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Relativistic Collapse to a Schwarzschild Sphere*

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Analytical models of spherically symmetric, nonstatic processes, including relativistic collapse to, and explosion from, a central body, are constructed by joining three simple space-time metrics across surfaces of discontinuity. The tetrad or dyadic formulation of the junction conditions, implied by the postulate of admissible coordinates, is employed to match the solutions.

I. INTRODUCTION

THE interior Schwarzschild solution describing a static (rigid, nonrotating) sphere of perfect fluid at constant density is the simplest analytic metric known for a finite gravitating body. It has not usually been noticed that the field equations allow an arbitrary function of time to be introduced into this metric, without destroying the properties of simplicity, rigidity, and constant density, but resulting in a time-dependent pressure field.[1] We have exploited this time dependence, together with the Oppenheimer-Snyder solution for a collapsing "dust cloud," to construct idealized models of spherically symmetric, nonstatic processes such as accretion or relativistic collapse onto a central body. The models are of sufficient simplicity to permit exact analytical solution of the entire gravitational problem, making comparison of the predictions of general relativity with Newtonian and post-Newtonian theory for the dynamics of the models unambiguous, even in extreme relativistic regimes.

The complete solutions are obtained by piecing together three different, well-known, space-time metrics across surfaces of discontinuity. All junction conditions required by the Lichnerowicz postulate of admissible coordinates[2] are satisfied. The metrics, however, are expressed in "nonadmissible" co-moving coordinates and the fitting is actually accomplished using these coordinates and the tetrad or dyadic formulation of the junction conditions.[3] Derivation of the metrics in dyadic form and application of the dyadic continuity conditions are given in Secs. II and III.

The three metrics employed are: I, the interior Schwarzschild with constant density ρ_s but time-dependent pressure; II, the Oppenheimer-Snyder solutions[4] for the spherically symmetric motion of incoherent matter; and III, the exterior Schwarzschild solution. Figure 1 is a schematic space-time diagram of the simplest combination of these. It represents a contracting spherical dust cloud (region II, outer boundary Σ_1), condensing and accreting on the surface Σ_3 of a growing Schwarzschild sphere (region I, below dashed line) to build the final configuration, a static Schwarzschild sphere of radius r_m (region I, above dashed line, bound-

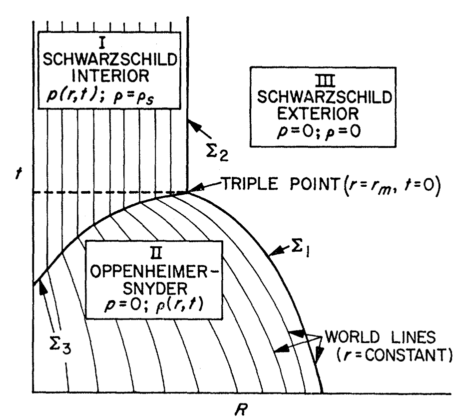

FIG. 1. Contraction of dust cloud to Schwarzschild sphere. The notation is explained in the text.

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[1] See, however: A. H. Taub, *Recent Developments in General Relativity* (The Macmillan Company, New York, 1962), p. 449.

[2] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Électromagnétisme* (Masson et Cie., Paris, 1955).

[3] F. B. Estabrook and H. D. Wahlquist, J. Math. Phys. (to be published).

[4] J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).

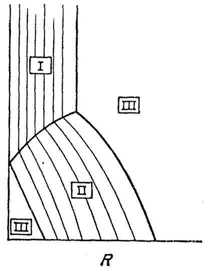

FIG. 2. Contracting or exploding shell.

ary Σ_2). On Σ_3 the matter undergoes a phase transformation from the incoherent state of zero pressure and varying density to the interior state of varying pressure and constant density ρ_s. In Sec. IV we give a complete summary of the metrics appropriate to Fig. 1. Some other possible mosaics of these metrics are shown in Figs. 2–5; their analytical descriptions can also be constructed using the methods described.

