Relative-Angular-Momentum-Zero Part of Two-Nucleon Wave Functions*

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A method is given for finding the relative-angular-momentum-zero part of the wave function of two particles moving in a finite single-particle potential. The results are applied to form factors for two-nucleon transfer reactions and to two-nucleon interaction matrix elements.

I. INTRODUCTION

MANY calculations in nuclear physics involve the use of two-particle shell-model wave functions. The most familiar example is the calculation of the matrix elements of a two-particle interaction. Another example is the distorted-wave Born-approximation (DWBA) treatment of direct two-particle transfer processes, such as the (t,p) or (He^3,p) reactions.

These shell-model wave functions are products of functions of the variables of the separate particles. However, the calculation can usually be performed more simply if the wave function is expressed in terms of the relative and center-of-mass (c.m.) variables of the two particles. Talmi[1] and Moshinsky[2] have developed methods for effecting this transformation to relative and center-of-mass variables when the single-particle wave functions are characteristic of an infinite harmonic oscillator well. These methods, especially when used in conjunction with the transformation brackets tabulated by Brody and Moshinsky,[3] have enormously facilitated many calculations in nuclear physics.

However, there are some circumstances in which one must use wave functions characteristic of a finite potential well. This is the case in Hartree-Fock calculations, in which the single-particle wave functions are eigenfunctions in the self-consistent potential, which is of course finite. Also, Drisko and Rybicki[4] have emphasized that the proper treatment of transfer processes, which are sensitive to the nuclear wave functions in the vicinity of the nuclear surface and beyond, requires wave functions with the asymptotic behavior of a finite well.

This problem of performing the transformation to relative and c.m. coordinates with the single-particle wave functions of a finite well has been treated by first expanding the finite-well wave functions in terms of harmonic-oscillator wave functions of varying numbers of nodes, and then performing the Talmi-Moshinsky transformation on the individual harmonic oscillator components.[4,5] The number of these components required depends upon the radius out to which one wishes to approximate the wave function, and upon the accuracy one desires. In this paper, we describe a method for performing the transformation which works directly with the finite-well wave functions, and does not involve the harmonic oscillator expansion. It is thus free from uncertainties about how many terms to include. It is limited, however, to that part of the wave function in which the two-particles have relative angular momentum zero. This is sufficient for many purposes.

II. DERIVATION OF THE FORMULAS

We begin with a normalized two-particle shell-model wave function

$$\Psi_M{}^{n_1l_1j_1;n_2l_2j_2;I,T}(\mathbf{r}_1\sigma_1,\mathbf{r}_2\sigma_2)=\frac{[\psi^{n_1l_1j_1}(\mathbf{r}_1,\sigma_1)\psi^{n_2l_2j_2}(\mathbf{r}_2,\sigma_2)]_M{}^I+(-1)^T[\psi^{n_1l_1j_1}(\mathbf{r}_2,\sigma_2)\psi^{n_2l_2j_2}(\mathbf{r}_1,\sigma_1)]_M{}^I}{[2(1+\delta_{n_1n_2}\delta_{l_1l_2}\delta_{j_1j_2})]^{1/2}}. \quad (1)$$

Here T is the isobaric spin, zero if the state is symmetric in the two-particles and unity if it is antisymmetric. The bracket notation indicates vector coupling to total angular momentum I and its z component M. The single-particle states $\psi_m{}^{nlj}(\mathbf{r},\sigma)$ have the form

$$\psi_m{}^{nlj}(\mathbf{r},\sigma)\equiv[\phi^l(\mathbf{r})\chi^{1/2}(\sigma)]_m{}^j, \quad (2a)$$

$$\phi_m{}^l(\mathbf{r})\equiv u_{nlj}(r)Y_m{}^l(\hat{r}). \quad (2b)$$

For convenience, we have suppressed the n and j dependence of the orbital functions $\phi_m{}^l(\mathbf{r})$. The radial functions $u_{nlj}(r)$ are normalized solutions of the radial Schrödinger equation with the chosen single-particle potential. We use the Condon-Shortley[6] phase convention for the spherical harmonics.

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Our aim is to find the part of the wave function (1) in which the two-particles have zero relative angular momentum. To this end we expand in an L-S-coupling representation

$$\Psi_M{}^{n_1 l_1 j_1; n_2 l_2 j_2; I, T}(\mathbf{r}_1\sigma_1, \mathbf{r}_2\sigma_2) = \sum_{L,S} ((l_1\tfrac{1}{2})_{j_1}(l_2\tfrac{1}{2})_{j_2} | (l_1 l_2)_L(\tfrac{1}{2}\tfrac{1}{2})_S)_I \times \frac{\{([\varphi^{l_1}(\mathbf{r}_1)\phi^{l_2}(\mathbf{r}_2)]^L + (-1)^{1+S+T}[\phi^{l_1}(\mathbf{r}_2)\phi^{l_2}(\mathbf{r}_1)]^L)[\chi^{1/2}(\sigma_1)\chi^{1/2}(\sigma_2)]^S\}_M{}^I}{[2(1+\delta_{n_1 n_2}\delta_{l_1 l_2}\delta_{j_1 j_2})]^{1/2}}. \quad (3)$$

The part of the wave function in which the two particles have zero relative angular momentum is permutation-symmetric in $\mathbf{r}_1$ and $\mathbf{r}_2$. Thus if $T=1$ we keep only the $S=0$ term and if $T=0$ we keep only the $S=1$ term. We then introduce relative and center-of-mass coordinates defined by

$$\mathbf{r} \equiv (\mathbf{r}_2 - \mathbf{r}_1)/\sqrt{2}, \quad (4a)$$

$$\mathbf{R} \equiv (\mathbf{r}_2 + \mathbf{r}_1)/\sqrt{2}, \quad (4b)$$

and envisage an expansion of the form

$$\frac{[\phi^{l_1}(\mathbf{r}_1)\phi^{l_2}(\mathbf{r}_2)]_M{}^L + [\phi^{l_1}(\mathbf{r}_2)\phi^{l_2}(\mathbf{r}_1)]_M{}^L}{[2(1+\delta_{n_1 n_2}\delta_{l_1 l_2}\delta_{j_1 j_2})]^{1/2}} = \sum_{\lambda,\Lambda} \frac{f_{\lambda,\Lambda}{}^L(r,R)}{rR}[Y^\lambda(\hat{r})Y^\Lambda(\hat{R})]_M{}^L. \quad (5)$$

