High-Energy Approximation and the Nonlocal Nucleon-Nucleus Watson Potential*

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(Received 6 September 1966)

The high-energy approximation is used to evaluate an effective local optical potential to describe nucleon-nucleus scattering in terms of soft-core two-body potentials.

INTRODUCTION

THE nonlocality of the Watson[1] potential has been shown by Mulligan[2] to be an important effect in describing nucleon-nucleus scattering in terms of a two-body t matrix $T_E(\mathbf{k},\mathbf{k}')$. One may derive an effective local potential if one knows the value of a certain derivative of the t matrix.[1-3] It is the purpose of this paper to indicate how such a derivative may be calculated for not-too-singular potentials in the high-energy approximation.

A typical potential $V(\mathbf{r},\mathbf{r}')$ is given by

$$V(\mathbf{r},\mathbf{r}')=-\frac{\hbar^2N}{(2\pi)^5m}\int e^{+i\mathbf{k}\cdot\mathbf{r}}T_E(\mathbf{k},\mathbf{k}')\times G(\mathbf{k}-\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{r}}d^3kd^3k'\,,$$

where G is the nucleon form factor of an assumed independent-particle nucleus with N nucleons. Assuming the nuclear radius to be much greater than the range of the nucleon-nucleon force gives

$$\begin{aligned}V(\mathbf{r},\mathbf{r}')&=-\frac{\hbar^2N}{(2\pi)^2m}\rho\left(\frac{\mathbf{r}+\mathbf{r}'}{2}\right)\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}T_E(\mathbf{k},\mathbf{k})d^3k\\&=-\frac{2\pi\hbar^2N}{m}\rho\left(\frac{\mathbf{r}+\mathbf{r}'}{2}\right)U(\mathbf{r}-\mathbf{r}')\\&=V\left(\frac{\mathbf{r}+\mathbf{r}'}{2}\right)U(\mathbf{r}-\mathbf{r}')\approx V(r)U(\mathbf{r}-\mathbf{r}')\,,\end{aligned}\tag{1}$$

where the last step follows because U is of very short range compared to V. The nuclear density is ρ and $U(\mathbf{R})$ is given by

$$U(\mathbf{R})=\frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\mathbf{R}}T_E(\mathbf{k},\mathbf{k})d^3k\,.$$

* This work is supported in part through funds provided by the U. S. Atomic Energy Commission under Contract No. A.T. (30-1) 2098.

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[1] B. W. Reisenfeld and K. N. Watson, Phys. Rev. **102**, 1157 (1956); A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N. Y.) **8**, 551 (1959).

[2] B. Mulligan, Ann. Phys. (N. Y.) **26**, 159 (1964).

[3] W. E. Frahn, Nucl. Phys. **66**, 357 (1965).

This "superseparable" form of the nonlocal potential is so peculiarly easy to handle that one might assume from reading some of the physics literature[4] that this is the only form a nonlocality can take. This is certainly not the correct form to describe nucleon-nucleus scattering at low energies, but at high energies, when we neglect antisymmetrization and correlations, it happens fortunately to be the correct one.

Following the procedure of Ref. 4 we solve the scattering problem by defining a phase S such that the wave function ψ, describing particles of momentum $\mathbf{k}_E$ scattering from $V(\mathbf{r},\mathbf{r}')$, is given by

$$\psi=\exp(i\mathbf{k}_E\cdot\mathbf{r}+iS)\,.$$

In the high-energy approximation[5] we then have

$$-\hbar v\frac{\partial S}{\partial z}=\exp[-i\mathbf{k}_E\cdot\mathbf{r}-iS(\mathbf{r})]\times\int V(\mathbf{r},\mathbf{r}')\exp[i\mathbf{k}_E\cdot\mathbf{r}'+iS(\mathbf{r}')]d^3r'\,.\tag{2}$$