II. THE DYADIC EQUATIONS FOR SPHERICAL SYMMETRY

The general dyadic equations for timelike congruences[5] are specialized to the present case by imposing several conditions. The matter is to be nonrotating ($\mathbf{\Omega}=0$), and we shall adopt nonrotating axes ($\boldsymbol{\omega}=0$). Spherical symmetry is imposed by representing the only distinguished spatial direction at every point with a unit, radial, 3-vector $\mathbf{u}$, and expressing all vectors and dyadics in terms of it. In particular then the spatial gradient of any scalar must be expressible as

$$\nabla\xi=\xi'\mathbf{u}, \tag{1}$$

with

$$\xi'\equiv\mathbf{u}\cdot\nabla\xi, \tag{2}$$

and the gradient of $\mathbf{u}$ itself as

$$\nabla\mathbf{u}=\eta(\mathbf{I}-\mathbf{uu}). \tag{3}$$

The remaining kinematical variables of the matter take the forms

$$\mathbf{a}=a\mathbf{u},\quad \mathbf{S}=\tfrac{1}{3}\theta\mathbf{I}+\sigma(\mathbf{I}-3\mathbf{uu}), \tag{4}$$

where a is the absolute acceleration, θ is the expansion, and σ is the shear. Similarly, the electric and magnetic components of the Weyl tensor, represented by two symmetric traceless dyadics, must have the forms

$$\mathbf{A}=\alpha(\mathbf{I}-3\mathbf{uu}),\quad \mathbf{B}=\beta(\mathbf{I}-3\mathbf{uu}), \tag{5}$$

and it can be shown from the equations that, in fact, in the present case $\beta=0$. Finally, for a co-moving frame of reference, the momentum density vanishes ($\mathbf{t}=0$) while the remaining components of the Einstein tensor will include ρ, the local proper energy density, and the stress dyadic for a perfect fluid,

$$\mathbf{T}=-p\mathbf{I}. \tag{6}$$

[5] F. B. Estabrook and H. D. Wahlquist, J. Math. Phys. **5**, 1629 (1964).

With these assumptions the set of dyadic equations reduces to the following (relativistic units, $c=4\pi K=1$, are used):

$$\begin{aligned}&\dot\eta=(\tfrac13\theta+\sigma)(a-\eta),\quad \dot\alpha=-3(\tfrac13\theta+\sigma)\alpha-\sigma(p+\rho),\\ &\dot\sigma=-\theta\sigma+\sigma(\tfrac13\theta+\sigma)-\alpha-\tfrac13(a'+a^2-\eta a),\\ &(\tfrac13\theta+\sigma)^{\cdot}=-(\tfrac13\theta+\sigma)^2-(\tfrac13\rho+\alpha)-p+\eta a,\\ &(\tfrac13\rho+\alpha)^{\cdot}=-3(\tfrac13\theta+\sigma)(\tfrac13\rho+\alpha+\tfrac13 p),\end{aligned} \tag{7}$$

where the dot signifies proper-time derivative; and

$$\begin{aligned}&(\tfrac13\theta+\sigma)'=-3\eta\sigma,\quad (\tfrac13\rho+\alpha)'=-3\eta\alpha,\\ &\eta'=-\eta^2+(\tfrac13\theta+\sigma)^2-3\sigma(\tfrac13\theta+\sigma)-2(\tfrac13\rho+\alpha)+3\alpha,\\ &p'=-(p+\rho)a.\end{aligned} \tag{8}$$