To obtain $f_{0,L}{}^L(r,R)$ we set $\hat{R}=\hat{z}$ and integrate (5) over $d\hat{r}$. This integration picks out the $\lambda=0$, $\Lambda=L$ term on the right-hand side. Since

$$Y_M{}^\Lambda(\hat{z}) = \delta_{M,0}\left(\frac{2\Lambda+1}{4\pi}\right)^{1/2}, \quad Y_0{}^0(\hat{r}) = \frac{1}{(4\pi)^{1/2}},$$

we get

$$f_{0,L}{}^L(r,R) = \frac{rR}{(2L+1)^{1/2}} \times \int_{(\hat{R}=\hat{z})} \frac{[\phi^{l_1}(\mathbf{r}_1)\phi^{l_2}(\mathbf{r}_2)]_M{}^L + [\phi^{l_1}(\mathbf{r}_2)\phi^{l_2}(\mathbf{r}_1)]_M{}^L}{[2(1+\delta_{n_1 n_2}\delta_{l_1 l_2}\delta_{J_1 J_2})]^{-/2}} d\hat{r}. \quad (6)$$

The "distribution function" $f_{0,L}{}^L(r,R)$ obtained by performing the integral in (6) contains all the information about the relative-angular-momentum-zero part of the two-particle wave function (1).

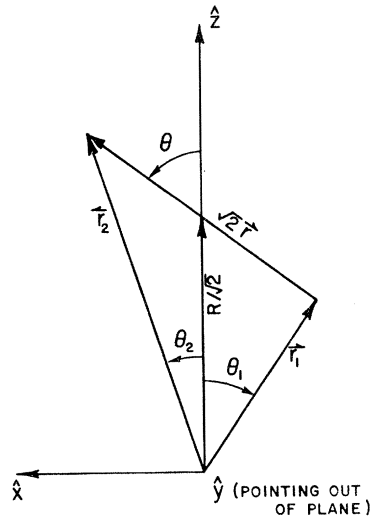

FIG. 1. The integration of equation of Eq. (6) is performed with $\mathbf{R}=R\hat{z}$ and $\mathbf{r}_1$ and $\mathbf{r}_2$ in the x-z plane.

To integrate over $\hat{\mathbf{r}}(=\theta,\phi)$, we set $M=0$ in (6) since $f_{0,L}{}^L(r,R)$ is independent of M. With $M=0$ and $\mathbf{R}=R\hat{\mathbf{z}}$ the integrand in (6) is independent of ϕ. It is sufficient then to allow the $\mathbf{r}_1$, $\mathbf{r}_2$ plane to coincide with the $\hat{\mathbf{x}}$, $\hat{\mathbf{z}}$ plane, as shown in Fig. 1. Since

$$\mathbf{r} = r\cos\theta\,\hat{\mathbf{z}} + r\sin\theta\,\hat{\mathbf{x}},$$

it follows that

$$\mathbf{r}_1 = \frac{\mathbf{R}-\mathbf{r}}{\sqrt{2}} = \frac{(R - r\cos\theta)\hat{\mathbf{z}} - r\sin\theta\hat{\mathbf{x}}}{\sqrt{2}},$$

$$\mathbf{r}_2 = \frac{\mathbf{R}+\mathbf{r}}{\sqrt{2}} = \frac{(R + r\cos\theta)\hat{\mathbf{z}} + r\sin\theta\hat{\mathbf{x}}}{\sqrt{2}}.$$

Thus we can express $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of r, R, and θ as follows:

$$r_1 = \left(\frac{(R - r\cos\theta)^2 + (r\sin\theta)^2}{2}\right)^{1/2} = \left(\frac{R^2 + r^2 - 2rR\cos\theta}{2}\right)^{1/2}, \quad (7a)$$

$$\cos\theta_1 = (R - r\cos\theta)/\sqrt{2}r_1, \quad \phi_1 = \pi$$

$$r_2 = \left(\frac{(R + r\cos\theta)^2 + (r\sin\theta)^2}{2}\right)^{1/2} = \left(\frac{R^2 + r^2 + 2rR\cos\theta}{2}\right)^{1/2}, \quad (7b)$$

$$\cos\theta_2 = (R + r\cos\theta)/\sqrt{2}r_2, \quad \phi_2 = 0.$$

The explicit expressions for the spherical harmonics are

$$Y_m{}^l(\theta,\phi) = e^{im\phi} d_{m,0}{}^l(\theta)\left(\frac{2l+1}{4\pi}\right)^{1/2}, \quad (8a)$$

$$d_{m,0}{}^l(\theta) = \sum_{n=m}^{l} (-1)^n \frac{l![(l+m)!(l-m)!]^{1/2}}{n!(l+m-n)!(l-n)!(n-m)!} \times (\cos\tfrac{1}{2}\theta)^{2l+m-2n}(\sin\tfrac{1}{2}\theta)^{2n-m}, \quad (8b)$$

in terms of which we write

$$[Y^{l_1}(\hat{r}_1)Y^{l_2}(\hat{r}_2)]_0{}^L=\sum_m (l_1l_2m-m\,|\,L0)Y_m{}^{l_1}(\theta_1,\pi)Y_{-m}{}^{l_2}(\theta_2,0)$$

$$=\left[\frac{(2l_1+1)(2l_2+1)}{(4\pi)^2}\right]^{1/2}\sum_m (l_1l_2m-m\,|\,L0)e^{im\pi}d_{m,0}{}^{l_1}(\theta_1)d_{-m,0}{}^{l_2}(\theta_2) \tag{9}$$

$$=\left[\frac{(2l_1+1)(2l_2+1)}{(4\pi)^2}\right]^{1/2}\sum_m (l_1l_2m-m\,|\,L0)d_{m,0}{}^{l_1}(\theta_1)d_{m,0}{}^{l_2}(\theta_2)\,.$$