The z direction is taken as the direction of $\mathbf{k}_E$. The scattering amplitude $f(\mathbf{k}_E',\mathbf{k}_E)$ is given by

$$\begin{aligned}f(\mathbf{k}_E',\mathbf{k}_E)&=\frac{-m}{2\pi\hbar^2}\int\exp[-i\mathbf{k}_E'\cdot\mathbf{r}+i\mathbf{k}_E\cdot\mathbf{r}'+iS(\mathbf{r}')]\times V(\mathbf{r},\mathbf{r}')d^3rd^3r'\\&=\frac{k_E}{2\pi i}\int\exp[-i(\mathbf{k}_E'-\mathbf{k}_E)\cdot\mathbf{b}]\times[e^{iS(\mathbf{b},\infty)}-1]d^2b\,,\end{aligned}$$

where $\mathbf{r}=(\mathbf{b},z)$ in cylindrical coordinates. This is the same expression as one would find for a local potential, $V_e(r)$, when

$$V_e(r)=-\hbar v\partial S/\partial z\tag{3}$$

and v is the velocity of the particle. Thus, if we can solve the integral equation (2) for $\partial S/\partial z$ we can find an effective local potential $V_e(r)$ which will reproduce the scattering amplitude f.

It is convenient at this point to introduce a new variable $\mathbf{R}$;

$$\mathbf{R}=\mathbf{r}'-\mathbf{r}\,.$$

[4] J. F. Reading, Phys. Letters **20**, 518 (1966).

[5] R. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1958), Vol. 1.

We then rewrite Eq. (2) as

$$-\hbar v\frac{\partial S}{\partial z}=V(r)\int U(\mathbf{R})\times\exp[i\mathbf{k}_E\cdot\mathbf{R}+iS(\mathbf{R}+\mathbf{r})-iS(\mathbf{r})]d^3R.$$

Expanding $S(\mathbf{r}+\mathbf{R})$ about the point $\mathbf{r}$, and keeping only the first derivative of S gives

$$-\hbar v\frac{\partial S}{\partial z}\approx V(r)\int U(\mathbf{R})e^{i\mathbf{k}_E\cdot\mathbf{R}}(1+i\mathbf{R}\cdot\nabla S)d^3R. \quad (4)$$

With $T_E(\mathbf{k},\mathbf{k})$ a function of the scalar k (which is the case if we assume the two-body potential W is spherically symmetric) U is a function of the scalar R. Thus, the only contribution from the second term in the integration of Eq. (4) comes when $\mathbf{R}$ is pointing in the z or $\mathbf{k}_E$ direction. Therefore, we have

$$-\hbar v\frac{\partial S}{\partial z}=V(r)T_E(\mathbf{k}_E,\mathbf{k}_E)+V(r)\frac{\partial S}{\partial z}T_E'(\mathbf{k}_E,\mathbf{k}_E),$$

where

$$T_E'(\mathbf{k},\mathbf{k})=\frac{\partial}{\partial k}T_E(\mathbf{k},\mathbf{k}),$$

$$|\mathbf{k}|\neq|\mathbf{k}_E|,$$

$$\hbar^2k_E^2=2Em.$$

This gives

$$-\hbar v\frac{\partial S}{\partial z}=V_e(r)=V(r)T_E(\mathbf{k}_E,\mathbf{k}_E)\times\left[1+\frac{V(r)}{\hbar v}T_E'(\mathbf{k}_E,\mathbf{k}_E)\right]^{-1}, \quad (5)$$

which completes the calculation of $V_e(r)$. We only need the derivative of $T_E(\mathbf{k},\mathbf{k})$ evaluated on the mass shell. This is a nice simplification but, of course, the knowledge of the derivative implies knowledge of the off-mass-shall behavior in the immediate neighborhood. Thus, in $V_e(r)$ which is experimentally measurable, we have a method of exploring the off-mass-shell behavior of the two-body t matrix over a whole range of energies. The only off-mass-shell effects that have been obtainable directly from experiment before, have been those from bound states or negative energies.[6] As we can go up to 300 MeV for E with no difficulty at all, and considerably higher than this if we are prepared to treat pion production, we have a tool which should be of great value in the study of the nucleon-nucleon interaction. This is well known, of course, and was the original motivation for studying nucleon-nucleus scattering.[7] The Watson potential we have considered so far has ignored effects due to correlations both dynamic and kinematic.[8] The extraction of $T_E'(\mathbf{k}_E,\mathbf{k}_E)$ from the experimental data is somewhat complicated by these effects, but it is hoped that these difficulties may be circumvented by comparing deuteron-nucleus scattering to nucleon-nucleus scattering.[9] Thus, we need two nuclear experiments (deuteron and nucleon) and a knowledge of $T_E(\mathbf{k}_E,\mathbf{k}_E)$ in order to calculate $T_E'(\mathbf{k},\mathbf{k})$, an involved program; one, however, that is well worth attempting, considering the valuable insight it can give us into the proposed two-body interactions.

