Shell-Model Calculations with Separable Potentials

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Shell-model calculations are performed on O^{18}, Ca^{42}, Zr^{90}, Po^{210}, and F^{18} using separable potential forms. In most cases, the potentials have been fitted to low- and intermediate-energy two-body data. The over-all agreement is quite good, and the calculations are considerably simpler than similar ones using hard-core potentials. Replacing the potential matrix by an approximate reaction matrix offers no general improvement and produces results which are comparatively insensitive to the potential parameters.

1. INTRODUCTION

VELOCITY-DEPENDENT potentials have been studied by several authors, primarily in search of an interaction which can replace the hard core, and ease computations. Whether such a potential can actually simulate a hard core still seems to be in question,[1] but its ability to reproduce scattering data is good. Formal relations between hard cores and velocity-dependent potentials have been examined by Bell[2] and Baker.[3] They indicate that the form of an equivalent velocity-dependent potential is more complex than those employed to date.

Although a nonlocal interaction may appear unrealistic in certain aspects, there is no *a priori* reason why the two-body interaction should be restricted to local forms involving a central core. Folk and Bonnem[4] have investigated the ground-state properties of the three- and four-nucleon system using several potentials, and have concluded that the hard core can indeed be replaced by a velocity-dependent potential.

Short of finding a direct relation between a hard core and a nonlocal interaction, one must proceed by analyzing experimental data using a nonlocal form, and determine if such an analysis can at least give as good a fit as the local form.

The separable potential is a limiting case of a velocity-dependent interaction. Such a form reduces computation considerably and has been found capable of reproducing the low-energy data quite well for bound and continuum spectra.[5–10] It is advantageous in that the Schrödinger equation is exactly soluble. Some of its more general properties have also been investigated recently.[11]

This study is concerned with a shell-model analysis of elements having two nucleons outside a closed shell, using a separable nonlocal interaction. The analysis is based on the assumption that the residual interaction can be derived from the free nucleon-nucleon potential as determined by the scattering data and low-energy properties. As a preliminary study of the potential form, calculations are carried out using the free interaction directly, and also using an approximate reaction matrix in which the Pauli principle has been ignored.

Recently,[12] a more general separable potential[13] has been applied to a shell-model analysis of a few nuclei. The free potential was taken as the residual interaction, producing results in poor agreement with experiment. The potential forms employed here offer considerable improvement.

Calculations based on the Hamada-Johnstone potential[14] have recently been performed on O^{18} and F^{18} using the reaction matrix and including some core excitation. The resulting spectra were in good agreement with experiment, emphasizing that the residual interaction can indeed be obtained from the free interaction.

The results obtained here are very encouraging. Energy-level agreement for the $T=1$ cases is good with the exception of Ca^{42}. They are in accord with similar calculations using realistic hard-core interactions. Good agreement is also obtained with the $T=0$ potential in the case of F^{18}. Only in a few cases does introduction of the approximate reaction matrix improve agreement.

In Sec. 2, the various forms of the potential examined are discussed. Section 3 outlines briefly the method of performing the shell-model calculations, and Sec. 4 introduces and discusses the form of the reaction matrix. The results of the calculations are presented and discussed in Sec. 5.

2. THE SEPARABLE POTENTIAL

The separable form of the nonlocal potential has received considerable attention in recent years. The first calculations performed using such a potential were those of Yamaguchi.[5] He employed the form

$$(\mathbf{p}|V|\mathbf{p}')=-(\lambda/M)g(\mathbf{p})g(\mathbf{p}') \qquad (1)$$

[1] M. Razavy, Nucl. Phys. **50**, 465 (1964); R. M. May, *ibid.* **62**, 177 (1965); B. H. J. McKellar and R. M. May, *ibid.* **65**, 289 (1965).
[2] J. S. Bell, in *Proceedings of the Rutherford Jubilee International Conference, Manchester, 1961*, edited by J. B. Birks (Heywood and Company, Ltd., London, 1961), p. 373.
[3] G. A. Baker, Phys. Rev. **128**, 1485 (1962).
[4] R. Folk and E. Bonnem, Nucl. Phys. **63**, 513 (1965).
[5] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954); **95**, 1635 (1954).
[6] A. N. Mitra and V. L. Narasimham, Nucl. Phys. **14**, 407 (1959), hereafter referred to as MN.
[7] A. N. Mitra and J. H. Naqvi, Nucl. Phys. **25**, 307 (1961).
[8] V. L. Narasimham, S. K. Shah, and S. P. Pandya, Nucl. Phys. 33, 529 (1962); S. K. Shah and S. P. Pandya, *ibid.* **38**, 420 (1962).
[9] V. L. Narasimham, Nucl. Phys. **35**, 95 (1962); V. L. Narasimham and S. K. Shah, *ibid.* **69**, 204 (1965).
[10] J. H. Naqvi, Nucl. Phys. **36**, 578 (1962).
[11] W. H. Nichols, Am. J. Phys. **33**, 474 (1965); G. C. Ghirardi and A. Rimini, J. Math. Phys. **5**, 722 (1964).
[12] C. W. Lee and E. Baranger, Nucl. Phys. **79**, 835 (1966).
[13] F. Tabakin, Ann. Phys. (N. Y.) **10**, 51 (1964).
[14] T. T. S. Kuo and G. E. Brown, Nucl. Phys. **85**, 40 (1966).

to study the bound-state properties of the deuteron and low-energy neutron-proton scattering. With the functional form

$$g(\mathbf{p})=C(p)+8^{-1/2}T(p)S(\hat{p}), \quad (2)$$

where $S(\hat{p})$ is the tensor operator and $C(p)$ and $T(p)$ are functions of $p=|\mathbf{p}|$ (Appendix A), parameters were found which gave good agreement with the deuteron properties, and with the low-energy triplet eigen-S scattering data.

Potential forms similar to (1) have been applied to other spin-isospin states. If $g(p)$ depends only on $p=|\mathbf{p}|$, then (1) acts only in the S state. For angular momentum l, a potential of the form

$$(\mathbf{p}|V_l|\mathbf{p}')=-\frac{\lambda}{M}\sum_{m=-l}^{l}Y_l^{m*}(\theta_p,\phi_p) \times Y_l^m(\theta_{p'},\phi_{p'})g(p)g(p') \quad (3)$$

may be assumed. V_l causes scattering only for the lth partial wave, i.e., the potential is completely "separable." Thus a general potential can be formed by carrying out a summation over possible l values:

$$(\mathbf{p}|V|\mathbf{p}')=\sum_l(-\lambda_l/M)(2l+1)v_l(p)v_l(p')P_l(\hat{p}\cdot\hat{p}'). \quad (4)$$

A very general form similar to this, but which uses a combination of two product terms,

$$-g_l(p)g_l(p')+h_l(p)h_l(p'),$$

for each l value, has been studied by Tabakin.[13] Thirty-seven parameters were employed for $l=0$, 1, 2 to fit the high-energy scattering data and to achieve saturation in nuclear matter.

A list of the different potential forms studied here, with a brief description of the experimental results used to obtain parameter values, is given in Appendix A. It should be noted here that all forms have been adjusted to fit low- and intermediate-energy two-body data with the exception of the singlet potentials due to Naqvi,[15] which have been fitted to high-energy (30–300 MeV) phase-shift data. For example, the triplet-even potential used here, which has explicit tensor and spin-orbit terms, does not reproduce the high-energy eigen-phase shifts. However, it gives good agreement with the low-energy properties, and calculations presently being carried out show that its behavior in nuclear matter is very similar to Tabakin's triplet-even form.

The triplet-odd potential of Mitra and Naqvi[7] is limited to a spin-orbit term. Calculations are also performed using Tabakin's form. Also given in Appendix A are the forms of the associated reaction matrices, which will be discussed in Sec. 4.

3. SHELL-MODEL CALCULATION

The energy levels of certain nuclei having two nucleons outside a closed shell have been calculated. These calculations were performed in the shell model with configuration mixing, using harmonic-oscillator basis functions.