If we now introduce co-moving spatial coordinates (r,ψ,χ), the spherical metric can be written in the form

$$ds^2=-\phi^{-2}dt^2+\delta^{-2}dr^2+R^2(d\psi^2+\sin^2\psi\, d\chi^2), \tag{9}$$

where R denotes the usual radial curvature coordinate and ϕ, δ, and R are functions of r and t only. It follows from the general discussion of co-moving coordinates in the dyadic formalism[6] that the metric coefficients are here related to the dyadic variables by the equations

$$\begin{aligned}&\dot\delta/\delta=-(\tfrac13\theta+\sigma)+3\sigma,\quad \dot R/R=\tfrac13\theta+\sigma,\\ &\phi'/\phi=-a,\qquad R'/R=\eta,\\ &R^2[\eta^2-(\tfrac13\theta+\sigma)^2+2(\tfrac13\rho+\alpha)]=1,\end{aligned} \tag{10}$$

while for the coordinates themselves we have, of course,

$$\dot r=\dot\psi=\dot\chi=0,\quad t'=\psi'=\chi'=0$$

and

$$r'=\delta,\quad \dot t=\phi, \tag{11}$$

so that for any scalar, say $\xi(r,t)$,

$$\xi'=\frac{\partial\xi}{\partial r}r'=\delta\frac{\partial\xi}{\partial r},\quad \dot\xi=\frac{\partial\xi}{\partial t}\dot t=\phi\frac{\partial\xi}{\partial t}. \tag{12}$$

Each of the three regions used in these models is characterized by different further specializations. In region I we want to describe a rigid fluid with constant density ($\theta=\sigma=\dot\rho=\rho'=0$), and since the space-time of the interior Schwarzschild solution is conformally flat,

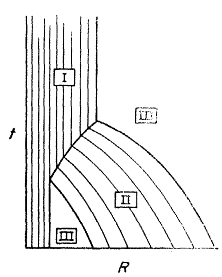

FIG. 3. Accretion or partial explosion.

[6] H. D. Wahlquist and F. B. Estabrook, J. Math. Phys. **7**, 894 (1966); also, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California, Report No. 32-868, 1966 (unpublished).

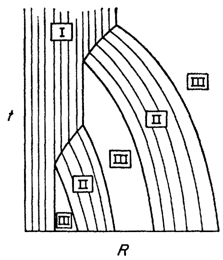

FIG. 4. Multiple bursts of accretion or explosion.

we also have $\alpha=0$. From Eq. (10) we find $\dot{R}=0$ and, since the scale of the co-moving r coordinate is as yet unspecified, we may pick $R=r$. The entire set of equations is now readily integrated and leads to the following results:

$$R=r,\quad \delta=(1-\tfrac{2}{3}\rho_s r^2)^{1/2},\quad \phi=2/[3F(t)-\delta],$$
$$a=\frac{\tfrac{2}{3}\rho_s r}{3F(t)-\delta},\quad p=\rho_s\left[\frac{\delta-F(t)}{3F(t)-\delta}\right], \tag{13}$$

where ρ_s is the constant energy-density and $F(t)$ is an arbitrary function of time.

In like manner for the incoherent matter of region II, we put $p=a=0$, set the scale of the time coordinate by putting $\phi=1$ (so that t_{II} is proper time on the matter world lines), set the scale of the r coordinate by picking $r=R$ on the surface Σ_3, and find

$$\phi=1,\quad \delta=f(r)\left(\frac{\partial R}{\partial r}\right)^{-1},\quad \eta=\frac{f(r)}{R},$$
$$\sigma=-\frac{1}{3}\frac{(\delta R)^{\cdot}}{\delta R},\quad \tfrac{1}{3}\theta+\sigma=\frac{\dot{R}}{R},\quad \rho=\frac{h'(r)}{f(r)R^2},\quad \tfrac{1}{3}\rho+\alpha=\frac{h(r)}{R^3},$$
$$R'=f(r),\quad \dot{R}=\pm[-1+f^2(r)+2h(r)/R]^{1/2}, \tag{14}$$

where $f(r)$ and $h(r)$ are arbitrary functions of r only. Partially integrating the expression for $\dot{R}$ we may write

$$t_{\text{II}}=\pm\int_r^{R(r,t_{\text{II}})}\left[-1+f^2(r)+2\frac{h(r)}{x}\right]^{-1/2}dx-g(r), \tag{15}$$

thereby introducing one more function $g(r)$ which determines the surface Σ_3, for having chosen $r=R$ on Σ_3, the equation of Σ_3 in terms of the coordinates of region II becomes