Expression (6) then becomes

$$f_{0,L}{}^L(r,R)=rR\left[\frac{(2l_1+1)(2l_2+1)}{(4\pi)^2(2L+1)}\right]^{1/2}2\pi\int_0^\pi \sin\theta d\theta\sum_m (l_1l_2m-m\,|\,L0)$$

$$\times\frac{u_l\,(r_1)u_{l_2}(r_2)d_{m,0}{}^{l_1}(\theta_1)d_{m,0}{}^{l_2}(\theta_2)+u_{l_1}(r_2)u_{l_2}(r_1)d_{m,0}{}^{l_1}(\theta_2)d_{m,0}{}^{l_2}(\theta_1)}{[2(1+\delta_{n_1n_2}\delta_{l_1l_2}\delta_{j_1j_2})]^{1/2}}. \tag{10}$$

According to (7a) and (7b), replacing θ by $\pi-\theta$ interchanges $(r_1, \cos\theta_1)$ and $(r_2, \cos\theta_2)$. Since $\frac{1}{2}\theta_{1,2}$ in (8b) are confined to the interval between 0 and $\frac{1}{2}\pi$, so that $\cos\frac{1}{2}\theta_{1,2}$ and $\sin\frac{1}{2}\theta_{1,2}$ remain positive, this operation will also interchange $d_{m,0}{}^l(\theta_1)$ and $d_{m,0}{}^l(\theta_2)$. Thus the two terms in the integrand make the same contribution to the integral. Furthermore, since

$$(l_1l_2m-m\,|\,L0)=(-1)^{l_1+l_2-L}(l_1l_2-mm\,|\,L0)\,, \tag{11a}$$

$$d_{-m,0}{}^l(\theta)=(-1)^m d_{m,0}{}^l(\theta)\,, \tag{11b}$$

the m and $-m$ terms in the integrand of (10) differ only by a factor of $(-1)^{l_1+l_2-L}$. Finally, then, we get

$$f_{0,L}{}^L(r,R)=\frac{1+(-1)^{l_1+l_2-L}}{2}\left[\frac{(2l_1+1)(2l_2+1)}{2(2L+1)(1+\delta_{n_1n_2}\delta_{l_1l_2}\delta_{j_1j_2})}\right]^{1/2}\times r\times R\times\int_{-1}^1 dx\; u_{l_1}(r_1)u_{l_2}(r_2)$$

$$\times[(l_1l_200\,|\,L0)d_{0,0}{}^{l_1}(\theta_1)d_{0,0}{}^{l_2}(\theta_2)+2\sum_{m>0}(l_1l_2m-m\,|\,L0)d_{m,0}{}^{l_1}(\theta_1)d_{m,0}{}^{l_2}(\theta_2)], \tag{12}$$

in which

$$r_1=\left(\frac{R^2+r^2-2rRx}{2}\right)^{1/2},$$

$$r_2=\left(\frac{R^2+r^2+2rRx}{2}\right)^{1/2}, \tag{13a}$$

$$\left.\begin{matrix}\cos\frac{1}{2}\theta_1\\ \sin\frac{1}{2}\theta_1\end{matrix}\right\}=\left(\frac{1}{2}\pm\frac{R-rx}{2\sqrt{2}r_1}\right)^{1/2},$$

$$\left.\begin{matrix}\cos\frac{1}{2}\theta_2\\ \sin\frac{1}{2}\theta_2\end{matrix}\right\}=\left(\frac{1}{2}\pm\frac{R+rx}{2\sqrt{2}r_2}\right)^{1/2}. \tag{13b}$$

The x integration in (12) was performed numerically in a manner described in Sec. III.

III. NUMERICAL CALCULATION OF THE DISTRIBUTION FUNCTION

We evaluate the integral in (12) by means of the Gauss-Legendre quadrature formula.[7] This approximates a definite integral by a sum of N_0 terms.

[7] Zdenek Kopal, *Numerical Analysis* (Chapman and Hall Ltd., London, 1955).

$$\int_{-1}^1 g(x)dx\cong\sum_{i=1}^{N_0}\omega_i g(x_i)\,. \tag{14}$$

The weight coefficients ω_i and the abscissas x_i are tabulated for various values of N_0.

In an actual calculation we usually need $f_{0,L}{}^L(r,R)$ at many (r,R) points. Since we must perform a separate x integration at each (r,R) point, we would like to minimize the number of terms N_0 in the approximation (14) without sacrificing too much accuracy. We can be guided in our choice of the minimum acceptable N_0 by considering the case in which the radial wave functions in (2b) are those appropriate to the 3-dimensional harmonic oscillator potential. We will show that the integrand in (2) [equivalently in (6) with $M=0$] is then a polynomial in x, in which case it is possible to find a value of N_0 for which (14) ceases to be an approximation and becomes exact. This value of N_0 should then also serve well when the radial wave functions are those of a finite potential.