The necessary matrix elements are determined in the momentum representation. The harmonic-oscillator wave functions can be obtained by a Fourier transform of the general expression of Talmi.[16] The result is given by[17]

$$\chi_{nlm}(p,\theta,\phi)=\Pi_{nl}(p)Y_{lm}(\theta,\phi), \quad (5)$$

where

$$\Pi_{nl}(p)=A_{nl}p^l e^{-\frac{1}{2}a^2p^2}\,{}_1F_1(-n;l+\tfrac{3}{2};a^2p^2), \quad (6)$$

$$A_{nl}=a^{3/2}(-)^n(-ia)^l\left[\frac{2(l+\frac{3}{2})_n}{n!\Gamma(l+\frac{3}{2})}\right]^{1/2}. \quad (7)$$

Here a is the oscillator parameter

$$a=(m\omega)^{-1/2} \quad (8)$$

(note that $\hbar=1$ throughout). ${}_1F_1(a;b;x)$ is the Kummer confluent hypergeometric function, and $(a)_n \equiv a(a+1)\cdots(a+n-1)$. The quantum numbers nl are those employed by Talmi, with $n=0$ as the lowest state. The following expressions are particularly useful:

$$\begin{aligned}&{}_1F_1(0;m;a^2p^2)=1,\\ &{}_1F_1(-1;m;a^2p^2)=1-a^2p^2/m,\\ &{}_1F_1(-2;m;a^2p^2)=1-2a^2p^2/m+a^4p^4/m(m+1).\end{aligned} \quad (9)$$

The matrix elements are calculated in j-j coupling by the usual transformation from the LS-coupling results. Following Moshinsky,[18] the transformation

$$\begin{aligned}\mathbf{r}&=2^{-1/2}(\mathbf{r}_1-\mathbf{r}_2), \quad \mathbf{R}=2^{-1/2}(\mathbf{r}_1+\mathbf{r}_2),\\ \mathbf{p}&=2^{-1/2}(\mathbf{p}_1-\mathbf{p}_2), \quad \mathbf{P}=2^{-1/2}(\mathbf{p}_1+\mathbf{p}_2)\end{aligned} \quad (10)$$

to relative and center-of-mass (r.c.m.) coordinates, which leaves the Hamiltonian and wave functions invariant, has been used.

LS-coupling wave functions are readily built up from the single-particle wave functions in the usual manner:

$$\begin{aligned}&|n_1l_1,n_2l_2;\lambda\mu\rangle\\ &\quad=\sum_{m_1m_2}\langle l_1m_1l_2m_2|\lambda\mu\rangle\Pi_{n_1l_1}(p_1)\Pi_{n_2l_2}(p_2)\\ &\qquad\times Y_{l_1m_1}(\hat{p}_1)Y_{l_2m_2}(\hat{p}_2), \quad (11)\\ &|n_1l_1,n_2l_2;JM\rangle\\ &\quad=\sum_{m_\lambda m_S}\langle\lambda m_\lambda Sm_S|JM\rangle|n_1l_1,n_2l_2;\lambda\mu\rangle\chi_{Sm_S}.\end{aligned}$$

The Moshinsky brackets then give the wave function in r.c.m. coordinates:

$$|n_1l_1,n_2l_2,\lambda\rangle=\sum_{nlNL}\langle n_1l_1,n_2l_2,\lambda|nl,NL,\lambda\rangle|nl,NL,\lambda\rangle. \quad (12)$$

[15] J. H. Naqvi, Nucl. Phys. **58**, 289 (1964).

[16] I. Talmi, Helv. Phys. Acta. **25**, 185 (1952).

[17] B. H. J. McKellar (unpublished); see also Phys. Rev. **134**, B1190 (1964).

[18] M. Moshinsky, Nucl. Phys. **13**, 104 (1959); T. A. Brody and M. Moshinsky, *Tables of Transformation Brackets* (Monografias del Institutio de Fisica, Mexico, 1960).

For the antisymmetrized wave function, the transformation coefficient can be determined by

$$\langle nl,NL,\lambda|n_1l_1,n_2l_2,\lambda\rangle$$
$$=\tfrac{1}{2}(1-(-)^{T+S+l})\sqrt{2}\langle nl,NL,\lambda|n_1l_1,n_2l_2,\lambda\rangle, \quad n_1l_1\neq n_2l_2$$
$$=\tfrac{1}{2}(1-(-)^{T+S+l})\langle nl,NL,\lambda|n_1l_1,n_2l_2,\lambda\rangle, \quad n_1l_1=n_2l_2. \tag{13}$$

As an example, consider a central potential of the form

$$(\mathbf{p}|V|\mathbf{p}')=-\lambda g(p')g(p)P_l(\hat{p}\cdot\hat{p}'),$$
$$P_l(\hat{p}\cdot\hat{p}')=\frac{4\pi}{2l+1}\sum_m Y_{lm}^*(\hat{p})Y_{lm}(\hat{p}'). \tag{14}$$

Matrix elements of similar potentials have been evaluated elsewhere.[8] The LS-coupling result can be written

$$\langle n_1l_1,n_2l_2;JM|V|n_1'l_1',n_2'l_2';JM\rangle$$
$$=-\lambda\frac{4\pi}{2l+1}\sum_{nn'NL}\langle nl,NL,\lambda|n_1l_1,n_2l_2,\lambda\rangle$$
$$\times\langle n'l,NL,\lambda|n_1'l_1',n_2'l_2',\lambda\rangle\langle nl\|g(p)g(p')\|n'l\rangle. \tag{15}$$

Here,

$$\langle nl\|g(p)g(p')\|n'l'\rangle$$
$$=\int_0^\infty \Pi_{nl}^*(p)g(p)p^2dp\int_0^\infty \Pi_{n'l'}(p')g(p')p'^2dp'. \tag{16}$$

These integrals can be expressed in terms of the following three forms:

$$X_{2n}(\beta)=\int_0^\infty p^{(2n+2)}(p^2+\beta^2)^{-1}e^{-\frac{1}{2}a^2p^2}dp,$$
$$Y_{2n}(\beta)=\int_0^\infty p^{(2n+4)}(p^2+\beta^2)^{-2}e^{-\frac{1}{2}a^2p^2}dp, \tag{17}$$
$$Z_{2n}(\beta)=\int_0^\infty p^{(2n+1)}(p^2+\beta^2)^{-1}e^{-\frac{1}{2}a^2p^2}dp.$$

The first two of these integrals can be expressed analytically in terms of the complementary error function, the third in terms of the exponential integral.

Evaluating matrix elements for the tensor and spin-orbit forces is more involved, and the results are summarized in Appendix B.

Since the transformation of Moshinsky has been employed in carrying out the calculations, one cannot carry over the potential forms and parameters directly as given in Appendix A. Rather, one must make the transformation $p\rightarrow p/\sqrt{2}$, and replace the coefficient λ by $\lambda/2\sqrt{2}$.

In performing the actual calculations, it is necessary to know the value of the nuclear size parameter a for the various nuclei. Goldhammer[19] gives a value of $a=1.71$ F for O^{16}, deduced from electron scattering. This value has been used for the $A=18$ basis functions.

Moszkowski[20] has given a relation for estimating a, namely,

$$\omega=41A^{-1/3}\,\text{MeV}, \tag{18}$$

with $a=(m\omega)^{-1/2}$. This has been used to determine a values for larger A. For O^{16}, this estimate gives a lower result than that of Goldhammer, and it might be expected to underestimate a for larger A values as well.

4. THE REACTION MATRIX

The many-body reaction matrix theory of Brueckner[21] and Bethe[22] was developed to study potentials with hard cores in nuclear matter. Replacing the potential matrix V by its associated reaction matrix for the potential forms considered here presents little difficulty if an approximate form is used. Calculations[23] of the doublet splitting in O^{17} and Ca^{41}, using Yamaguchi's potential and replacing the nuclear mass M by an effective mass M^*, demonstrated that neglect of the Pauli principle produced results only slightly different from those obtained with it taken into account. On this basis, assuming that neglect of the Pauli principle does not alter the reaction matrix drastically,[24] the following form was employed:

$$(\mathbf{p}|G|\mathbf{p}')=(\mathbf{p}|V|\mathbf{p}')+2M^*\int\frac{(\mathbf{p}|V|\mathbf{k})(\mathbf{k}|G|\mathbf{p}')}{p'^2-k^2}d^3k. \tag{19}$$

With a separable potential, this form is exactly soluble. The resulting expressions for the various potentials are supplied in Appendix A.

Current work in progress shows that saturation can best be achieved using the even-l potential forms given in Appendix A, combined with the odd-l potentials of Tabakin.[13] This occurs for a self-consistent effective-mass value of approximately $0.6M$. All calculations have been done using this value.

In most cases, the nuclear energy levels have also been determined by using the approximate reaction matrix in place of the potential matrix. This enables one to study the degree of improvement, if any, offered by such a replacement. If there were little change, one

[19] P. Goldhammer, Rev. Mod. Phys. **35**, 40 (1963).
[20] S. A. Moszkowski, in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39.
[21] K. A. Brueckner and W. Wada, Phys. Rev. **103**, 1008 (1956); K. A. Brueckner and J. L. Gammel, *ibid.* **109**, 1023 (1958); K. A. Brueckner, A. M. Lockett, and M. Rotenberg, *ibid.* **121**, 255 (1961).
[22] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).
[23] B. P. Nigam and M. K. Sundersan, Can. J. Phys. **36**, 571 (1958).
[24] E. L. Lomon and M. McMillan, Ann. Phys. (N. Y.) **23**, 439 (1963).