$$t_{\text{II}}+g(r)=0. \tag{16}$$

In region III, which being empty, offers no physically preferred timelike congruence, we adopt the timelike isometry for our reference frame so that, in addition to $p=\rho=0$, we have $\theta=\sigma=0$, all time derivatives vanish, and we may again set $r=R$. The equations then lead directly to the usual exterior Schwarzschild metric with

$$r=R,\quad \delta=(1-2M/R)^{1/2},\quad \phi=1/\delta,$$
$$\eta=\delta/R,\quad a=M/\delta R^2,\quad \alpha=M/R^3. \tag{17}$$

III. THE DYADIC JUNCTION CONDITIONS FOR SPHERICAL SYMMETRY

Having obtained the metrics and physical quantities for each region, expressed in terms of intrinsic co-moving coordinates, the remaining task is to join them together properly at the three surfaces of discontinuity Σ_1, Σ_2, and Σ_3. This problem is usually formulated theoretically in terms of admissible coordinates,[2] for which the metric tensor itself and its first derivatives are *everywhere* continuous. When such coordinates can be discovered, the matching problem is quite simple. In general, of course, the co-moving coordinates used here will not satisfy the requirements of admissibility. To proceed, however, by searching for a set of admissible coordinates valid for all three regions leads to considerable difficulties, not the least of which arise from the facts that the metrics of regions I and II are not explicit, but involve several functions as yet arbitrary, and that the forms of the boundary surfaces themselves are not yet completely specified. In practice this problem can be handled quite straightforwardly by turning to a tetrad or dyadic formulation of the junction conditions, which can be specially adapted to the matching of metrics expressed in co-moving coordinates. The general formalism of this approach is developed in Ref. 3, where it is shown that the continuity conditions used below are equivalent to, and in fact guarantee, the *existence* of admissible coordinates without explicitly employing them.

When the 29 general dyadic junction conditions of Ref. 3 for motion normal to spatial boundaries are written for spherical symmetry and co-moving frames of reference, the following set of seven continuity conditions results:

$$(\cot\psi)/R, \tag{18a}$$
$$\gamma[\eta+v(\tfrac{1}{3}\theta+\sigma)], \tag{18b}$$
$$\gamma[v\eta+\tfrac{1}{3}\theta+\sigma], \tag{18c}$$
$$\gamma[\gamma^2 vv'+\gamma^2\dot{v}+a+v(\tfrac{1}{3}\theta-2\sigma)], \tag{18d}$$
$$\gamma^2 v(p+\rho), \tag{18e}$$
$$\gamma^2[v^2\rho+p], \tag{18f}$$
$$\tfrac{1}{3}\rho+\alpha, \tag{18g}$$

where each of these expressions must be continuous across the boundary. The quantity v is the proper,

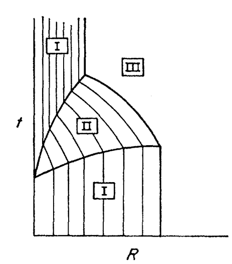

FIG. 5. Collapse to smaller Schwarzschild sphere (supernova process).

radial 3-velocity of the moving spherical 2-surface as observed from the co-moving frame of reference on each side and $\gamma=(1-v^2)^{-1/2}$. We assume throughout that the angular coordinates ψ and χ are propagated continuously across all boundaries, so the first expression simply requires that R also be everywhere continuous. Equations (18a)–(18d) ensure that the intrinsic first and second fundamental forms of the boundary are unique. Equations (18e) and (18f) are the usual relativistic Rankine-Hugoniot relations in a symmetric form, and Eq. (18g) requires continuity of a certain combination of curvature components involving both Einstein and Weyl tensors.

