sigma_A P|\Psi_i(t_a)\rangle|^2,$$

$$\mathcal{P}(1,1,0)=\sum_{zpq} |\langle zpq0|K(t_c-t_b)\sigma_B QK(t_b-t_a) \times\sigma_A P|\Psi_i(t_a)\rangle|^2,$$

$$\mathcal{P}(1,1,1)=\mathcal{P}(1,1)-\mathcal{P}(1,1,0) =\sum_{z} |\langle z001|K(t_c-t_b)\sigma_B QK(t_b-t_a) \times\sigma_A P|\Psi i(t_a)\rangle|^2.$$

The possible states of the system $S+A+B+C$ after t_c are

$$|\Psi_{000}(t_c)\rangle=[\mathcal{P}(0,0,0)]^{-1/2}(1-R)\sigma_C U(t_c-t_b)(1-Q) \sigma_B U(t_b-t_a)(1-P)\sigma_A|\Psi_i(t_a)\rangle,$$

$$|\Psi_{001}(t_c)\rangle=[\mathcal{P}(0,0,1)]^{-1/2}R\sigma_C U(t_c-t_b)(1-Q) \times\sigma_B U(t_b-t_a)(1-P)\sigma_A|\Psi_i(t_a)\rangle,$$

$$|\Psi_{010}(t_c)\rangle=[\mathcal{P}(0,1,0)]^{-1/2}(1-R)\sigma_C U(t_c-t_b) \times Q\sigma_B U(t_b-t_a)(1-P)\sigma_A|\Psi_i(t_a)\rangle,$$

$$|\Psi_{011}(t_c)\rangle=[\mathcal{P}(0,1,1)]^{-1/2}R\sigma_C U(t_c-t_b) \times Q\sigma_B U(t_b-t_a)(1-P)\sigma_A|\Psi_i(t_a)\rangle, \quad (26)$$

$$|\Psi_{100}(t_c)\rangle=[\mathcal{P}(1,0,0)]^{-1/2}(1-R)\sigma_C U(t_c-t_b) \times(1-Q)\sigma_B U(t_b-t_a)P\sigma_A|\Psi_i(t_a)\rangle,$$

$$|\Psi_{101}(t_c)\rangle=[\mathcal{P}(1,0,1)]^{-1/2}R\sigma_C U(t_c-t_b)(1-Q) \sigma_B U(t_b-t_a)P\sigma_A|\Psi_i(t_a)\rangle,$$

$$|\Psi_{110}(t_c)\rangle=[\mathcal{P}(1,1,0)]^{-1/2}(1-R)\sigma_C U(t_c-t_b) \times Q\sigma_B U(t_b-t_a)P\sigma_A|\Psi_i(t_a)\rangle,$$

$$|\Psi_{111}(t_c)\rangle=[\mathcal{P}(1,1,1)]^{-1/2}R\sigma_C U(t_c-t_b) \times Q\sigma_B U(t_b-t_a)P\sigma_A|\Psi_i(t_a)\rangle.$$

If in these expressions one sets $t_c=t_b$ or $t_b=t_a$, thus reducing one U (or both) to the identity (and sets the indeterminate vectors equal to zero), one gets the correct probabilities and state vectors for the corresponding cases of two simultaneous observations with one other before or after, or of three simultaneous observations.

III. THE RELATIVISTIC DESCRIPTION OF OBSERVATIONS MADE BY COUNTERS

In attempting to make the foregoing analysis relativistically causal and covariant, one immediately encounters two problems which arise even in the case of a single instantaneous observation. If one really applied to a one-body system an instantaneous interaction with another system, with space dependence given by a projection operator like P, then S would not in general be a one-body system afterward, and the discussion of later observations would become extremely complicated. Since we are trying to analyze problems involving only one particle S, we must assume that the rise and fall times of the interaction are long enough (presumably of the order of $\hbar/mc^2$), and the boundaries of the volume of A are fuzzy enough ($\sim\hbar/mc$), to preclude production of new S and $\tilde{S}$ particles. It may be overelaborate even to mention this matter, since actual counters do not produce particle pairs by virtue of being turned on and off. In any case, such smearing of the edges of the space-time region of observation does not seem to vitiate seriously any of the relationships already arrived at. No harm is done by replacement of the δ function in Eq. (9) by a smoother function; and although smeared-out operators P, Q, and R are no longer projection operators, they should act enough like projection operators in σ_A, σ_B, and σ_C [if not in $\exp(-iJP/\hbar)$] to be treated as such.