Thus we now assume that we have harmonic oscillator radial functions. To see that the integrand in (6) is then a polynomial in x, we expand it with the aid of the Brody-Moshinsky transformation brackets into

vector-coupled products of functions of $\mathbf{r}$ and $\mathbf{R}$;

$$\left(\frac{[\varphi^{n_1l_1}(\mathbf{r}_1)\phi^{n_2l_2}(\mathbf{r}_2)]_0{}^L+[\phi^{n_1l_1}(\mathbf{r}_2)\phi^{n_2l_2}(\mathbf{r}_1)]_0{}^L}{[2(1+\delta_{n_1n_2}\delta_{l_1l_2}\delta_{j_1j_2})]^{1/2}}\right)_{\mathbf{R}=R\hat{z}}=\sum_{\nu\lambda N\Lambda}\frac{(n_1l_1,n_2l_2|\,\nu\lambda,N\Lambda)_L}{[2(1+\delta_{n_1n_2}\delta_{l_1l_2}\delta_{j_1j_2})]^{1/2}}$$
$$\times([\phi^{\nu\lambda}(\mathbf{r})\phi^{N\Lambda}(R\hat{\mathbf{z}})]_0{}^L+[\phi^{\nu\lambda}(-\mathbf{r})\phi^{N\Lambda}(R\hat{\mathbf{z}})]_0{}^L)=\sum_{\nu\lambda N\Lambda}\frac{(n_1l_1,n_2l_2|\,\nu\lambda,N\Lambda)_L}{[2(1+\delta_{n_1n_2}\delta_{l_1l_2}\delta_{j_1j_2})]^{1/2}}$$
$$\times[1+(-1)^\lambda]u_{\nu\lambda}(r)u_{N\Lambda}(R)[Y^\lambda(\hat{\mathbf{r}})Y^\Lambda(\hat{\mathbf{r}})]_0{}^L=\sum_{\nu\lambda N\Lambda}\left(\frac{2\Lambda+1}{4\pi}\right)^{1/2}(\lambda\Lambda00|\,L0)$$
$$\times\frac{(n_1l_1,n_2l_2|\,\nu\lambda,N\Lambda)_L}{[2(1+\delta_{n_1n_2}\delta_{l_1l_2}\delta_{j_1j_2})]^{1/2}}[1+(-1)^\lambda]u_{\nu\lambda}(r)u_{N\Lambda}(R)Y_0{}^\lambda(\hat{\mathbf{r}}). \quad (15)$$

Since $Y_0{}^\lambda(\hat{r})=[(2\lambda+1)/4\pi]^{1/2}P_\lambda(x)$ is a polynomial of degree λ in x, the sum in (14) is a polynomial in x whose degree equals the greatest value of λ ($\lambda_{\max}$) occurring in the sum. This will occur when ν and N are both zero, and Λ is as small as it can be, consistent with the requirement that (λ,Λ,L) form a triangle, namely, $\Lambda=\lambda_{\max}-L$. Since the total number of quanta is preserved in the Brody-Moshinsky transformation, we then have

$$2n_1l_1+2n_2l_2=\lambda_{\max}+\lambda_{\max}-L,$$
$$\lambda_{\max}=n_1+n_2+\tfrac{1}{2}(l_1+l_2+L),$$

for the degree in x of the polynomial integrand (12). Under these circumstances, the expression (14) is exact provided that N_0 is sufficiently large that

$$2N_0-1\geqslant\lambda_{\max}, \quad (16)$$
$$N_0\geqslant\tfrac{1}{4}[2(n_1+n_2)+l_1+l_2+L+2].$$

For example, for two-particles in the $i_{13/2}$ orbital near the Fermi surface of Pb^{208}, we would need

$$N_0\geqslant\tfrac{1}{4}(14+L),$$

so that an N_0 of 7 would cover all possible cases. Equations (12) and (14) would then yield exact expressions for the distribution function for harmonic oscillator wave functions, and good approximations for the distribution function when other reasonable radial wave functions are used. Using Woods-Saxon radial functions in this case, we found that changing N_0 form 4 to 8 had very little effect on the calculated $f_{L,0}(r,R)$. Numbers of the order of 10^{-1} were changed by about 2×10^{-6}, whereas numbers of the order of 10^{-10} were changed by about 2×10^{-13}.

IV. FORM FACTORS FOR A 2-PARTICLE TRANSFER REACTION

In a zero-range DWBA treatment of pickup from or stripping into the two-particle state (1), we need to evaluate form factors $F_{L,S}(r_3)$ defined by

$$\int d\mathbf{r}_1d\mathbf{r}_2\delta(\mathbf{r}_3-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2))\Phi_{\text{He}^3\text{ or He}^3}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$$
$$\Psi_M{}^{n_1l_1j_1,n_2l_2j_2;I}(\mathbf{r}_1\sigma_1,\mathbf{r}_2\sigma_2)=\sum_{L,S}F_{L,S}(r_3)$$
$$\times\{Y^L(\hat{\mathbf{r}}_3)[\chi^{1/2}(\sigma_1)\chi^{1/2}(\sigma_2)]^S\}_M{}^I. \quad (17)$$

We take the three-particle (triton or He^3) internal orbital wave function to be

$$\Phi_{\text{H}^3\text{ or He}^3}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=\left(\frac{216\kappa^6}{\pi^3}\right)^{1/2}e^{-\kappa^2[(\mathbf{r}_1-\mathbf{r}_2)^2+(\mathbf{r}_2-\mathbf{r}_3)^2+(\mathbf{r}_3-\mathbf{r}_1)^2]}$$
$$=\left(\frac{216\kappa^6}{\pi^3}\right)^{1/2}e^{-3\kappa^2[r^2+\rho^2]}, \quad (18)$$

where r was defined in (4a) and ρ is defined by

$$\boldsymbol{\rho}=\frac{2\mathbf{r}_3-\mathbf{r}_1-\mathbf{r}_2}{\sqrt{6}}=(\sqrt{\tfrac{2}{3}})\left[\mathbf{r}_3-\frac{\mathbf{R}}{\sqrt{2}}\right].$$

The mean-square radius of the triton or He^3 described by (18) would be $1/6\kappa^2$. The normalization in (18) is chosen so that

$$\int|\Phi_{\text{H}^3\text{ or He}^3}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)|^2d\mathbf{r}d\boldsymbol{\rho}=1. \quad (19)$$