The simplest situation occurs at Σ_2, between regions I and III. Writing Eq. (18e), for instance, we have

$$\gamma_{\rm I}^2 v_{\rm I}(p_{\rm I}+\rho_{\rm I})|_{\Sigma_2}=\gamma_{\rm III}^2 v_{\rm III}(p_{\rm III}+\rho_{\rm III})|_{\Sigma_2}, \tag{19}$$

but since $p_{\rm III}=\rho_{\rm III}=0$ and $(p_{\rm I}+\rho_{\rm I})>0$, we find $v_{\rm I}=0$, $\gamma_{\rm I}=1$. Thus, this boundary must be fixed in the co-moving frame of region I and its equation can be written $r=$constant, or specifically $r=r_m$. Similarly writing out both sides of Eq. (18f) we find that

$$p_{\rm I}|_{\Sigma_2}=0, \tag{20}$$

so from Eq. (13)

$$F(t_{\rm I})|_{\Sigma_2}=\delta(r_m)=(1-\tfrac{2}{3}\rho_s r_m^2)^{1/2}. \tag{21}$$

However, since F may depend only on time, this result actually determines F throughout the part of region I above the dashed line of Fig. 1. Remembering that θ and σ vanish in both regions I and III, Eq. (18c) requires that also $v_{\rm III}=0$, while Eqs. (18b), (18d), and (18g) all reduce to the single condition

$$M=\tfrac{1}{3}\rho_s r_m^3. \tag{22}$$

Turning to Σ_1 and again writing out both sides of the junction conditions, this time for regions II and III, we first find $v_{\rm II}=0$ [Eq. (18f)], so that $r=r_m$ is also the equation of Σ_1 in region II. Using the expression for $\dot{R}$ [Eq. (14)], other junction conditions give

$$h(r_m)=\tfrac{1}{3}\rho_s r_m^3=M, \tag{23}$$

and

$$f(r_m)=\gamma_{\rm III}\delta_{\rm III}, \tag{24}$$

and thus solve for $v_{\rm III}$ as

$$v_{\rm III}=\pm\frac{1}{f(r_m)}\left[-1+f^2(r_m)+2\frac{M}{R}\right]^{1/2}. \tag{25}$$

The remaining condition Eq. (18d) vanishes identically in II, so that it simply becomes the equation of motion of the collapsing spherical 2-surface in region III; it is in fact just the equation for a timelike radial geodesic as would be expected.

When the junction conditions are applied at the internal boundary Σ_3, we find that we are still left with two functions on this boundary which remain arbitrary. In other words for each choice of two such functions and the two parameters ρ_s and r_m, we obtain a unique solution for a model of the type depicted in Fig. 1. The physical significance of this is perhaps most clearly seen by interpreting the functions as (1) the amount of matter crossing Σ_3 and accreting in region I per unit time, and (2) the velocity of impact of this matter at Σ_3. Since we impose no thermodynamic constraints at Σ_3, other than the local conservation of energy-momentum given by the Rankine-Hugoniot relations, both of these quantities can be specified. Recalling the definition of the velocities appearing in the junction conditions, the two quantities can be written as $\rho_s v_{\rm I}$ and $(v_{\rm I}-v_{\rm II})/(1-v_{\rm I}v_{\rm II})$, respectively; so it is convenient to adopt as arbitrary functions the two velocities

$$u(r)\equiv v_{\rm I}, \tag{26}$$

the velocity of the surface of the growing sphere, and

$$v(r)\equiv\frac{v_{\rm I}-v_{\rm II}}{1-v_{\rm I}v_{\rm II}}, \tag{27}$$

the velocity of impact of the dust, both relative to the static matter of region I. Since the time coordinates of I and II will not agree on Σ_3, whereas we have arranged that the r coordinates do coincide there, we express the velocities as functions of r along Σ_3 as indicated. Note that for either accretion or explosion (Fig. 1 upside down) we have $uv\leqq 0$.