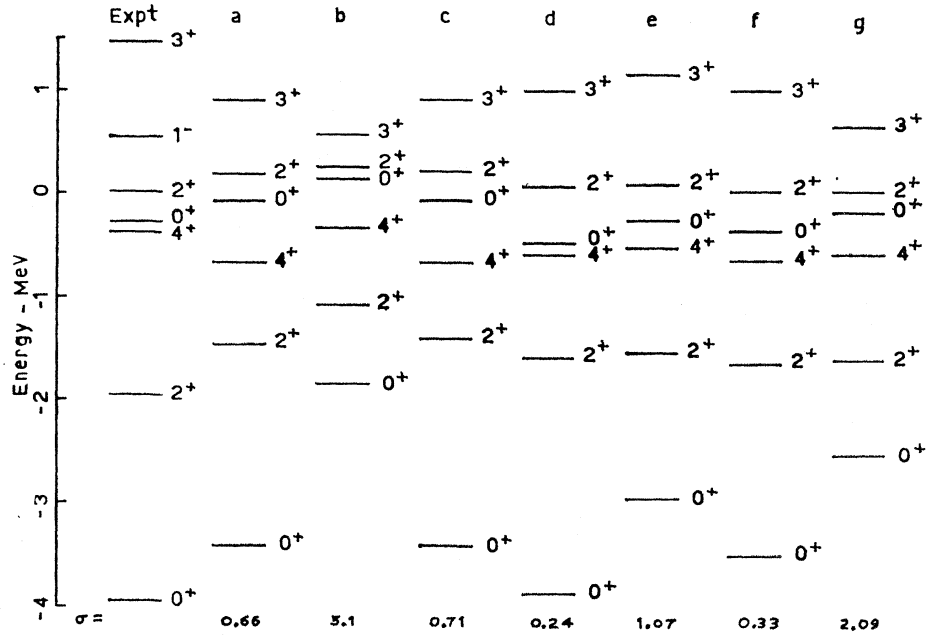

FIG. 1. Energy levels of O^{18} determined using the potential matrix. For comparison are the results of (a) Engeland and Kallio (Ref. 26) and (b) McKellar (Ref. 17). (c) Yamaguchi's singlet potential only. (d) SE and TO: Mitra-Naqvi; (e) SE: Mitra-Naqvi, TO: Tabakin; (f) SE: Naqvi, TO: Mitra-Naqvi; (g) SE: Naqvi, TO: Tabakin. (Here SE=singlet-even, TO=triplet-odd.) Experimental levels are from J. L. Wiza, R. Middleton, and P. V. Hewka, Phys. Rev. 141, 975 (1966).

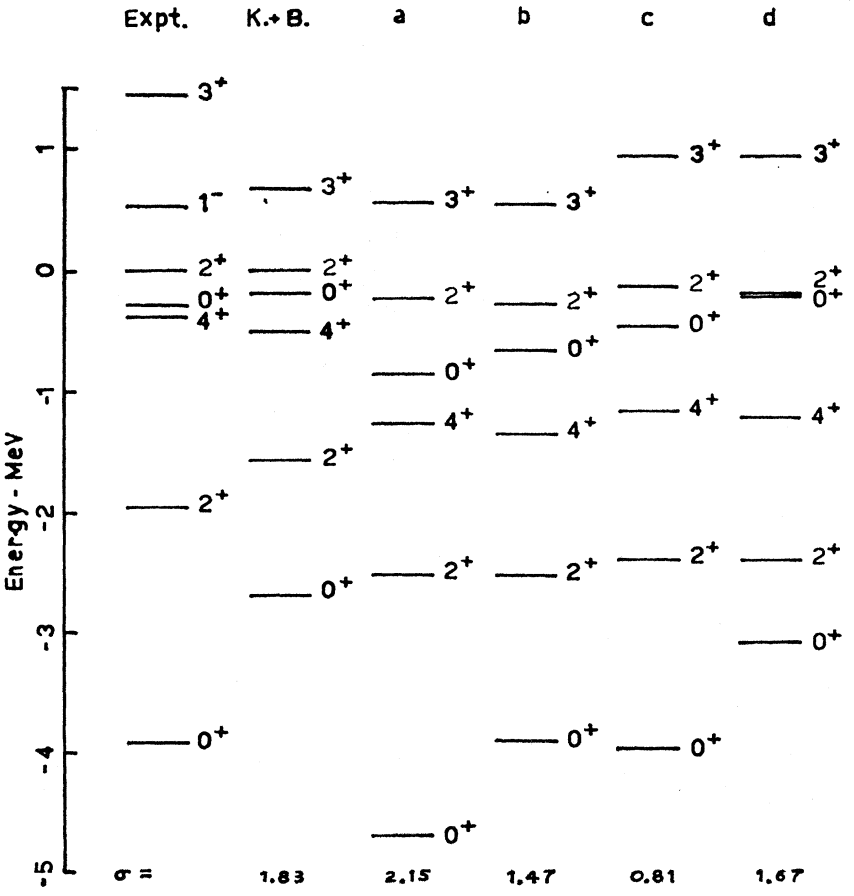

FIG. 2. Energy levels of O^{18} determined using the approximate reaction matrix. The reaction matrix results of Kuo and Brown (Ref. 14) using the Hamada-Johnstone potential are included. (a) SE and TO: Mitra-Naqvi; (b) SE: Naqvi, TO: Mitra-Naqvi; (c) SE: Mitra-Naqvi, TO: Tabakin; (d) SE: Naqvi, TO: Tabakin.

could conclude that the potential form used is a good approximation to the reaction matrix.

The actual mechanics of this work proceed along the same lines as outlined above for the potential matrix. However, as well as encountering the integrals X_{2n}, Y_{2n}, and Z_{2n}, one also encounters the following:

$$X_{2n}'=\int_0^\infty A(p)p^{(2n+2)}(\beta^2+p^2)^{-1}e^{-\frac{1}{2}a^2p^2}dp,$$
$$Y_{2n}'=\int_0^\infty A(p)p^{(2n+4)}(\beta^2+p^2)^{-2}e^{-\frac{1}{2}a^2p^2}dp, \quad (20)$$
$$Z_{2n}'=\int_0^\infty A(p)p^{(2n+1)}(\beta^2+p^2)^{-1}e^{-\frac{1}{2}a^2p^2}dp.$$

Here $A(p)$ is used to refer to the function encountered in defining the appropriate reaction matrix. The integrals were evaluated numerically.

5. CALCULATIONS AND RESULTS

The energy levels of five nuclei having two nucleons outside a closed shell have been calculated using the potentials described in Appendix A. For the isospin triplet ($T=1$) case, the levels of O^{18}, Ca^{42}, Zr^{90}, and Po^{210} are examined. Only F^{18} is considered for $T=0$.

A. $T=1$

This case involves only the singlet-even and triplet-odd potentials.

O^{18}

This nucleus consists of two neutrons outside the closed-shell (O^{16}) core. These are considered to occupy the $0d_{5/2}$, $1s_{1/2}$, and $0d_{3/2}$ levels, and their interaction with the core is taken empirically from the level positions of O^{17}. The O^{18} ground state is placed at B.E.(O^{18})$-$B.E.(O^{16})-2[B.E.(O^{17})$-$B.E.(O^{16})], where B.E. means binding energy.

In an attempt to study the effect of "improving" the potential, various approximations were employed. The simplest case was the singlet potential of Yamaguchi. Here,

$$(\mathbf{p}|V|\mathbf{p}')=-(\lambda_s/M)g(p)g(p'),$$
$$g(p)=2/(p^2+\beta_s^2), \quad (21)$$
$$\beta_s=1.254\sqrt{2}\times10^{13}\ \text{cm}^{-1}, \quad \beta_s^3/(\pi^2\lambda_s)=8.536.$$

These parameters were taken from Puff.[25] The energy levels obtained are displayed in Fig. 1.

Following this, various combinations of potentials were employed. The singlet potential of Naqvi[15] and the triplet potential of Tabakin[13] were combined with the forms of Mitra and Naqvi.[7] These results are also shown in Fig. 1, and the matrix elements for Mitra and Naqvi's potentials are presented in the first row of Table I.

[25] R. D. Puff, Ann. Phys. (N. Y.) **13**, 317 (1961).