The second problem is one alluded to by Wigner[1]: An interaction which is turned on simultaneously over the whole of a finite volume presupposes a particular Lorentz frame (perhaps the rest frame of the counter), and in different frames the turning-on will not be simultaneous, but will occur at different times in different parts of the volume. Thus the description of the interaction as a function of t, multiplied by a (smeared-out) projection operator into a space volume, can be valid only in one Lorentz frame, and our equations will be valid as they stand only if applied in that frame. However, they can be Lorentz-transformed, and it would appear that their gross features should persist in different frames; even though there is no upper limit on the spatial or temporal extension of a given region of observation, the largest invariant interval within this region has an invariant ratio to the interval between one observation and another, and, if this ratio is very small, each region of observation can presumably be approximated as a space-time point. There can be

[1] E. P. Wigner, Am. J. Phys. 31, 6 (1963).

ambiguities due to changing time orders of the observations—a matter that will be discussed below. In any case, it *should* be all right to calculate state vectors and probabilities (which are scalars) in the rest frame of the counters, and then to transform the results to other frames.

A difficulty which seems more serious than these is related to the change of the state of $S+A$ from a product to a sum of products. In using the Schrödinger picture of time dependence—and arguments like the following one can be made in the Heisenberg picture too—one associates a state vector with each three-dimensional surface $t=\text{const}$. Then the Schrödinger equation describes how the vector changes from one such surface to another. In integrating the equation from initial conditions, in a situation in which an interaction is turned on at some time t_a (say, in the rest frame of a counter), one calculates a vector that is unaffected by the interaction at all times before t_a. Then immediately after t_a one encounters state vectors that have been affected by the interaction. In a relatively simple case like that of a potential which is turned on at t_a, a particle's wave function $\langle \mathbf{x} | \mathcal{S}(t) \rangle$ remains a pure state and is affected only inside and on the future light cone of the space-time region of interaction. But in a case like that of a counter, the state vector of the whole system $S+A$ was a product before the interaction; the fact that the P factor in σ_A multiplies the part of the wave function of S *inside* the counter by states $|A_{1n}\rangle$ means that the part *outside* the counter, though still multiplied by the same states $|A_{0n}\rangle$ as before, is now part of a mixture instead of being part of a pure state. The state of S has changed instantaneously (and noncovariantly) from a pure state to a mixture, along a surface $t=\text{const}$. This situation does not appear to be changed in any important way by the use of some other set of spacelike 3-surfaces in lieu of the surfaces $t=\text{const}$, for in all such cases the wave function of S suddenly becomes a mixture all over a given surface.

Clearly this behavior of the wave function of S depends on there being a system for S to interact with that is more complicated than a potential; it must be a system that has dynamical variables of its own, and the interaction must be one that can be turned on and off (at least if our present analysis is to be applicable). In short, it must be a system which can be used, along with an "observer" A', as a particle detector capable of "instantaneous" operation.

The most important feature of a mixture like the one in question is that the components of it cannot interfere; i.e., there can be no later interference of the part of the wave function that is inside the counter at time t_a with the part that is outside the counter at that time. Since such interference could never occur outside the future light cone of the region of observation anyway, it would not seem to matter when and where, outside this cone, the wave function of S became a mixture.

But problems arise if one imposes two requirements which seem to be quite orthodox: (1) that every particle has, at any given space-time point, a unique wave function, whether pure state or mixture, which transforms under an irreducible representation of the Lorentz group— or at least each pure state entering the mixture must transform thus; (2) that one and only one component of the mixture in a case like ours is *the* wave function of S after the interaction, even though A' may delay a long time in finding out which component this is.

The difficulty becomes manifest if one compares the state of S as calculated in the rest frame of the counter and then transformed to another frame, with the state of S as it would have been calculated from the beginning in the other frame. In this comparison the wave function of S is ambiguous throughout the region outside the counter and between the planes $t=t_a$ and $t'=t_a'$; there will be two wave functions at each point in this region, one affected and the other unaffected by the observation. In particular, if the counter detects the particle, one of the two wave functions is zero and the other is not, at any given point in the region.

This contradiction could be avoided if the transition from pure state to mixture somehow took place along a light cone or an invariant hyperboloid; but Schrödinger equations do not seem to lead to such behavior. Possibly an equation with more than one time variable (such as the Bethe-Salpeter equation) could resolve the problem, but the author does not see how such a resolution would proceed; nor does he know of a many-time equation that applies to systems as complicated as particle detectors.

Since the real use of "collapsed" wave functions is the prediction of later observations, let us look again at the probabilities and state vectors worked out in Sec. II, for cases in which two or three observations are made at different places and times. If counters A and B act in two space-time regions a and b which have a timelike separation with $t_a<t_b$, their time order is unambiguous. Although the condition $PQ=0$ is noninvariant, this fact should not affect the validity of Eqs. (15)–(20), provided the equations are properly Lorentz-transformed, and the range of summation of the eigenvalue q is extended to cover the possibility that p and q may both differ from zero at the same time. The limit $t_a=t_b$ now refers to a single observation with a single counter, since the two counters have been assumed similar. Equations (15)–(20) do have the proper limiting forms for this case.