The result of applying the junction conditions is now to express the unknown physical variables and the previous arbitrary functions in terms of $u(r)$ and $v(r)$. From Eqs. (18e) and (18f) we get the energy-momentum relations in more familiar form

$$p_{\rm I}|_{\Sigma_3}=-\rho_s uv \tag{28}$$

and

$$\rho_{\rm II}|_{\Sigma_3}=\rho_s u(1-v^2)/(u-v), \tag{29}$$

while Eqs. (18b) and (18c) determine $f(r)$ as

$$f(r)=\left[\frac{1-2h(r)/r}{1-v^2(r)}\right]^{1/2}, \tag{30}$$

and Eq. (18g) gives

$$h(r)=\tfrac{1}{3}\rho_s r^3. \tag{31}$$

Inserting $p_{\rm I}$ from Eq. (13) into Eq. (28) we get

$$F(t_{\rm I})|_{\Sigma_3}=\delta_{\rm I}\left[\frac{1-u(r)v(r)}{1-3u(r)v(r)}\right], \tag{32}$$

but by using the equation of Σ_3 [Eq. (36)], the right side of this equation can be converted to a function of $t_{\rm I}$ only which then gives $F(t_{\rm I})$ throughout the time-dependent part of region I (below dashed line of

Fig. 1). Similarly from Eq. (14) we have

$$\frac{\partial R}{\partial r}\bigg|_{\Sigma_3}=\frac{\rho_s}{\rho_{\rm II}}\bigg|_{\Sigma_3}, \tag{33}$$

and using Eqs. (15) and (29) we get

$$\frac{dg(r)}{dr}=-\frac{(1-uv)}{u(1-v^2)^{1/2}(1-\frac{2}{3}\rho_s r^2)^{1/2}}. \tag{34}$$

Integrating and adjusting the constant so that $t_{\rm II}=0$ at the triple point of Fig. 1, we have

$$g(r)=\int_r^{r_m}\frac{[1-u(x)v(x)]}{u(x)[1-v^2(x)]^{1/2}[1-\frac{2}{3}\rho_s x^2]^{1/2}}dx. \tag{35}$$

Finally, the equation for Σ_3 in region I is

$$dt_{\rm I}=\frac{\phi_{\rm I}}{\delta_{\rm I}}\bigg|_{\Sigma_3}\frac{dr}{u(r)},$$

so that, also setting $t_{\rm I}=0$ at the triple point, we have

$$t_{\rm I}=-\int_r^{r_m}\frac{[1-3u(x)v(x)]}{u(x)[1-\frac{2}{3}\rho_s x^2]}dx. \tag{36}$$

IV. SUMMARY

In this section we collect the previous results to give a summary of the metrics and physical quantities for the type of models depicted in Fig. 1. The solutions are specified by the two velocity functions $u(r)$ and $v(r)$ on Σ_3, which are arbitrary to within some broad constraints discussed at the end, and it is convenient to define

$$u_m=u(r_m),\quad v_m=v(r_m). \tag{37}$$

The equations as written are equally valid for either contracting or expanding models.

The space-time metrics and physical quantities in the three regions of Fig. 1 are the following.

Region I

$$ds^2=-\tfrac{1}{4}[3F(t_{\rm I})-(1-\tfrac{2}{3}\rho_s r^2)^{1/2}]^2dt_{\rm I}^2+[1-\tfrac{2}{3}\rho_s r^2]^{-1}dr^2+r^2d\Omega^2, \tag{38}$$

with

$$F(t_{\rm I})=(1-\tfrac{2}{3}\rho_s r^2)^{1/2}\left\{\frac{1-u(r)v(r)}{1-3u(r)v(r)}\right\}\bigg|_{r=r_{\Sigma_3}(t_I)}\quad (t_{\rm I}<0)$$
$$=(1-\tfrac{2}{3}\rho_s r_m^2)^{1/2}\quad (t_{\rm I}>0). \tag{39}$$

The inequalities here would reverse for an expanding motion. The function of r in the first expression for F is to be converted to a function of $t_{\rm I}$ by means of the equation of the boundary Σ_3 which is

$$t_{\rm I}=-\int_r^{r_m}\frac{[1-3u(x)v(x)]}{u(x)[1-\frac{2}{3}\rho_s x^2]}dx. \tag{40}$$