$T_E'(k_E,k_E)$ IN THE HIGH-ENERGY APPROXIMATION

We have stated above that in $T_E'(\mathbf{k}_E,\mathbf{k}_E)$ we have a measure of W which is not obtainable in a normal two-body experiment. We now show exactly what measure of W we have in a simple model situation, where we assume that for a range a of W,

$$E\gg W,\quad k_Ea\gg 1. \quad (6)$$

These conditions will hardly be applicable in practice (considering the hard-core fits of most interactions) below 300 MeV which is our upper limit in energy without discussing pion production. However, soft-core fits to the two-body data are now available and in any case it is nice to have an analytic expression to understand the general features of the problem and to obtain a deeper understanding of the factors involved. The square-well hard core in any case can be treated exactly.[2]

The generalized off-mass-shell wave function satisfies the following equation:

$$\psi_E(\mathbf{r},\mathbf{k})=\exp(i\mathbf{k}\cdot\mathbf{r})-\frac{m}{2\pi\hbar^2}\int\frac{\exp[ik_E|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}\times W(r')\psi_E(\mathbf{r}',\mathbf{k})d^3r', \quad (7)$$

where $|\mathbf{k}|\neq|\mathbf{k}_E|$ and the t matrix $T_E(\mathbf{k}',\mathbf{k})$ is defined by

$$T_E(\mathbf{k}',\mathbf{k})=-\frac{m}{2\pi\hbar^2}\int\exp(-i\mathbf{k}'\cdot\mathbf{r})W(r)\psi_E(\mathbf{r},\mathbf{k}_E)d^3r.$$

Thus, we have

$$T_E'(\mathbf{k}_E,\mathbf{k}_E)=-\frac{m}{2\pi\hbar^2}\int\exp(-i\mathbf{k}_E\cdot\mathbf{r})W(r)\times\left[\frac{\partial}{\partial k_E}\psi_E(\mathbf{r},\mathbf{k}_E)-iz\psi_E(\mathbf{r},\mathbf{k}_E)\right]d^3r. \quad (8)$$

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[6] See, however, M. I. Sobel, Phys. Rev. **138**, B1517 (1965).
[7] H. A. Bethe, Ann. Phys. (N. Y.) **3**, 190 (1958).
[8] J. F. Reading, Ph.D. thesis, Birmingham, United Kingdom, 1964 (unpublished); following paper, Phys. Rev. **156**, 1120 (1967).
[9] J. F. Reading, preceding paper, Phys. Rev. **156**, 1110 (1967).

We calculate $\partial\psi(\mathbf{r},\mathbf{k}_E)/\partial k_E$ in the high-energy approximation. Following Glauber,[5] we define Q by

$$\psi_E'(\mathbf{r})=\partial\psi_E(\mathbf{r},\mathbf{k}_E)/\partial k_E=Qe^{i\mathbf{k}_E\cdot\mathbf{r}}.$$

Then differentiating Eq. (7) gives

$$Q(\mathbf{r})=iz-\frac{m}{2\pi\hbar^2}\int\exp[ik_Er'-i\mathbf{k}_E\cdot\mathbf{r}']\times W(\mathbf{r}-\mathbf{r}')Q(\mathbf{r}-\mathbf{r}')d_1{}^3r'.$$

The exponential in the integrand is oscillating rapidly. Using the method of stationary phase gives

$$Q(\mathbf{r})\approx iz-\frac{i}{\hbar v}\int_{-\infty}^{z}W(\mathbf{b},z')Q(\mathbf{b},z')dz'.$$