If the regions a and b have spacelike separation, one is tempted to treat the two observations as simultaneous from the outset, since there can be no causal influence of one on the other, and it would seem a precondition of the analysis that the counters cannot both detect the same particle. However, this formulation incurs contradictions when applied to three observations in small regions a, b, and c, with a and c having timelike separation, and b having spacelike separation from both a and c. Since a and c have to be treated as consecutive, b cannot be taken simultaneous with both.

Thus it seems necessary to treat observations in regions a and b that have spacelike separation as consecutive observations. One is encouraged to treat them so by the fact that, nonrelativistically, simultaneous observations are a special case of consecutive ones. Because $QK(t_b-t_a)P=0$, the probability $\mathcal{P}(1,1)$ does vanish, as it should, and all the probabilities in Eqs. (19) will be scalars if the operators and vectors in those equations are assigned the proper transformation properties, as long as $t_b>t_a$.

The failure of covariance comes from the ambiguity of the time order of the space-time regions a and b.[1] Their time order can be reversed, but the positive-energy part of $K(t_b-t_a)$ vanishes when $t_b<t_a$. If one calculates the probabilities (19) in some frame of reference in which $t_b>t_a$ and then performs a proper Lorentz transformation which reverses the time order of the observations, one finds that all the probabilities go to zero. Thus, in order to get invariant probabilities, one must calculate them from different formulas in these two Lorentz frames, making the substitutions $a\leftrightarrow b$, $A\leftrightarrow B$, $P\leftrightarrow Q$, $p\leftrightarrow q$. This procedure is not necessary in most relativistic calculations that involve propagators between points with spacelike separation, because such propagators vanish regardless of the time order that the points have in a particular Lorentz frame, and the discontinuity at $t_b=t_a$ occurs only for timelike-separated points, whose time order is unambiguous anyway. The peculiarity of the situation contemplated here is that the probabilities of some possible results of observations in regions that have spacelike separation contain propagators from everywhere *outside* A at t_a, to B (or everywhere outside B) at t_b, and parts of *these* regions have timelike separation.

The calculation of probabilities would be unambiguous if one were speaking of a set of counters filling all space, all turned on briefly over a spacelike surface and then again over a later spacelike surface. Then these two surfaces would have a physical significance that is not possessed by the surfaces $t=t_a$ and $t=t_b$, and there would be no question of reversing their "time" order by proper Lorentz transformations. Each set of observations could be treated as simultaneous. Expressions for probabilities would contain propagators from a given part of one surface to all parts of the other lying in the future light cone of that part of the first, and there would never be any need to consider propagation into the past. Our two regions a and b *could* lie on two such spacelike surfaces (or on a single surface); one can even argue that an observation with a 100%-efficient counter A is equivalent to an observation with two counters, A and another which fills all the rest of space and has projection operator $1-P$. But if the second counter is not really there, one cannot say whether it is turned on before or after B.

The foregoing complication in the calculation of probabilities may not be regarded as serious; one merely has to add a proviso to the formulas (19) to the effect that t_a, σ_A, etc. by definition pertain to the observation that is earlier in the particular Lorentz frame that is being used. But this proviso does not resolve the ambiguity of the state vectors (16) and (20).

If $t_a<t_b$, a vector such as $|\Psi_{10}(t)\rangle$ appears to develop in a fairly normal way, considering the discontinuities attending the process of observation. The initial wave function behaves normally until t_a, at which time it collapses into the volume of counter A; thence it spreads in accordance with the wave equation of S alone. It does not "collapse" out of counter B, for it does not propagate that far from A by the time t_b. Thus it is nonzero over a large region of space before t_a, and throughout the future light cone of the region a.

But in a Lorentz frame in which $t_b<t_a$ (by which we mean that every point in the region b is earlier than any point in the region a), if one uses Eqs. (16) and (20) as they stand, the wave function is ambiguously given between t_b and t_a, being the initial wave function, *and* the one that has been affected by counter A, and the one that follows observation b. The probability that B counts is correspondingly ambiguous. If, on the other hand, one calculates the wave function in this case (in which counter A counts and counter B does not) by treating observation b as earlier, one finds that the initial state persists until t_b, at which time the wave function suddenly departs from the volume of counter B and becomes correspondingly larger elsewhere; then it propagates normally until, at time t_a, it collapses into counter A, and afterward proceeds without discontinuities. This wave function is unambiguous and corresponds properly to the events; the initial wave function for the observation at t_a correctly gives the probability that A detects the particle in view of the fact that B did not detect it. But clearly it is not what one would get by applying a Lorentz transformation to the wave function that describes the same physical situation in a frame in which $t_a<t_b$. It is nonzero over a large region of space before t_b; for a very short time it vanishes inside counter B; after t_a it is nonzero only inside the future light cone of region a. It is easy to think of space-time points at which the two functions are markedly different, in particular, points in and near the counter B, shortly before and after t_b. If one wished to use these functions to "predict" the outcome of an observation, say, in a region c such that b lies at the midpoint of a straight-line segment from a to c, the predictions from the two functions would be entirely different. However, our earlier discussion of probabilities indicates that the right way to predict results obtained at c is to use the time order that the three regions a, b,