Letting $d\Sigma_3$ represent interval in the surface Σ_3 we find

$$d\Sigma_3^2=-\frac{(1-u^2)}{u^2(1-\frac{2}{3}\rho_s r^2)}dr^2+r^2d\Omega^2. \tag{41}$$

The equation for the boundary Σ_2 is simply $r=r_m$ and has the intrinsic metric

$$d\Sigma_2^2=-(1-\tfrac{2}{3}\rho_s r_m^2)dt_{\rm I}^2+r_m^2d\Omega^2. \tag{42}$$

The pressure field in I is given by

$$p(r,t_{\rm I})=\rho_s\left\{\frac{(1-\frac{2}{3}\rho_s r^2)^{1/2}-F(t_{\rm I})}{3F(t_{\rm I})-(1-\frac{2}{3}\rho_s r^2)^{1/2}}\right\}, \tag{43}$$

and on the boundaries

$$p_{\Sigma_3}=-\rho_s uv,\quad p_{\Sigma_2}=0. \tag{44}$$

Region II

$$ds^2=-dt_{\rm II}^2+\frac{(1-v^2)}{(1-\frac{2}{3}\rho_s r^2)}\left(\frac{\partial R}{\partial r}\right)^2dr^2+R^2d\Omega^2, \tag{45}$$

with $R(r,t_{\rm II})$ given implicitly by

$$t_{\rm II}=\frac{v}{|v|}\int_r^R\left[\frac{v^2(r)-\frac{2}{3}\rho_s r^2}{1-v^2(r)}+\tfrac{2}{3}\rho_s\frac{r^3}{x}\right]^{-1/2}dx-\int_r^{r_m}\frac{1-u(x)v(x)}{u(x)[1-v^2(x)]^{1/2}(1-\frac{2}{3}\rho_s x^2)^{1/2}}dx \tag{46}$$

and

$$\frac{\partial R}{\partial r}=\left[\frac{v^2-\frac{2}{3}\rho_s r^2}{1-v^2}+\tfrac{2}{3}\rho_s\frac{r^3}{R}\right]^{1/2}\left\{\frac{u-v}{|v|u(1-v^2)^{1/2}(1-\frac{2}{3}\rho_s r^2)^{1/2}}+\int_r^R\left[\frac{v^2-\frac{2}{3}\rho_s r^2}{1-v^2}+\tfrac{2}{3}\rho_s\frac{r^3}{x}\right]^{-3/2}\times\left[\rho_s\frac{r^2}{x}+\frac{1}{2}\frac{d}{dr}\left(\frac{v^2-\frac{2}{3}\rho_s r^2}{1-v^2}\right)\right]dx\right\}. \tag{47}$$

The equations for the boundaries of region II are, for Σ_3, $r=R$, or from Eq. (46),

$$t_{\rm II}=-\int_r^{r_m}\frac{[1-u(x)v(x)]}{u(x)[1-v^2(x)]^{1/2}(1-\frac{2}{3}\rho_s x^2)^{1/2}}dx \tag{48}$$

giving Eq. (41) again for $d\Sigma_3^2$; and for Σ_1, $r=r_m$, so that

$$d\Sigma_1^2=-dt_{\rm II}^2+R^2(r_m,t_{\rm II})d\Omega^2$$

or, again using Eq. (46),

$$d\Sigma_1^2=-\left[\frac{v_m^2-\frac{2}{3}\rho_s r_m^2}{1-v_m^2}+\frac{2}{3}\rho_s\frac{r_m^3}{R}\right]^{-1}dR^2+R^2d\Omega^2. \quad (49)$$