We use the expansions (3) and (5) to write (17) as

$$\sum_{L,S}((l_1\tfrac{1}{2})_{j_1}(l_2\tfrac{1}{2})_{j_2}|\,(l_1l_2)_L(\tfrac{1}{2}\tfrac{1}{2})_S)_I\int d\mathbf{r}d\mathbf{R}\,\delta\left(\mathbf{r}_3-\frac{\mathbf{R}}{\sqrt{2}}\right)\left(\frac{216\kappa^6}{\pi^3}\right)^{1/2}\exp\left\{-3\kappa^2\left[r^2+\frac{2}{3}\left(\mathbf{r}_3-\frac{\mathbf{R}}{\sqrt{2}}\right)^2\right]\right\}$$
$$\times\sum_{\lambda,\Lambda}\frac{f_{\lambda,\Lambda}{}^L(r,R)}{rR}\{[Y^\lambda(\hat{\mathbf{r}})Y^\Lambda(\hat{\mathbf{R}})]^L[\chi^{1/2}(\sigma_1)\chi^{1/2}(\sigma_2)]^S\}_M{}^I=\left(\frac{864\kappa^6}{\pi^3}\right)^{1/2}\sum_{L,S}((l_1\tfrac{1}{2})_{j_1}(l_2\tfrac{1}{2})_{j_2}|\,(l_1l_2)_L(\tfrac{1}{2}\tfrac{1}{2})_S)_I$$
$$\times\int d\mathbf{r}\,e^{-3\kappa^2r^2}\frac{f_{0,L}{}^L(r,\sqrt{2}r_3)}{rr_3}\{[Y^0(\hat{\mathbf{r}})Y^L(\hat{\mathbf{r}}_3)]^L[\chi^{1/2}(\sigma_1)\chi^{1/2}(\sigma_2)]^S\}_M{}^I.$$

Comparison with (17) then yields

$$F_{L,S}(r_3)=(1728)^{1/2}\frac{\kappa^3}{\pi r_3}((l_1\tfrac{1}{2})_{j_1}(l_2\tfrac{1}{2})_{j_2}|(l_1l_2)_L(\tfrac{1}{2}\tfrac{1}{2})_S)_I \times\int_0^\infty rdre^{-3\kappa^2r^2}f_{0,L}{}^L(r,\sqrt{2}r_3). \quad (20)$$

To evaluate the integral in (20) we make use of the Gauss-Hermite approximation

$$\int_{-\infty}^{\infty}e^{-r^2}g(r)dr\cong\sum_{i=1}^{N/2}H_i[g(a_i)+g(-a_i)], \quad (N \text{ even}) \quad (21)$$
$$\cong\sum_{i=1}^{(N-1)/2}H_i[g(a_i)+g(-a_i)]+H_{(N+1)/2}g(0), \quad (N \text{ odd}).$$

The H_i and a_i are tabulated[7] for various values of N. The approximation (21) becomes an equality if $g(r)$ is a polynomial whose degree does not exceed $2N-1$.

Reference to (13) will show that the integrand in (12) is unchanged if the signs of x and r are both reversed. Thus $f(r,R)=-f(-r,R)$ and the integrand in (20) is an even function of r. If $g(r)=g(-r)$, the approximation (21) becomes

$$\int_0^\infty e^{-r^2}g(r)dr\cong\sum_{i=1}^{N/2}H_ig(a_i), \quad (N \text{ even}) \quad (22)$$
$$\cong\sum_{i=1}^{(N-1)/2}H_ig(a_i)+\tfrac{1}{2}H_{(N+1)/2}g(0), \quad (N \text{ odd})$$

which is the form of the approximation that we need.

Now suppose that the radial wave functions belong to a harmonic oscillator well with range parameters $t\,[\equiv(mw/\hbar)^{1/2}]$. Then

$$f_{0,L}{}^L(r,\sqrt{2}r_3)=e^{-\frac{1}{2}(tr)^2}C(r,\sqrt{2}r_3), \quad (23)$$

where $C(r,\sqrt{2}r_3)$ is a polynomial in r of degree $2(n_1+n_2)+l_1+l_2-L+1$. In this case the integral in (20) can be written

$$\int_0^\infty rdr\,e^{-[3\kappa^2+\frac{1}{2}t^2]r^2}C(r,\sqrt{2}r_3)=\frac{1}{[3\kappa^2+\frac{1}{2}t^2]}\times\sum_{i=1}^{m}H_ia_iC\left(\frac{a_i}{(3\kappa^2+\frac{1}{2}t^2)^{1/2}},\sqrt{2}r_3\right), \quad (24)$$

where m is an integer satisfying

$$m\geqslant\tfrac{1}{2}(n_1+n_2)+\tfrac{1}{4}(l_1+l_2-L+3). \quad (25)$$

Thus for the previously mentioned case of the $(0i)^2$ configuration in Pb, an m of 4 would give exact results if harmonic oscillator wave functions were used in the construction of $f_{0,L}{}^L(r,\sqrt{2}r_3)$.

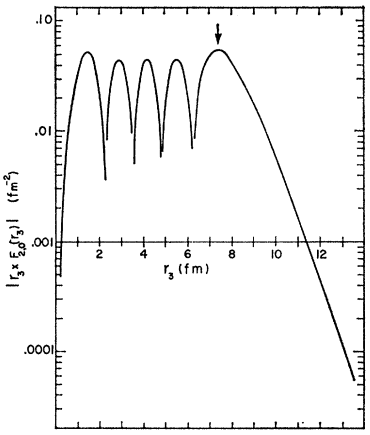

FIG. 2. Form factor for the reaction $Pb^{208}(p,t)Pb^{206}$ leading to the 0.8-MeV 2^+ state in Pb^{206}. The mean-square radius of the triton is taken to be (1.7 F)^2. The arrow indicates the location of the nuclear radius.