and c have in the Lorentz frame that one happens to be using. Thus, if one has $t_a<t_b$, then $t_b<t_c$, and one uses the vector $|\Psi_{10}\rangle$ in Eq. (20) to find that C cannot detect the particle. On the other hand, if one has $t_a<t_b$, then $t_c<t_b$, and the vectors (20) are not right; if C has detected the particle, all these vectors vanish except $|\Psi_{00}\rangle$. Such cases of three observations are manageable by equations like the set (25), which can be used to give correct probabilities if a, b, c are defined as coming in the time order $t_a<t_b<t_c$. But then the state vectors (26) have the same sort of ambiguity as do those of Eqs. (16) and (20).

So wave functions seem to be ambiguous in the neighborhood of such observations, but they are ambiguous in such a way as not to affect any predicted probabilities for the observations whose effects on states have produced the ambiguity. If one calculates probabilities and state vectors for some given set of observations (say, at a and b), the state vectors will not necessarily give the correct probabilities for a third observation at c, but if one includes c in the set of observations from the beginning, one can then work out correct and consistent probabilities, and state vectors which, though ambiguous, are still always consistent with the experimental results that they embody. Wave functions of S are ambiguous, but not in such a way as to make any difference. The ambiguity is gone at all times after both t_a and t_b (e.g., at all points that lie in the future light cones of both a and b), so that wave functions in this region can be used in the orthodox way to predict observations that might be made there, without its being necessary to decide in advance whether an observation actually *will* be made.

It might still appear desirable, if only on aesthetic grounds, to contrive covariant and unique wave functions which correctly correspond to given observational results. Apparently, though, if counter A detects the particle S in a small space-time region a, a unique and covariant wave function of S which would permit correct calculation of probabilities for other detection experiments would have to be one that vanishes everywhere outside *both* the past and the future light cones of a. This function would obviously imply zero probability of detection of S outside these light cones—as it should—and it would not matter whether one specified in advance which observations were to be made in this region. But such a function would be noncausal, and could not possibly be calculated from a differential equation with initial conditions; at all times before t_a, the function would already be influenced by the result of the observation a, which is not even made until t_a. This single function would partake of the properties of both the predictive and the retrodictive states discussed by Aharonov *et al.*,[2] and yet it would be intended as *the* wave function of S. Furthermore, if S is looked for in both regions a and b which have spacelike separation, the counter (if any) which detects S is the only one whose actions determine the earlier behavior of the wave function; the other counter plays no part.

Such functions clearly would be useless in predicting *a priori* probabilities, for they would have been predetermined by certainties. To calculate probabilities, one would still have to use a wave equation to determine an ambiguous wave function whence one could derive probability formulas like those exhibited above. One could, if one pleased, then define a teleological wave function corresponding to any given counter's having counted (or counting in the future), which would correctly give conditional probabilities: "If A counts at t_a, then the probabilities of B, C, etc. counting at times t_b, t_c, etc., are as follows: . . . ." Such a procedure appears to have little to recommend it.

In any case it appears that either causality or Lorentz covariance of wave functions must be sacrificed in situations like those contemplated. Covariance seems the smaller sacrifice, since it is apparently not required for the calculation of invariant probabilities.

If arguments like these are to have any application in the improvement of relativistic quantum theory, presumably their value will come from their relevance to the determination of state vectors of interacting systems. Even if two systems as simple as single particles interact, one of the particles can be regarded as the system A which, when observed by a macroscopic system A', can reveal the state of the other particle. Much of the foregoing analysis can be applied directly to such a "counter," but one important change would have to be made: the "counter" is "on" all the time, not just during a very short time interval at t_a. The corresponding problem involving actual counters is one (common in practice) in which several counters are turned on for long times, and each is capable of registering not only the fact of a count, but also the time at which the count occurs. If such a counter does detect a particle at some time t_a, the effect of such detection on the particle's wave function ought to be the conventional one of collapsing the function into the counter volume; what seems less obvious is the effect on the wave function of *non*detection in a given time interval. Attempts to analyze this problem are currently under way. Such an analysis should also avoid what is perhaps the most unrealistic aspect of the analysis presented here: Here we have ignored the interaction between S and A that takes place even when the counter A is not turned on.

[2] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1401 (1964).