TABLE I. Matrix elements for O^{18}. The first line corresponds to the potential matrix of Mitra and Naqvi. The second line was calculated using the approximate reaction matrix with Naqvi's singlet and Tabakin's triplet potentials. Kuo and Brown's reaction matrix values are in the third line.

	$0d_{5/2}^2$	$1s_{1/2}^2$	$0d_{3/2}^2$
		$J=0$	
$0d_{5/2}^2$	−3.38 −1.53 −1.47	−0.88 −1.32 −0.76	−1.48 −4.37 −3.18
$1s_{1/2}^2$		−2.51 −3.17 −2.28	−0.72 −1.08 −0.62
$0d_{3/2}^2$			−2.78 +0.25 −0.17

	$d_{5/2}^2$	$d_{5/2}s_{1/2}$	$d_{5/2}d_{3/2}$	$s_{1/2}d_{3/2}$	$d_{3/2}^2$
			$J=2$		
$d_{5/2}^2$	−0.55 −1.00 −0.89	−0.61 −0.85 −0.52	−1.03 −0.56 −0.52	−0.48 −0.70 −0.51	−0.74 −0.93 −0.63
$d_{5/2}s_{1/2}$		−1.34 −1.50 −1.01	−0.65 −0.22 −0.17	−1.02 −1.92 −1.38	−0.23 −1.01 −0.69
$d_{5/2}d_{3/2}$			−0.10 −0.35 −0.21	−0.08 −0.96 −0.59	−0.27 −0.90 −0.74
$s_{1/2}d_{3/2}$				−0.93 −0.72 −0.45	−0.66 −0.08 −0.03
$d_{3/2}^2$					−0.62 −0.05 −0.23

	$d_{5/2}^2$	$d_{5/2}d_{3/2}$
	$J=4$	
$d_{5/2}^2$	−0.35 −0.51 −0.33	−1.00 −1.48 −0.97
$d_{5/2}d_{3/2}$		−1.84 −2.74 −1.79

	$d_{5/2}s_{1/2}$	$d_{5/2}d_{3/2}$
	$J=3$	
$d_{5/2}s_{1/2}$	0.19 0.20 −0.16	0.58 −0.70 −0.03
$d_{5/2}d_{3/2}$		−0.05 0.22 −0.32

Figure 2 shows the levels of O^{18} calculated using the reaction matrix. For comparison, the levels obtained by Kuo and Brown[14] using the reaction matrix without core excitation terms for the Hamada-Johnstone potential are also given. These matrix elements are also compared with the ones obtained in Table I.

In Fig. 1 comparison is made with the results of Engeland and Kallio,[26] and of McKellar.[17] The former

[26] T. Engeland and A. Kallio, Nucl. Phys. **59**, 211 (1964).

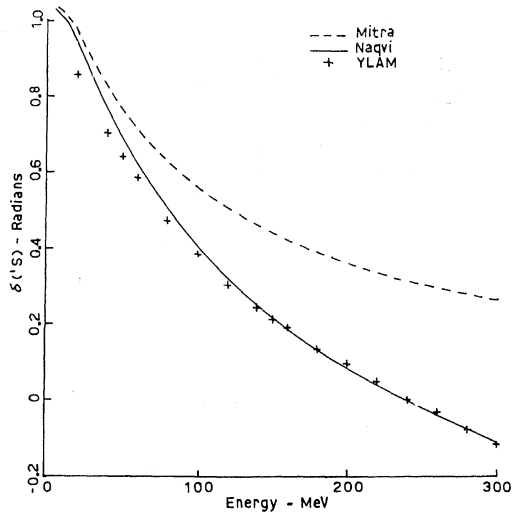

FIG. 3. The singlet-S phase shifts from Refs. 7 and 15. The points are from the YLAM fit; G. Breit *et al.*, Phys. Rev. **128**, 826 (1962).

employed a hard-core potential with short-range attraction acting only in S states, whereas the latter used a nonseparable form of velocity-dependent potential, both fitted to low-energy two-body data.

The over-all agreement is good. The best results are found using the potential matrix and those potentials which give good agreement with the low-energy two-body data, i.e., the singlet-even and triplet-odd potentials of Mitra and Naqvi. Replacing the singlet potential by that of Naqvi decreases agreement slightly. This should be expected since Naqvi's form was not adjusted to data below about 30 MeV. The two $^1S_0$ phase shifts are compared in Fig. 3.

One means of comparing the energy levels obtained with the experimental ones is to calculate the total square deviation σ. This has been done for O^{18}, considering only the lowest five energy levels (0^+, 2^+, and 4^+), which the simple shell model may be expected to fit. Values for σ are given in Figs. 1 and 2. Results obtained using the Brueckner-Gammel-Thaler potential[27] give a value of approximately 1.8 (MeV)^2 for σ, while the results of Lee and Baranger using Tabakin's potential and configuration mixing give $\sigma \simeq 2.3$ (MeV)^2 for the lowest three levels only.

The results obtained using the potential matrix and Tabakin's triplet-odd potential are in relatively poor agreement. This potential form was designed to reproduce the high-energy phase-shift data, but also contributes to the required P-wave repulsion in nuclear matter. Preliminary results with the potential of Mitra and Naqvi indicate that it may not be as suitable in this last respect. Furthermore, it is only capable of reproducing the low-energy $^3P_0$ eigen-phase shift.

[27] J. F. Dawson, I. Talmi, and J. D. Walecka, Ann. Phys. (N. Y.) **18**, 339 (1962).

Replacing the potential matrix with the reaction matrix decreases the level agreement in the cases where Mitra and Naqvi's triplet form is used, but improves it for Tabakin's. In fact, the results found using Tabakin's triplet and Mitra and Naqvi's singlet potentials are quite good except for the 4^+ level, which is low. This behavior is a function of the off-diagonal properties of the potential.

The appearance of a negative-parity state (1^-) suggests that near these energies core excitations begin to play a role. Recently, very good level agreement has been obtained for O^{18} by including that state formed by raising two protons from the $p_{1/2}$ orbital into the $d_{5/2}$ or $s_{1/2}$ orbitals.[28] Such analyses have correctly obtained three 0^+ and three 2^+ states below 6-MeV excitation.

Ca^{42}

The ground state of Ca^{41} is a $\frac{7}{2}^-$ level corresponding to the $f_{7/2}$ orbital. The first few excited states are found[29] to be $\frac{3}{2}^-(1.95)$, $\frac{3}{2}^+(2.02)$, $\frac{3}{2}^-(2.47)$, and $\frac{1}{2}^+(2.68)$. These are quite closely spaced. Note that there are two $\frac{3}{2}^-$ levels separated by only 0.52 MeV. This splitting of the $p_{3/2}$ orbital has not been successfully explained to date.[30]

Hence our shell-model assumption that one can consider only the $f_{7/2}$ and $p_{3/2}$ orbitals outside the closed shell is probably faulty, and one should take core excitation into account. With our assumption of no core excitation, calculations for Ca^{42} are analogous to those for O^{18}. The ground state was taken at -3.112 MeV, and 1.87 F was used as the value of the oscillator parameter.

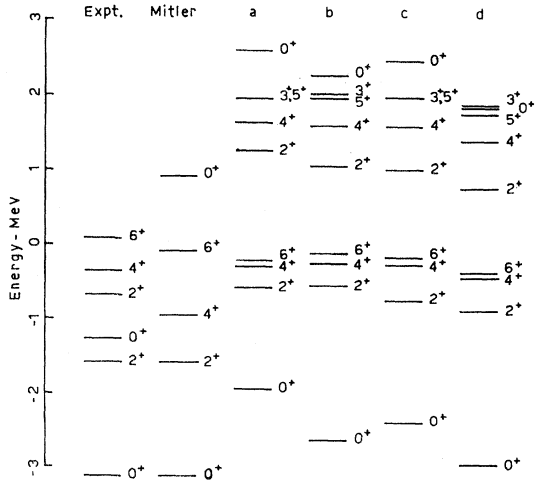

FIG. 4. Energy levels of Ca^{42}. Mitler's results taken from Ref. 31. (a) Yamaguchi's singlet potential only; (b) SE: Mitra-Naqvi, potential matrix; (c) SE: Naqvi, potential matrix; (d) SE: Mitra-Naqvi, reaction matrix. The last three use the triplet form of Mitra-Naqvi. Experimental results are from P. C. Rogers and G. E. Gordon, Phys. Rev. **129**, 2653 (1963).

[28] T. Engeland, Nucl. Phys. **72**, 68 (1965); P. Federman and I. Talmi, Phys. Letters **19**, 490 (1965).