If wave functions from a finite well were used, $C(r,\sqrt{2}r_3)$ as defined in (23) would not generally be a polynomial of finite degree in r. However, if the oscillator parameter t used in the definition of $C(r,\sqrt{2}r_3)$ is a reasonable one, then $C(r,\sqrt{2}r_3)$ differs only slightly from a polynomial of degree $2(n_1+n_2)+l_1+l_2-L+1$, and we would expect an accurate result from (24) and (25). Of course, we can improve the accuracy by taking higher values of m. For example, an m of 9 used in (24) would give exact results not only for $n=0$, $l=6$ harmonic oscillator radial wave functions, but for any linear combination of $l=6$ harmonic oscillator radial wave functions with $n\leqslant 5$.

Figure 2 shows the form factor $F_{2,0}(r_3)$ calculated in this way for the reaction $Pb^{208}(p,t)Pb^{206}$ ($E_x=0.803$ MeV). Pb^{208} is assumed to be composed of doubly-closed shells, while the 0.803-MeV 2^+ state in Pb^{206} is assumed to be the linear combination of two-hole states given by True and Ford[8]:

$$\Psi_M{}^2(Pb^{206},\,E_x=0.803)=0.7229\psi_M{}^2(2p_{1/2}{}^{-1},1f_{5/2}{}^{-1}) -0.6017\psi_M{}^2(2p_{1/2}{}^{-1},2p_{3/2}{}^{-1}) +0.2168\psi_M{}^2(1f_{5/2}{}^{-1},1f_{5/2}{}^{-1}) +0.1509\psi_M{}^2(2p_{3/2}{}^{-1},1f_{5/2}{}^{-1}) +0.2134\psi_M{}^2(2p_{3/2}{}^{-1},2p_{3/2}{}^{-1}). \quad (26)$$

The single-particle states[9] are calculated in the Woods-Saxon well[10]:

$$V=V_0\left[\frac{1}{1+\exp[(r-r_0)/a_0]}+\lambda\frac{\hbar^2}{4m^2c^2}\times\frac{\exp[(r-r_0)/a_0]}{\{1+\exp[(r-r_0)/a_0]\}^2a_0r}\times\begin{Bmatrix}l\\-l-1\end{Bmatrix}\right], \quad \text{for } J=\begin{Bmatrix}l+\frac{1}{2}\\l-\frac{1}{2}\end{Bmatrix} \quad (27)$$

[8] W. W. True and K. W. Ford, Phys. Rev. **109**, 1675 (1958).
[9] We are grateful to Professor E. Rost for giving us a computer program to determine these finite-well radial functions.
[10] A. A. Ross, Hans Mark, and R. D. Lawson, Phys. Rev. **102**, 1619 (1956).

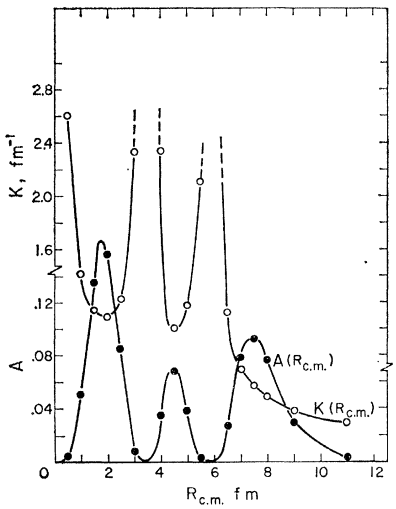

FIG. 3. $A(R_{c.m.})$ and $k(R_{c.m.})$ of Eq. (30) for the $I=0$ state of the $(2p_{3/2})^2$ configuration.

with $r_0=1.25\times(208)^{1/3}$ F, $a_0=0.7$ F, $\lambda=29.0$, and depths chosen to give each single-particle state a separation energy of -7.46 MeV. This gives the correct two-neutron separation energy of -14.92 MeV, which determines the slope of $\ln F_{2,0}(r_3)$ for large r_3. The individual terms in (26) lead to partial form factors with essentially the same shape as that of the total form factor.

The general features of these curves are determined by the exponential decay for large r_3, and the fact that all the two-particle states in (26) correspond to oscillator states with 10 quanta. There can thus be at most 10 quanta of c.m. motion, which, for $L=2$, leads to a c.m. wave function with 5 radial nodes.

V. TWO-PARTICLE INTERACTIONS

The fundamental problem in the theory of nuclear matter is that of two nucleons, moving in the presence of a Fermi sea, and interacting via the true nucleon-nucleon force. We will discuss only the singlet, relative angular-momentum-zero part of this force. The separation method of Scott and Moszkowski[11] has provided a very useful approximation to the solution of this problem. It expresses the nucleon-nucleon potential $v(r)$ as a sum of two terms;

$$v(r)=v_s(r)+v_l(r), \tag{28}$$

where

$$v_s(r)=0 \quad \text{for} \quad r>d,$$
$$v_l(r)=0 \quad \text{for} \quad r<d.$$

The "separation distance" d is a function of the relative momentum k. The separation distance is chosen so that a nucleon of momentum k scattering in the potential $v_s(r)$ has zero phase shift.