This we may solve, with the boundary condition that

$$Q(\mathbf{b},z)=iz,$$

neglecting scattering in the backward direction. This gives

$$Q(\mathbf{r})=i\exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{z}W(\mathbf{b},z')dz'\right)\times\left\{\int_{-\infty}^{z}\left[\exp\left(\frac{i}{\hbar v}\int_{-\infty}^{z'}W(\mathbf{b},z'')dz''\right)-1\right]dz'+z\right\}. \quad (9)$$

Combining the high-energy expression for $\psi_E(\mathbf{r},\mathbf{k}_E)$,

$$\psi_E(\mathbf{r},\mathbf{k}_E)=\exp\left(i\mathbf{k}_E\cdot\mathbf{r}-\frac{i}{\hbar v}\int_{-\infty}^{z}W(\mathbf{b},z')dz'\right),$$

with that for Q from Eq. (9) gives, on substitution into Eq. (8),

$$T_E'(\mathbf{k}_E,\mathbf{k}_E)=-\frac{k}{2\pi}\int\left[\exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{z}W(\mathbf{b},z')dz'\right)-1\right]\times\left[\exp\left(-\frac{i}{\hbar v}\int_{z}^{\infty}W(\mathbf{b},z')dz'\right)-1\right]. \quad (10)$$

From Eq. (10) we see that $T_E'(\mathbf{k}_E,\mathbf{k}_E)$ is at least second order in W. This is to be expected as in Born approximation $T_E(\mathbf{k},\mathbf{k}')$ is local (for a local potential) so that the nonlocality which is measured to some extent by $T_E'(k_E,k_E)$ should be at least second order in W. In fact, Eq. (10) represents an approximate summation of all terms in the Born series.[8]

The approximation satisfies unitarity (or at least a simple generalization) if we are prepared to make the usual small-angle approximation of Glauber.[5] We know that for all Hermitian potentials, and for W in particular, we have

$$T_E(\mathbf{k}',\mathbf{k})=W(\mathbf{k}'-\mathbf{k})+\frac{1}{2\pi^2}\int\frac{T_{E''}(\mathbf{k}',\mathbf{k}_{E''})T_{E''}{}^*(\mathbf{k},\mathbf{k}_{E''})}{k_{E''}{}^2-k_E{}^2-i\epsilon}d^3k_{E''},$$

where $W(\mathbf{k}'-\mathbf{k})$ is the Fourier transform of $W(r)$ and $\hbar^2k_{E''}{}^2=2E''m$.

Differentiating and taking the discontinuity across the cut in complex E, gives

$$\mathrm{Im}T_E'(\mathbf{k}_E,\mathbf{k}_E)=\frac{k_E}{4\pi}\int\left[T_E{}^*(\mathbf{k}_E,\mathbf{k})\frac{\partial T_E}{\partial k_E}(\mathbf{k}_E,\mathbf{k})+\frac{\partial T_E{}^*}{\partial k_E}(\mathbf{k}_E,\mathbf{k})T_E(\mathbf{k}_E,\mathbf{k})\right]d\Omega_k \quad (11)$$

with $|\mathbf{k}_E|=|\mathbf{k}|$.

In the high-energy approximation, we have

$$T_E(\mathbf{k}_E,\mathbf{k})=\frac{k_E}{2\pi i}\int\exp[i(\mathbf{k}-\mathbf{k}_E)\cdot\mathbf{b}]\times\left[\exp\left(-\frac{i}{\hbar v}\int_{-\infty}^{+\infty}W(\mathbf{b},z)dz\right)-1\right]d^2b$$

$$\frac{\partial}{\partial k_E}T_E(\mathbf{k}_E,\mathbf{k})\approx\frac{m}{2\pi\hbar^2}\int\exp[i(\mathbf{k}-\mathbf{k}_E)\cdot\mathbf{b}]izW(r)\times\exp\left[-\frac{i}{\hbar v}\int_{-\infty}^{z}W(\mathbf{b},z')dz'\right]d^3r. \quad (12)$$