[29] R. Bock, H. H. Duhm, and R. Stock, Phys. Letters **18**, 61 (1965).

[30] L. L. Lee *et al.*, Phys. Rev. Letters **10**, 496 (1963).

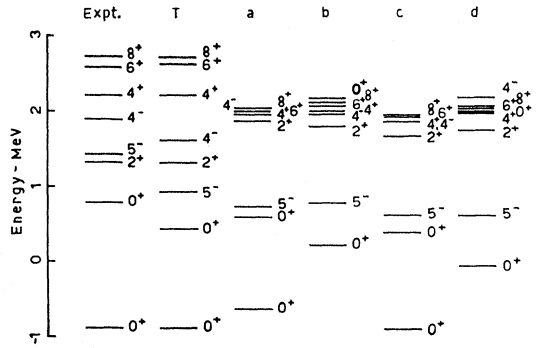

FIG. 5. Energy levels of Zr^{90}. T: results from Ref. 36; (a) and (c) SE and TO: Mitra-Naqvi; (b) and (d) SE: Mitra-Naqvi, TO: Tabakin. The reaction matrix was used in (c) and (d). Experimental values are from D. L. Hendrie and G. W. Farwell, Phys. Letters **9**, 321 (1964).

The results, using a few potential combinations, are presented in Fig. 4. For comparison, the results of Mitler[31] are also shown. Table II gives the matrix elements together with the effective interaction values of Engeland and Osnes.[32]

The levels obtained are in very poor agreement with experiment. This is a general feature of shell-model calculations with this nucleus. The low-lying 0^+(1.84 MeV) and 2^+(2.42 MeV) levels cannot be reproduced with any realistic potential. The levels obtained by Mitler were found by adjusting the potential parameters to reproduce the levels for several Ca isotopes. Good agreement could not be obtained. The inadequacy of the potential calculation is also reflected in the large discrepancy of the matrix elements with those of Engeland and Osnes.

The anomalies of Ca^{41} strongly indicate that core excitation will play a large role in the Ca^{42} spectrum. Indeed, recent investigations by Federman[33] have shown that the inclusion of collective core effects gives excellent results even with a Gaussian potential and Rosenfeld mixture. This is in line with experimental work[34] which indicates that the second 0^+ state is due to core excitation, and that there is considerable interaction between the two 2^+ states. Because of these results, it was not thought worthwhile pursuing a simple shell-model calculation with this nucleus.

Zr^{90}

This nucleus has two protons outside a closed shell (Sr^{88}). Only those configurations formed from the $2p_{1/2}$ and $1g_{9/2}$ orbitals are considered, with mixing. Single-particle levels are found from Y^{89}, with $\Delta(g_{9/2}-p_{1/2})=0.908$ MeV,[35] and a was taken as 2.13 F. The matrix elements of the Coulomb force were calculated in a straightforward manner following the procedure of Moshinsky.[18] The results are presented in Fig. 5 and Table III.

The first obvious feature is the poor level agreement obtained using Tabakin's triplet-odd potential. The second is the inability of Mitra and Naqvi's potential form to reproduce the level separation for the higher-energy levels and the correct positioning of the odd-parity levels. Only the correct ordering of the even-parity states is achieved. There was little difference between the results shown and those found using Naqvi's singlet potential. Furthermore, varying a produced no improvement.

The good results obtained by Thankappan *et al.*[36] are misleading, as the parameters of the local Gaussian potential used were adjusted to obtain the best separation for the 2^+, 4^+, and 8^+ levels.

TABLE II. Matrix elements for Ca^{42}.[a]

	Engeland and Osnes[b]	Potential matrix[c]	Reaction matrix[c]
E_0	−3.115	−2.50	−2.79
E_2	−0.887	−0.42	−0.69
E_4	−0.170	−0.26	−0.43
E_6	+0.166	−0.18	−0.40
E_0'	···	−1.76	−2.26
E_2'	···	−0.26	−0.56
F_0	···	−0.76	−0.96
F_2	···	−0.12	−0.21
G_2	−0.939	−0.43	−0.52
G_4	−0.560	−0.15	−0.26
G_2'	···	−0.14	−0.31
E_2''	−2.037	−1.02	−1.36
E_4''	−0.501	−0.37	−0.60

[a] $E_J'=\langle(p_{3/2})^2J\,|\,V\,|\,(p_{3/2})^2J\rangle$, $J=0,2$
$E_J=\langle(f_{7/2})^2J\,|\,V\,|\,(f_{7/2})^2J\rangle$, $J=0,2,4,6$
$F_J=\langle(f_{7/2})^2J\,|\,V\,|\,(p_{3/2})^2J\rangle$, $J=0,2$
$G_J=\langle(f_{7/2})^2J\,|\,V\,|\,(f_{7/2}p_{3/2})J\rangle$, $J=2,4$
$G_2'=\langle(p_{3/2})^22\,|\,V\,|\,(f_{7/2}p_{3/2})2\rangle$,
$E_J''=\langle(f_{7/2}p_{3/2})J\,|\,V\,|\,(f_{7/2}p_{3/2})J\rangle$, $J=2,3,4,5$.
[b] See Ref. 32.
[c] The singlet and triplet potentials are those of Mitra and Naqvi.

TABLE III. Matrix elements for Zr^{90}.[a]

	Auerbach and Talmi[b]	Potential matrix[c]	Reaction matrix[c]
E_0	−0.557	−0.49	−0.75
E_0'	−1.77	−1.37	−1.59
E_2'	−0.57	0.02	−0.16
E_4'	0.22	0.11	0.01
E_6'	0.55	0.16	0.07
E_8'	0.79	0.20	0.08
F_0	−0.863	0.37	0.41
E_4''	···	1.13	0.94
E_5''	···	−0.20	−0.32

[a] $E_0=\langle(p_{1/2})^20\,|\,V\,|\,(p_{1/2})^20\rangle$,
$E_J'=\langle(g_{9/2})^2J\,|\,V\,|\,(g_{9/2})^2J\rangle$, $J=0,2,4,6,8$
$F_0=\langle(p_{1/2})^20\,|\,V\,|\,(g_{9/2})^20\rangle$,
$E_J''=\langle(p_{1/2}g_{9/2})J\,|\,V\,|\,(p_{1/2}g_{9/2})J\rangle$, $J=4,5$.
[b] See Ref. 37.
[c] The singlet and triplet potentials are those of Mitra and Naqvi. The Coulomb elements are included.

[31] H. E. Mitler, Nucl. Phys. **23**, 200 (1961).
[32] T. Engeland and E. Osnes, Phys. Letters **20**, 424 (1966).
[33] P. Federman, Phys. Letters **20**, 174 (1966).
[34] J. H. Bjerregaard, H. R. Blieden, O. Hansen, G. Sidenius, and G. R. Satchler, Phys. Rev. **136**, B1348 (1964).
[35] D. M. Van Patter and S. M. Shafroth, Nucl. Phys. **50**, 113 (1964).
[36] V. K. Thankappan, Y. R. Waghmare, and S. P. Pandya, Progr. Theoret. Phys. (Kyoto) **26**, 22 (1961).

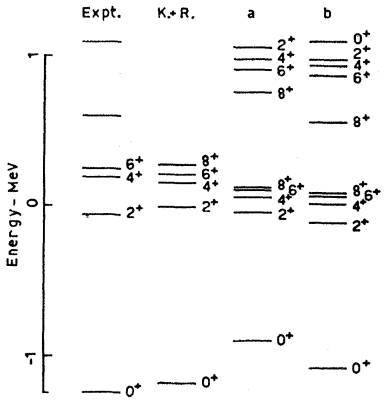

FIG. 6. Energy levels of Po^{210}. K.+R.: results from Ref. 38. The levels are determined using Mitra and Naqvi's potentials, with the reaction matrix used in (b). Experimental values are from F. Schima *et al.*, Phys. Rev. **132**, 2650 (1963).

On comparing the matrix elements with those of Auerbach and Talmi,[37] who performed an effective-interaction calculation in the Zr region, one sees that, with the exception of the sign of F_0, the general behavior is reproduced, but the potential elements are smaller in magnitude. The reaction matrix has the effect of increasing the magnitude of the negative elements in the right direction, but has the opposite effect on the positive elements.

Po^{210}

In this case the two external protons can occupy the $h_{9/2}$ and $f_{7/2}$ orbitals. Bi^{209} gives $\Delta(f_{7/2}-h_{9/2})=0.91$ MeV, and $a=2.45$ F was used. The required matrix elements are listed in Table IV.