If we wish to apply this method to finite nuclei, we must be able to ascribe an effective relative momentum to particles moving in specified shell-model states, at least when they are within about 1 F of each other. Reference 12 presents a natural way of doing this when the shell-model states are those of a harmonic oscillator potential. It is based on an approximate relationship between the harmonic oscillator wave function of relative motion and the spherical Bessel functions. For particles moving in a finite well, we may proceed as follows: Write (12) in terms of the variables

$$\mathbf{r}_{12}\equiv\mathbf{r}_2-\mathbf{r}_1=\sqrt{2}\mathbf{r}, \tag{29a}$$

$$\mathbf{R}_{c.m.}\equiv(\mathbf{r}_2+\mathbf{r}_1)/2=\mathbf{R}/\sqrt{2}, \tag{29b}$$

and for a given value of $R_{c.m.}$, try to express the de-

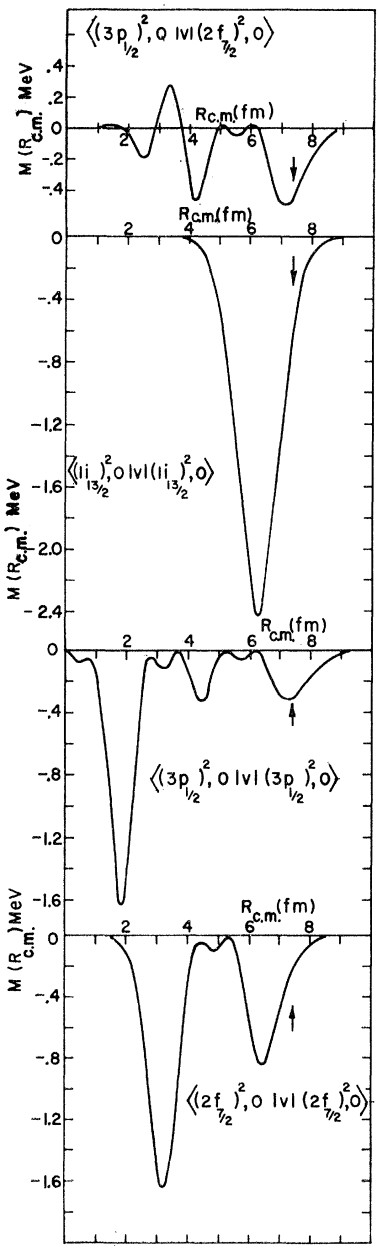

FIG. 4. $M(R_{c.m.})$, as defined in Eq. (31) for the $I=0$ states of various configurations near the Fermi surface of Pb^{208}. The arrows indicate the location of the nuclear radius.

[11] S. A. Moszkowski and B. L. Scott, Nucl. Phys. **29**, 665 (1962).

[12] A. Kallio, Phys. Letters **18**, 51 (1965).

pendence on r_{12} as follows:

$$f_{0,L^L}(r,R)=f_{0,L^L}(\tfrac{1}{2}\sqrt{2}r_{12},\sqrt{2}R_{\text{c.m.}}) \cong A(R_{\text{c.m.}})\sin[k(R_{\text{c.m.}})r_{12}]. \quad (30)$$

If the r_{12} dependence can be so expressed, at least for r_{12} less than about 1.2 F, then the coefficient $k(R_{\text{c.m.}})$ will be the effective relative momentum at this value of $R_{\text{c.m.}}$. Figure 3 shows the functions $A(R_{\text{c.m.}})$ and $k(R_{\text{c.m.}})$ so obtained for the $I=0$ state of the configuration $(2p_{3/2})^2$. It is apparent that $k(R_{\text{c.m.}})$ varies strongly as a function of $R_{\text{c.m.}}$. However, the expression for the interaction energy will be dominated by values of $R_{\text{c.m.}}$ at which the amplitude $A(R_{\text{c.m.}})$ is large, and for these $R_{\text{c.m.}}$, $k(R_{\text{c.m.}})$ varies much less. Using the $k(R_{\text{c.m.}})$ shown in Fig. 4, together with a curve of separation distance versus relative momentum such as that given in Ref. 11, we conclude that $d\cong 1.1$ F for $R_{\text{c.m.}}=2$ F, $d\cong 1.08$ F for $R_{\text{c.m.}}=4$ F, and $d\cong 1.04$ F for $R_{\text{c.m.}}=7.5$ F.

To understand the relationship between $k(R_{\text{c.m.}})$ and $A(R_{\text{c.m.}})$ exhibited in Fig. 3, let us write the independent-particle Hamiltonian in terms of the variables r_{12} and $R_{\text{c.m.}}$ defined in (29):

$$H=-\frac{\hbar^2}{2(2m)}\nabla_{R\text{c.m.}}{}^2-\frac{\hbar^2}{2(m/2)}\nabla_{r_{12}}{}^2 +U(|\mathbf{R}_{\text{c.m.}}-\tfrac{1}{2}\mathbf{r}_{12}|)+U(|\mathbf{R}_{\text{c.m.}}+\tfrac{1}{2}\mathbf{r}_{12}|). \quad (31)$$

A Taylor expansion of the potential energy terms about $r_{12}=0$ gives

$$2U(R_{\text{c.m.}})+(\tfrac{1}{2}\mathbf{r}_{12}\cdot\nabla_{R\text{c.m.}})^2U(R_{\text{c.m.}})+\cdots. \quad (32)$$

Let us suppose that r_{12} is small enough that $U(R_{\text{c.m.}})$ changes by a small fraction of itself in this distance. This will be true for $r_{12}<1.2$ F, except perhaps in the vicinity of the nuclear surface. Then the second term in (33) will be small compared to the first, and H becomes approximately separable in $R_{\text{c.m.}}$ and r_{12}. If we then seek an eigenfunction of (31) of the form

$$\Psi_M{}^L(\mathbf{R}_{\text{c.m.}},\mathbf{r}_{12})\approx\frac{u(r_{12})A(R_{\text{c.m.}})Y_M{}^L(\hat{\mathbf{R}}_{\text{c.m.}})}{r_{12}R_{\text{c.m.}}}, \quad (33)$$

the radial functions $U(r_{12})$ and $A(R_{\text{c.m.}})$ must satisfy

$$\frac{1}{A(R_{\text{c.m.}})}\left\{-\frac{\hbar^2}{2(2m)}\left[\frac{d^2}{dR_{\text{c.m.}}{}^2}-\frac{L(L+1)}{R_{\text{c.m.}}{}^2}\right]+2U(R_{\text{c.m.}})-\epsilon\right\}A(R_{\text{c.m.}})=\frac{1}{u(r_{12})}\left\{\frac{\hbar^2}{2(m/2)}\frac{d^2}{dr_{12}{}^2}\right\}u(r_{12}). \quad (34)$$