In Eq. (12) we have assumed that there is negligible contribution to the integral unless $\mathbf{k}_E$ is in the $\mathbf{k}$ or z direction. Using the closure property of the angular integration, it is then straightforward to show that Eq. (11) is satisfied. Perhaps the most interesting result of Eq. (10) is that $T_E'(\mathbf{k}_E,\mathbf{k}_E)$ is not zero in the high-energy approximation. Glauber[5] has derived the optical potential using the high-energy approximation and found it to be local, so we might expect at sufficiently high energies for soft-core potentials that the nonlocality of the Watson potential should disappear.

This apparently is not the case. The nonlocality of the potential is directly traceable to the occurrence in the calculation of off-the-mass-shell matrix elements. These occur because of multiple scattering. The wave incident on a target nucleon consists of a plane wave plus all the outgoing scattered waves. The latter immediately involves one in off-the-mass-shell matrix elements. The failure of the Glauber method is due to the absence in the approximate wave function of outgoing internally scattered waves. The correction for the nonlocality is so large,[2] that this must be considered a serious defect in the application of the high-energy approximation to the multiple-scattering problem.

APPLICATION TO A SQUARE WELL

As an example of the method, we calculate $T_E(\mathbf{k}_E,\mathbf{k}_E)$ and $T_E'(\mathbf{k}_E,\mathbf{k}_E)$ for a square well of radius a and strength W. It is straightforward to show that

$$T_E(\mathbf{k}_E,\mathbf{k}_E)=ka^2i[\tfrac{1}{2}-ie^{-i\alpha}/\alpha+(1-e^{-i\alpha})/\alpha^2] \quad (13)$$

and,

$$T_E'(\mathbf{k}_E,\mathbf{k}_E)$$
$$=-2ka^3\left[\frac{1}{3}+\frac{i}{\alpha}(e^{-i\alpha}+1)+\frac{4e^{-i\alpha}}{\alpha^2}+\frac{4i(1-e^{-i\alpha})}{\alpha^3}\right], \quad (14)$$

where

$$\alpha=2Wa/hv.$$

It is shown in Ref. 4 that the ratio $L=T_E'/T$ is the length involved in determining the correction to the Watson potential due to the nonlocality. It is also stated there that

$$L=O(k_Ea^2)$$

and is always negative. The proof of the latter statement was conditional on the real and imaginary parts of the nonlocality as typified by $U(R)$ having the same radius. From Eqs. (13) and (14) it is clear that neither of these statements is true for all values of W. If α is small, then

$$L=\left(-\frac{ka^3\alpha^2}{15}\right)\left(\frac{3}{ka^2\alpha}\right)=\frac{-Wa^2}{5hv}=-\frac{Wa^2m}{5h^2k}.$$

This is only negative if W is positive and is inversely proportional to k. Similarly, if α is large, then

$$L=(-\tfrac{1}{3}ka^32)(2/ka^2i)=\tfrac{4}{3}ai,$$

which is purely imaginary and proportional to a. Thus, the behavior of L which has been inferred in Ref. 4 from apparently quite general arguments, is completely wrong.

From Eq. (5) we see another important ratio is f:

$$f=[V(r)/hv]T_E'(k_E,k_E)$$

with

$$V_e(r)=V_L(r)/(1+f),$$

where $V_L(r)$ is the potential one would get if one ignored the locality. Assuming W is large, but E is larger, would give

$$f=+\left(\frac{4\pi N}{k}\rho(r)\right)\left(\frac{ka^32}{3}\right)=\frac{2a^3}{R_0^3}$$

which predicts therefore that $|V_e|<|V_L|$ and that the real and imaginary parts of V_e and V_L are proportional.

CONCLUSION

We have derived an expression for $T_E'(\mathbf{k}_E,\mathbf{k}_E)$ to be used for soft-core potentials in evaluating the optical potential to describe nucleon-nucleus scattering. The approximation satisfies a generalized unitarity condition. It is hoped that the analytic expression obtained for a square well may contain some of the features of an exact calculation. The length L is, in general, complex.