Once again, Tabakin's triplet potential produced very poor results, and only Mitra and Naqvi's levels are presented in Fig. 6. The level ordering of the first few levels is correct, but the separation is too small. They are compared with the results of Kim and Rasmussen,[38] who used a potential without a hard core, which was deduced from the free two-nucleon potentials of Blatt and Jackson, and Brueckner, Gammel, and Thaler.

TABLE IV. Matrix elements for Po^{210}.

$E_J=\langle(h_{9/2})^2J\mid V\mid(h_{9/2})^2J\rangle$,	$J=0, 2, 4, 6, 8$
$E_J'=\langle(f_{7/2})^2J\mid V\mid(f_{7/2})^2J\rangle$,	$J=0, 2, 4, 6$
$F_J=\langle(h_{9/2})^2J\mid V\mid(f_{7/2})^2J\rangle$,	$J=0, 2, 4, 6$
$G_J=\langle(h_{9/2})^2J\mid V\mid(h_{9/2}f_{7/2})J\rangle$,	$J=2, 4, 6, 8$
$G_J'=\langle(f_{7/2})^2J\mid V\mid(h_{9/2}f_{7/2})J\rangle$,	$J=2, 4, 6$
$E_J''=\langle(h_{9/2}f_{7/2})J\mid V\mid(h_{9/2}f_{7/2})J\rangle$,	$J=1, 2, \cdots, 8$

B. $T=0$

The only nucleus to which the singlet isospin potential was applied was F^{18}. With a neutron and a proton outside the O^{16} core, the ground-state energy was taken as

$$\text{B.E.}(F^{18})-\text{B.E.}(O^{16})-[\text{B.E.}(F^{17})-\text{B.E.}(O^{16})]-[\text{B.E.}(O^{17})-\text{B.E.}(O^{16})]=-5.023\ \text{MeV},$$

and the single-particle separation $\Delta(s_{1/2}-d_{5/2})$ is 0.50 MeV.

TABLE V. Matrix elements for F^{18} $(T=0)$.[a]

	Results for Naqvi's triplet potential				
	Central	Central-tensor	Tensor	Spin-orbit	Total
E_1	−1.856	0.530	−0.051	0.017	−1.360
E_3	−1.195	0.259	−0.022	−0.043	−1.000
E_5	−2.740	0	0	−0.027	−2.768
E_1'	−2.775	0	−0.103	0	−2.878
F_1	−0.816	0.206	−0.024	0	−0.635
F_3	−1.152	−0.044	0.015	−0.023	−1.204
E_2'	−0.888	0.084	−0.050	0.022	−0.832
E_3'	−2.219	−0.060	−0.010	−0.110	−2.399

[a] $E_J=\langle(d_{5/2})^2J\mid V\mid(d_{5/2})^2J\rangle$, $J=1, 3, 5$
$E_1'=\langle(s_{1/2})^21\mid V\mid(s_{1/2})^21\rangle$,
$F_1=\langle(d_{5/2})^21\mid V\mid(s_{1/2})^21\rangle$,
$F_3=\langle(d_{5/2})^23\mid V\mid(d_{5/2}s_{1/2})3\rangle$,
$E_J'=\langle(d_{5/2}s_{1/2})J\mid V\mid(d_{5/2}s_{1/2})J\rangle$, $J=2, 3$.

Figure 7 shows the results obtained using the triplet-even potential of Naqvi,[10] and the singlet-odd potential of Tabakin. These were calculated using the same configurations as for O^{18}. The correct ordering is obtained and the separation of the lowest three levels is in good agreement. These results are compared with the reaction matrix values of Kuo and Brown. Calculations performed using the approximate reaction matrix for the potentials used here gave very poor results, increasing the level separation considerably.

Table V lists the triplet-even matrix elements only for those configurations formed from the $d_{5/2}$ and $s_{1/2}$ orbitals. Configurations with $d_{3/2}$ were included in calculating the energy levels.

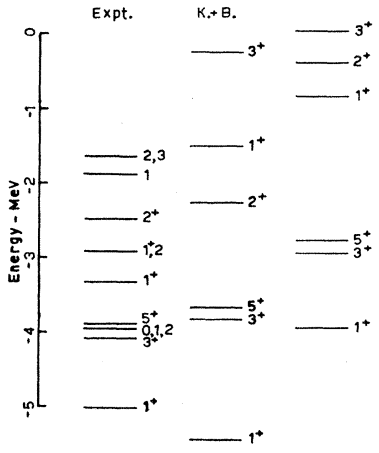

FIG. 7. Energy levels of F^{18}. K.+B.: reaction matrix values from Ref. 14. The levels are calculated using Naqvi's triplet-even and Tabakin's singlet-odd potentials, with the potential matrix. Experimental levels are taken from A. R. Polletti and E. K. Warburton, Phys. Rev. **137**, B595 (1965).

[37] N. Auerbach and I. Talmi, Nucl. Phys. **64**, 458 (1965); see also J. Vervier, *ibid.* **75**, 17 (1966).
[38] Y. E. Kim and J. O. Rasmussen, Nucl. Phys. **47**, 184 (1963).

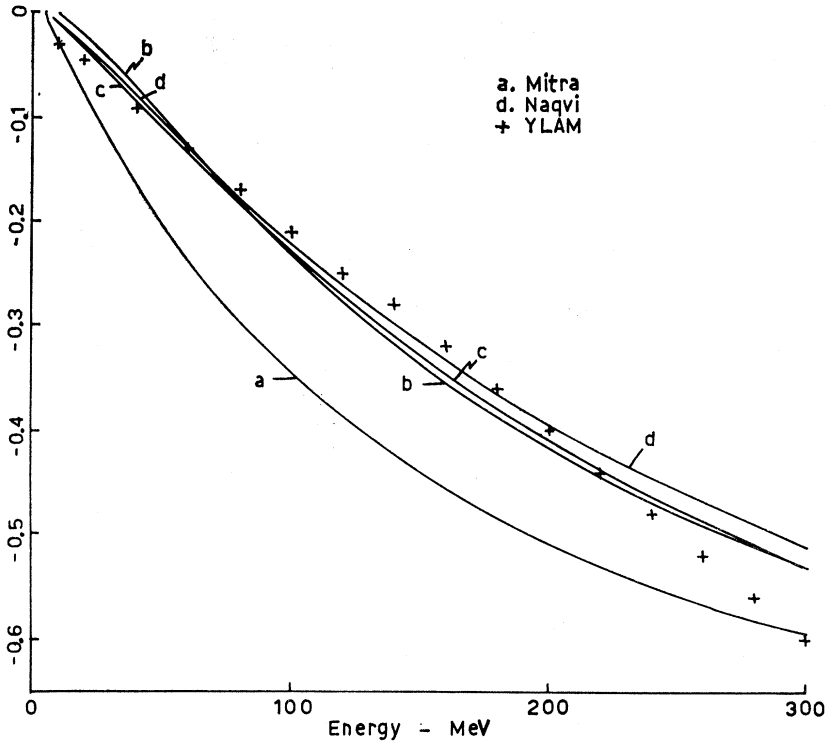

FIG. 8. The singlet-P phase shift, in radians, for various potentials. Curve b is due to the parameters of potential 5 in Table VI, and curve c is due to potential 4. The points refer to the YLAM experimental fit; G. Breit *et al.*, Phys. Rev. **128**, 830 (1962).

The singlet-odd potential form of Naqvi[15] produces excessively large matrix elements. This difficulty was also noted by Mitra and Pandya[39] with their form. One would expect from scattering data and the deuteron, for example, that the singlet-odd contribution would be small compared with the triplet-even. The potential itself seems to be poorly defined by the two-nucleon data, as has been found in the case of the hard-core potential.[40]

Table VI, part (a) illustrates the wide variation obtained for two matrix elements using Naqvi's form. The parameters all give quite reasonable $^1P_1$ phase shifts. Some of these are presented in Fig. 8.

TABLE VI. Singlet-odd matrix elements.

	$\lambda_1^{(0)}$	ν_1	t_1'	E_1	E_3
		(a) Potential matrix			
1.	-2.66α	5.8α	0.7	14.22	3.21
2.	-9.0α	5.85α	0.7	46.91	10.55
3.	-0.3α	4.6α	4.0	14.65	2.84
4.	-0.07α	4.2α	5.0	6.24	1.24
5.	-5.5α	5.8α	0.7	29.40	6.63
	Tabakin's potential[a]			2.05	0.35
		(b) Reaction matrix			
1.	-2.66α	5.8α	0.7	1.75	0.40
2.	-9.0α	5.85α	0.7	1.88	0.43
3.	-0.3α	4.6α	4.0	1.72	0.34
4.	-0.07α	4.2α	5.0	1.68	0.33
5.	-5.5α	5.8α	0.7	1.87	0.42
	Tabakin's potential[a]			2.23	0.40

[a] See Ref. 13.