Here ϵ is the sum of the two single-particle energies. Equation (34) implies that

$$u(r_{12})=\sin[k(R_{\text{c.m.}})r_{12}],$$

where

$$k(R_{\text{c.m.}})=\left\{\frac{1}{A(R_{\text{c.m.}})}\left[\frac{1}{4}\left(\frac{d^2}{dR_{\text{c.m.}}{}^2}-\frac{L(L+1)}{R_{\text{c.m.}}{}^2}\right)+\frac{m}{\hbar^2}(\epsilon-2U(R_{\text{c.m.}}))\right]A(R_{\text{c.m.}})\right\}^{1/2}. \quad (35)$$

Thus $k(R_{\text{c.m.}})$ becomes large near the zeros of $A(R_{\text{c.m.}})$. On the other hand, $k(R_{\text{c.m.}})$ has its minima near values of $R_{\text{c.m.}}$ making $A(R_{\text{c.m.}})^{-1}d^2A(R_{\text{c.m.}})/dR_{\text{c.m.}}{}^2$ most negative.

If we neglect the $R_{\text{c.m.}}$ dependence of d, the expression for the interaction matrix element is

$$\Delta E=\int_{R\text{c.m.}=0}^{\infty}dR_{\text{c.m.}}M(R_{\text{c.m.}}), \quad (36)$$

where

$$M(R_{\text{c.m.}})\equiv\int_{r_{12}=d}^{\infty}V_l(r_{12})\left[f_{0,L^L}\left(\frac{r_{12}}{\sqrt{2}},\sqrt{2}R_{12}\right)\right]^2dr_{12}. \quad (37)$$

Figure 4 shows $M(R_{\text{c.m.}})$ calculated for the $I=0$ states of various neutron configurations near the Fermi surface of lead. $V(R)$ was taken from Ref. 13, with $d=1.045$ F. The nodeless single-particle wave functions of the $(i_{13/2})^2$ configuration causes the effective interaction to be concentrated near the nuclear surface. For all other cases, the interaction is distributed throughout the nuclear volume.

Finally, we give in Table I a comparison of matrix elements of the force of Ref. 13, calculated between har-

TABLE I. Matrix elements for Pb^{206} in LS coupling. The oscillator parameter is $\hbar\omega=7$ MeV. The Woods-Saxon (W-S) wells use $r_0=1.25$ F, $a_0=0.65$ F, and $\lambda=20$.

		Matrix elements		
	I	Oscillator	W-S $V=44.7$	W-S $V=39.0$
$\langle 2p^2\|v_L\|2p^2\rangle$	0	−0.851	−0.978	−0.784
$\langle 1f^2\|v_L\|1f^2\rangle$	0	−1.22	−1.34	−1.23
$\langle 0i^2\|v_L\|0i^2\rangle$	0	−1.54	−1.71	−1.64
$\langle 2p^2\|v_L\|1f^2\rangle$	0	−0.477	−0.423	−0.380
$\langle 2p^2\|v_L\|0i^2\rangle$	0	0.455	0.357	0.324
$\langle 1f^2\|v_L\|0i^2\rangle$	0	0.835	0.684	0.656
$\langle 2p^2\|v_L\|2p^2\rangle$	2	−0.468	−0.546	−0.426
$\langle 1f^2\|v_L\|1f^2\rangle$	2	−0.347	−0.386	−0.351
$\langle 0i^2\|v_L\|0i^2\rangle$	2	−0.404	−0.451	−0.431
$\langle 2p^2\|v_L\|1f^2\rangle$	2	−0.162	−0.145	−0.130
$\langle 2p^2\|v_L\|0i^2\rangle$	2	0.149	0.117	0.107
$\langle 1f^2\|v_L\|0i^2\rangle$	2	0.244	0.184	0.175

[13] A. Kallio and K. Kolltveit, Nucl. Phys. **53**, 87 (1964).

monic oscillator wave functions, and wave functions from a Woods-Saxon well. The oscillator parameter was chosen from $\omega=41\ \mathrm{MeV}/A^{1/3}$. It is seen that the Woods-Saxon wave functions generally yield larger diagonal elements, but smaller off-diagonal elements, than do the harmonic oscillator wave functions. We also show in Table I that a rather large change in the depth of the Woods-Saxon potential has little effect on the interaction matrix elements. This change in potential depth has a sizeable effect on the single-particle energies, and on the tails of the single-particle wave functions. However, although these tails are important in the calculation of form factors for particle transfer reactions, they contribute little to interaction matrix elements.

VI. DISCUSSION

We have described a method of obtaining the relative-angular-momentum-zero part of a two-particle wave function. If this wave function is a product of single-particle wave functions, an alternative method is the expansion of the single-particle radial functions in terms of harmonic oscillator radial functions, and the subsequent use of Brody-Moshinsky transformation brackets. However, suppose the orbital part of the two-particle wave function has the more general form

$$u(r_1,r_2)[Y^{l_1}(\hat{r}_1)Y^{l_2}(\hat{r}_2)]_M{}^L + u(r_2,r_1)[Y^{l_1}(\hat{r}_2)Y^{l_2}(\hat{r}_1)]_M{}^L, \quad (38)$$

where the radial function $u(r_1,r_2)$ is not simply a product of one-particle radial functions. Terms of this form could arise from a solution of the two-body problem that did not start from an independent-particle basis. The double expansion of $u(r_1,r_2)$ in terms of harmonic oscillator products $u_{n_1l_1}(r_1)\times u_{n_2l_2}(r_2)$ would generally be very involved. But the more general wave function (38) poses no additional difficulty for our method. One need only replace the product $u_{l_1}(r_1)\times u_{l_2}(r_2)$ in the integrand in (12) by $u(r_1,r_2)$.