The proper density in region II is given by

$$\rho(r,t_{\rm II})=\rho_s\frac{r^2}{R^2}\left(\frac{\partial R}{\partial r}\right)^{-1}, \quad (50)$$

and on Σ_3,

$$\rho_{\Sigma_3}=\rho_s u(1-v^2)/(u-v). \quad (51)$$

Region III

$$ds^2=-\left(1-2\frac{M}{R}\right)dt_{\rm III}^2+\left(1-2\frac{M}{R}\right)^{-1}dR^2+R^2d\Omega^2, \quad (52)$$

with

$$M=\tfrac{1}{3}\rho_s r_m^3. \quad (53)$$

The boundary equations for this region are: for Σ_2, $R=r_m$, giving Eq. (42) for $d\Sigma_2^2$ (with $t_{\rm I}=t_{\rm III}$ on Σ_2); and for Σ_1,

$$t_{\rm III}=\frac{v}{|v|}\frac{(1-\frac{2}{3}\rho_s r_m^2)^{1/2}}{(1-v_m^2)^{1/2}}\int_{r_m}^{R}\left[\frac{v_m^2-\frac{2}{3}\rho_s r_m^2}{1-v_m^2}+\frac{2}{3}\rho_s\frac{r_m^3}{x}\right]^{-1/2}\times\left[1-\frac{2}{3}\rho_s\frac{r_m^3}{x}\right]^{-1}dx \quad (54)$$

giving Eq. (49) again for $d\Sigma_1^2$.

Certain constraints must be imposed on the parameters and velocities if these solutions are to be complete and everywhere regular with no further boundaries. First, to describe the situation of Fig. 1, rather than Fig. 2, we must choose $v(0)=0$. Second, to ensure that $0\leq p(r,t_{\rm I})<\infty$, we must have

$$0\leqq(-uv)<\frac{[3(1-\frac{2}{3}\rho_s r^2)^{1/2}-1]}{3[1-(1-\frac{2}{3}\rho_s r^2)^{1/2}]}, \quad (55)$$

and it follows from this that

$$\tfrac{2}{3}\rho_s r_m^2<8/9, \quad (56)$$

which is the usual limit for the interior Schwarzschild solution. Next, from Eq. (50) we must have $\partial R/\partial r>0$, and thus from the first factor of Eq. (47),

$$\tfrac{2}{3}\rho_s r^2\leqq v^2<1. \quad (57)$$

The left inequality here may be rewritten in an obvious notation as an energy constraint

$$\tfrac{1}{2}v^2-M(r)/r\geqq 0 \quad (58)$$

required to ensure that the matter of region II comes from (or goes to) infinite distance. The second factor of Eq. (47) also must be prevented from vanishing, which it might do if

$$\frac{d}{dr}\left[\frac{v^2-\frac{2}{3}\rho_s r^2}{1-v^2}\right] \quad (59)$$

became sufficiently negative (all other terms of the factor are positive). Such behavior of the energy on Σ_3 leads to shells of matter overtaking others (intersection of the world lines in region II), violating the assumption of incoherent matter in II.

It may be noted from Eq. (39) that, in general, a discontinuity of the pressure in region I occurs at the surface $t_{\rm I}=0$ (the dashed line of Fig. 1). Physically, this would appear as a shock wave of infinite speed, resulting from the sudden cessation of accretion at $r=r_m$. The discontinuity is removed by setting $u_m=0$, which corresponds simply to vanishing density at the outer boundary of the collapsing cloud as is evident from Eq. (51). Letting $u(r)=0$ anywhere, however, gives rise to some difficulties with the coordinate r in region II and on Σ_3 at such points. These can be handled by using instead a new coordinate $\bar{r}$ defined, for instance, by $d\bar{r}=(\rho_{\Sigma_3})^{-1}dr$.

For arbitrary choices of the functions $u(r)$ and $v(r)$, satisfying the foregoing constraints and $u_m=0$, the models may still have the acausal feature that the surfaces of constant pressure in I are spacelike. This is a not surprising result of the incompressibility of the matter in region I. For such fluids, in the words of Sommerfeld,[7] the pressure takes on the rather unphysical character of a "Lagrangian multiplier . . . a reaction against the condition of incompressibility . . . without energetic consequence". Nevertheless, here its contribution to the gravitational field is fully included.

[7] A. Sommerfeld, *Mechanics of Deformable Bodies* (Academic Press Inc., New York, 1950), p. 91.