Introduction of the reaction matrix leads to a considerable reduction of Naqvi's singlet matrix elements, as seen in Table VI, part (b). They are of the same order of magnitude as the triplet-even ones. Note the small deviation of these results with differing parameters, compared with the results using the potential matrix.

Thus, potentials which give consistently good results for the phase shift produce similar results for the reaction matrix, but wildly varying results for the potential matrix. Some of the reasons for this can be seen from the functions involved. Consider, for example,

$$(\mathbf{p}|V|\mathbf{p}')=g(p)g(p'),$$

for which the reaction matrix is given by

$$(\mathbf{p}|G|\mathbf{p}')=A(p')g(p)g(p'),$$

$$A(p')=\left[1+\int\frac{g^2(p'')}{p'^2-p''^2}d^3p''\right]^{-1}.$$

Note that $A(p)$ is a functional of $g(p)$, and as such tends to have a compensating effect on the variations of $g(p)$. For example, for fairly large values of p, multiplication of $g(p)$ by $A(p)$ decreases the value of the function. For small p, it has the opposite effect. $A(p)$ remains of the order of unity for the potential forms used, being slightly greater than 1 for small momenta, and less for large momenta.

[39] A. N. Mitra and S. P. Pandya, Nucl. Phys. **20**, 455 (1960); **29**, 352(E) (1962).
[40] N. Azziz and P. Signell, Nucl. Phys. **59**, 444 (1964).

It is thus seen that if $g(p)$ decreases slowly with increasing p, the potential elements can be quite large and sensitive to its behavior. However, the reaction matrix, and the phase shifts given by its diagonal elements, will necessarily be smaller and have much smaller variations.

This is borne out by the calculations performed using the $^1P_1$ potential of Tabakin. His functions fall off more rapidly with increasing momentum than those of Naqvi. As a result, his potential matrix elements are smaller than Naqvi's [see Table VI, part (a)], and his reaction matrix elements are of the same order.

It would naturally be expected that the level agreement for F^{18} could be improved by using more satisfactory potential forms; however, the results obtained with the inclusion of collective states for O^{18} and Ca^{42} suggest that a similar treatment be attempted on F^{18}.

6. CONCLUSION

The level calculations carried out using a separable potential and the two-body potential matrix as the residual interaction give quite reasonable results considering the limitations of the simple shell model employed. In those cases (O^{18}, F^{18}) where comparison has been made with results obtained using realistic hard-core potentials, favorable agreement is found for the low-lying levels. The discrepancy with experimental results found in each case is due to lack of consideration of collective states, insufficient consideration of configuration mixing, imprecise potential forms, or some combination of these effects. It would also be advantageous if one had more direct means of testing the off-diagonal behavior of the potential, in order, for example, that the ambiguities in the singlet-odd and triplet-odd states could be resolved.

The influence which the approximate reaction matrix had on the energy levels was dependent on the potential form used. When applied to the smooth odd-state potentials of Tabakin, the change in the matrix elements was small. On the other hand, considerable differences were found for Naqvi's $^1P_1$ form.

It would appear quite reasonable to expect that calculations using a modified separable form, similar to the form employed here, and including collective motion from core excitation, could give excellent results for energy-level calculations. The calculations involved offer an extreme saving of computation over similar work using hard cores.

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APPENDIX A

Listed below are the potential forms employed by various authors, along with the parameter values determined by experimental data. Included in most cases are the corresponding reaction matrices. Note that these are given for the common representation where the relative momentum is given by $\mathbf{p}=\frac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)$.

Triplet-Even

(1)

$$(\mathbf{p}|\,V^{(0)}\,|\mathbf{p}')=-\lambda^{(0)}/Mg(\mathbf{p})g(\mathbf{p}'). \tag{A1}$$

Yamaguchi[5] introduced two forms. The simplest used $g(p)=(p^2+\beta^2)^{-1}$ with $\beta=6.255\alpha$, $\lambda^{(0)}=33.37\alpha^3$, and $\alpha^2/M=2.225$ MeV. The other form included the tensor force:

$$\begin{gathered} g(\mathbf{p})=C(p)+8^{-1/2}T(p)S(\hat{p}), \\ S(\hat{p})=3(\boldsymbol{\sigma}_1\cdot\hat{p})(\boldsymbol{\sigma}_2\cdot\hat{p})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2), \\ C(p)=(p^2+\beta^2)^{-1}, \quad T(p)=-tp^2/(\gamma^2+p^2)^2, \end{gathered} \tag{A2}$$

with $\beta=5.759\alpha$, $\gamma=6.771\alpha$, $t=1.784$, and $\lambda^{(0)}=20.04\alpha^3$. In the first case, the parameters were determined by the experimental scattering length and effective range. Values of the deuteron D-state probability and quadrupole moment were included in the parameter evaluation in the second case.

(2) To the Yamaguchi tensor form (A2), Mitra and Narasimham[6] added a spin-orbit term

$$\begin{gathered} -5\lambda_2^{(0)}/Mv_2^{(0)}(p)v_2^{(0)}(p')(\mathbf{L}\cdot\mathbf{S})P_2(\hat{p}\cdot\hat{p}'), \\ v_2^{(0)}(p)=p^2/(\delta^2+p^2)^2. \end{gathered} \tag{A3}$$

By assuming that the tensor and spin-orbit forces have the same "range" (i.e., $\gamma=\delta$), exact expressions for the scattering length, effective range, deuteron binding energy, quadrupole and magnetic moments, and the D-state probability are employed to determine useful parameter values. They give

$$\begin{aligned} &\lambda^{(0)}=24.1\alpha^3, \quad \beta=5.8\alpha, \quad t=0.75, \\ &\lambda_2^{(0)}=-20.1\alpha^3, \quad \gamma=\delta=6.0\alpha. \end{aligned}$$

By dropping the assumption that $\gamma=\delta$, Naqvi[10] has taken $\gamma=5.8\alpha$ and $\delta=12\alpha$, and assumed that $\beta=\gamma$. These values, together with $\lambda^{(0)}=22.9\alpha^3$ and $\lambda_2^{(0)}=-119.15\alpha^3$, give a good account of the low-energy scattering parameters, deuteron binding energy, quad-

rupole moment, and magnetic moment with relativistic effect included. The spin-orbit force is of short range.

(3) The reaction matrix for the last form is given by MN.

Singlet-Even

(1) Mitra and Naqvi[7] employ the form

$$(\mathbf{p}|V^{(1)}|\mathbf{p}')=\sum_{l=0,2}(2l+1)(-\lambda_l{}^{(1)}/M)\times v_l{}^{(1)}(p)v_l{}^{(1)}(p')P_l(\hat{p}\cdot\hat{p}') \quad \text{(A4)}$$

using

$$\lambda_0{}^{(1)}=18.6\alpha^3,\quad \beta_0=5.8\alpha,\quad v_0{}^{(1)}(p)=(p^2+\beta_0{}^2)^{-1},$$
$$\lambda_2{}^{(1)}=10.5\alpha^3,\quad \beta_2=5.8\alpha,\quad v_2{}^{(1)}(p)=p^2(p^2+\beta_2{}^2)^{-2}.$$

These were determined by examining the low-energy scattering parameters and the p-p cross sections, polarizations, and phase shifts at various energies.

Corresponding to this,

$$(\mathbf{p}|G_l{}^{(1)}|\mathbf{p}')=\bar{A}_l{}^{(1)}(\mathbf{p}|V^{(1)}|\mathbf{p}'),$$
$$\bar{A}_l{}^{(1)}(p')=[1-(M^*/M)\lambda_l{}^{(1)}J_l(p')]^{-1},$$
$$J_l(p')=\int\frac{v_l{}^{(1)2}(p'')}{p''^2-p'^2}d^3p''. \quad \text{(A5)}$$

(2) Naqvi[14] fits the singlet phase-shift data, using for $l=0$:

$$(p|V_0{}^{(1)}|p')=(-\lambda_0{}^{(1)}/M)[v_0{}^{(1)}(p)v_0{}^{(1)}(p')-t'v^{(1)}(p)v^{(1)}(p')],$$
$$v_0{}^{(1)}(p)=(p^2+\beta_0{}^2)^{-1},$$
$$v^{(1)}(p)=p^2(p^2+\bar{\beta}_0{}^2)^{-2}, \quad \text{(A6)}$$
$$\lambda_0{}^{(1)}=62.4\alpha^3,\quad t'=4.986,$$
$$\beta_0=\bar{\beta}_0=8\alpha,$$

and for $l=2$,

$$(\mathbf{p}|V_2{}^{(1)}|\mathbf{p}')=-5(\lambda_2{}^{(1)}/M)\times v_2{}^{(1)}(p)v_2{}^{(1)}(p')P_2(\hat{p}\cdot\hat{p}'),$$
$$v_2{}^{(1)}(p)=p\beta_2{}^{(1)-1}(p^2+\beta_2{}^{(1)2})^{-1}, \quad \text{(A7)}$$
$$\beta_2{}^{(1)}=8\alpha,\quad \lambda_2{}^{(1)}=12.48\alpha^3.$$

For $l=2$, the reaction matrix takes the same form as in 1, whereas for $l=0$,

$$(\mathbf{p}|G_0{}^{(1)}|\mathbf{p}')=\left(-\frac{\lambda_0{}^{(1)}}{M}\right)D^{-1}\{[1-(M^*/M)t'\mu_{01}]v_0{}^{(1)}(p)v_0{}^{(1)}(p')$$
$$-t'[1+(M^*/M)\mu_0]v^{(1)}(p)v^{(1)}(p')+(M^*/M)t'\mu_{02}$$
$$\times(v^{(1)}(p)v_0{}^{(1)}(p')+v_0{}^{(1)}(p)v^{(1)}(p'))\}, \quad \text{(A8)}$$

with

$$\mu_0(p')=\lambda_0{}^{(1)}\int\frac{v_0{}^{(1)2}(p)}{p'^2-p^2}d^3p,$$
$$\mu_{01}(p')=\lambda_0{}^{(1)}\int\frac{v^{(1)2}(p)}{p'^2-p^2}d^3p, \quad \text{(A9)}$$
$$\mu_{02}(p')=\lambda_0{}^{(1)}\int\frac{v_0{}^{(1)}(p)v^{(1)}(p)}{p'^2-p^2}d^3p.$$

(3) Alternatively, Puff[25] gives parameters for Yamaguchi's simple form:

$$\beta_s=5.42\alpha,\quad \lambda_s=15.12\alpha^3.$$

Triplet-Odd

Mitra and Naqvi[7] propose the form

$$(\mathbf{p}|V_t{}^{(1)}|\mathbf{p}')=-\frac{4\pi\lambda_1{}^{(1)}}{M}v_1{}^{(1)}(p)v_1{}^{(1)}(p')\times\sum_{m=-1}^{1}Y_{1m}{}^*(\hat{p})(\mathbf{L}\cdot\mathbf{S})Y_{1m}(\hat{p}'), \quad \text{(A10)}$$
$$v_1{}^{(1)}(p)=p(p^2+\beta_1{}^2)^{-1},$$

with $\lambda_1{}^{(1)}=0.288\alpha$, $\beta_1=6\alpha$, determined together with the parameters for the singlet-even form above.

Singlet-Odd

$$(\mathbf{p}|V_s{}^{(0)}|\mathbf{p}')=-\frac{3\lambda_1{}^{(0)}}{M}v_1{}^{(0)}(p)v_1{}^{(0)}(p')P_1(\hat{p}\cdot\hat{p}'). \quad \text{(A11)}$$

With this form, Mitra and Pandya[39] employ

$$v_1{}^{(0)}(p)=p(p^2+\beta_1{}^2)^{-1},$$

with

$$\lambda_1{}^{(0)}=-1.7\alpha,\quad \beta_1=5\alpha;$$

whereas Naqvi[14] fits the ${}^1P_1$ phase shift using $v_1{}^{(0)}(p)=p(\nu_1{}^2+p^2)^{-1}+t_1'p^3(p^2+\bar{\nu}_1{}^2)^{-2}$, $\lambda_1{}^{(0)}=-2.66\alpha$, $t_1'=0.7$, and $\nu_1=\bar{\nu}_1=5.8\alpha$.

The reaction matrix takes the same form as for Mitra and Naqvi's singlet-even form.

APPENDIX B

Here are summarized the matrix elements for the tensor and spin-orbit potentials. The potential form (A1) is expanded as

$$g(\mathbf{p})g(\mathbf{p}')=C(p)C(p')+8^{-1/2}[C(p)T(p')S(\hat{p}')+T(p)S(\hat{p})C(p')]+8^{-1}T(p)S(\hat{p})T(p')S(\hat{p}'). \quad \text{(B1)}$$

The evaluation of the central-tensor terms is carried out in the usual manner, making use of

$$8^{-1/2}S(\hat{p})\chi_{m_s}=(4\pi)^{1/2}\sum_{\nu}Y_2{}^{\nu}(\hat{p})\chi_{m_s-\nu}\times\langle 2\nu 1m_s-\nu|1m_s\rangle. \quad \text{(B2)}$$

This leads to the result, in LS coupling,

$$\langle n_1l_1,n_2l_2,\lambda;S;JM|-\frac{\lambda}{M\sqrt{8}}[C(p)T(p')S(\hat{p}')+T(p)S(\hat{p})C(p')]|n_1'l_1',n_2'l_2',\lambda';S;JM\rangle$$

$$=-4\sqrt{3}\pi\frac{\lambda}{M}\begin{Bmatrix}\lambda & J & 1\\ 1 & 2 & \lambda'\end{Bmatrix}[(-)^{\lambda'+J+1}(2\lambda'+1)^{1/2}\sum_{nn'N}\langle n0N\lambda,\lambda|n_1l_1,n_2l_2,\lambda\rangle\langle n'2,N\lambda,\lambda'|n_1'l_1',n_2'l_2',\lambda'\rangle$$

$$\times\langle n0\|C(p)T(p')\|n'2\rangle+(-)^{\lambda+J+1}(2\lambda+1)^{1/2}\sum_{nn'N}\langle n2,N\lambda',\lambda|n_1l_1,n_2l_2,\lambda\rangle\langle n'0,N\lambda',\lambda'|n_1'l_1',n_2'l_2',\lambda'\rangle$$

$$\times\langle n2\|T(p)C(p')\|n'0\rangle]. \quad \text{(B3)}$$

The same procedure for the tensor matrix element results in

$$\langle n_1l_1,n_2l_2,\lambda;S;JM|-\frac{\lambda}{8M}T(p)S(\hat{p})T(p')S(\hat{p}')|n_1'l_1',n_2'l_2',\lambda';S;JM\rangle$$

$$=-12\pi(\lambda/M)(2\lambda+1)^{1/2}(2\lambda'+1)^{1/2}(-)^{\lambda'+J}\sum_{n'nNL}\begin{Bmatrix}1 & J & L\\ \lambda & 2 & 1\end{Bmatrix}\begin{Bmatrix}1 & J & L\\ \lambda' & 2 & 1\end{Bmatrix}\langle n2,NL,\lambda|n_1l_1,n_2l_2,\lambda\rangle$$

$$\times\langle n'2,NL,\lambda'|n_1'l_1',n_2'l_2',\lambda'\rangle\langle n2\|T(p)T(p')\|n'2\rangle. \quad \text{(B4)}$$

For the spin-orbit term, use of the identity

$$\sum_{\nu}\chi_{Sm_S}{}^*Y_{L^\nu}{}^*(\hat{p})(\mathbf{L}\cdot\mathbf{S})Y_{L^\nu}(\hat{p}')\chi_{Sm_S'}=\sum_{\nu}(-)^{m_S'-m_S}[L(L+1)S(S+1)]^{1/2}\langle Sm_S1m_S'-m_S|Sm_S'\rangle$$

$$\times\langle L,\nu+m_S'-m_S,1,m_S-m_S'|L\nu\rangle Y_{L^\nu}{}^*(\hat{p})Y_{L^{\nu+m_S'-m_S}}(\hat{p}') \quad \text{(B5)}$$

gives, for the LS matrix element of (A3),

$$-24(\sqrt{5})\pi\frac{\lambda_2{}^{(0)}}{M}(2\lambda+1)^{1/2}(2\lambda'+1)^{1/2}\sum_{nn'NL}(-)^{J+L+1}\begin{Bmatrix}\lambda & 1 & \lambda'\\ 2 & L & 2\end{Bmatrix}\begin{Bmatrix}\lambda & 1 & \lambda'\\ 1 & J & 1\end{Bmatrix}\langle n2,NL,\lambda|n_1l_1,n_2l_2,\lambda\rangle$$

$$\times\langle n'2,NL,\lambda'|n_1'l_1',n_2'l_2',\lambda'\rangle\langle n2\|v_2{}^{(0)}(p)v_2{}^{(0)}(p')\|n'2\rangle. \quad \text{(B6)}$$