THE
PHYSICAL REVIEW
A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 156, NO. 4 20 APRIL 1967

Nuclear Photodisintegration in the 1s Shell: A Perturbative Approach to the Dipole Sum Rules*

P. O. DAVEY
Department of Physics, State University College, Fredonia, New York

AND

H. S. VALK
Behlen Laboratory of Physics, University of Nebraska, Lincoln, Nebraska

(Received 8 November 1966)

The integrated and bremsstrahlung-weighted $E1$ photoabsorption cross sections, σ_{int} and σ_b, have been calculated for the lightest nuclei, H^2, H^3, He^3, and He^4, within the framework of a second-order perturbation procedure. For the purpose of comparison, two Gaussian nucleon-nucleon potentials were employed: one containing a repulsive core and central attractive and tensor components, the other only central attractive and tensor components. The results for σ_{int} indicate that while reasonable over-all agreement with experiment may be achieved with either potential, the component contributions differ widely because of the admixture of the repulsive core. Furthermore, the results for σ_b for the deuteron seem to indicate a deficiency in the present choice of oscillator basis functions when applied to the loosely bound system. Considerable improvement for σ_b is noted for H^3, He^3, and He^4, where the nuclei are more tightly bound and less extended structures. It is found by analyzing exact and approximate third-order contributions to both the deuteron binding energy and integrated cross section that the use of a wave function containing parameters determined by minimization of the perturbation expansion through second order is probably not acceptable, at least for this nucleus. This is further substantiated by comparison with the binding-energy results obtained from an exact numerical solution of the coupled S and D radial differential equations.

I. INTRODUCTION

IT is the intent of this paper to present a reasonably consistent perturbation procedure for computing via sum rules the total integrated (σ_{int}) and bremsstrahlung-weighted (σ_b) electric dipole ($E1$) photoabsorption cross sections for H^2, H^3, He^3, and He^4.

The perturbative approach permits one to introduce relatively realistic interaction operators and to obtain acceptable values for the $E1$ cross sections. More importantly, perhaps, it also possesses the advantage of uniformity in treatment. With the use of the same pair interaction operators and types of wave function throughout, the method not only demonstrates that most of the sum-rule results for the 1s shell nuclei can be reproduced, but also permits conclusions to be drawn regarding the relative contributions of the different interaction components, such as the repulsive core and tensor force, with increasing nucleon number.

The format of the present paper is as follows: Section II develops the general perturbative formalism to be applied in Secs. III and IV to the alpha particle, the trinucleon, and the deuteron. Section V is then concerned with a more detailed discussion of applications to the deuteron, while Sec. VI is devoted to a discussion of the advantages and disadvantages of the perturbation technique.

II. GENERAL PERTURBATIVE TECHNIQUE AND INTEGRATED CROSS SECTION

The vehicle adopted for our program of calculations is the second-order perturbation procedure devised by Bolsterli and Feenberg[1] for their calculations of nuclear binding energies in the 1s shell. This particular formulation of Brillouin-Wigner perturbation theory appears especially suitable for photoabsorption cross-section calculations.

* Work supported in part by a grant from the National Science Foundation.

[1] M. Bolsterli and E. Feenberg, Phys. Rev. **101**, 1349 (1956).

In this procedure, the nuclear Hamiltonian is written as

$$H=(H_0'+U)+W=H_0+W$$
$$=[\tfrac{1}{2}\hbar\omega\sum_{i=1}^{A}(p_i^2+q_i^2)+U]$$
$$+\Big[\sum_{i<j}V_{ij}-\frac{1}{2}\frac{\hbar\omega}{A}\sum_{i<j}(\mathbf{q}_i-\mathbf{q}_j)^2-U\Big], \quad (1)$$

where the notation is that of Ref. 1, and the shift function U is chosen so that $W_{00}\equiv\langle 0|W|0\rangle=0$.

In second order, the eigenvalue E corresponding to H is given implicitly by the first two terms in the Brillouin-Wigner energy series generated by the wave function

$$\Psi_0=\frac{1}{N}\Big[\psi_0+\sum_{n\neq 0}\frac{W_{0n}\psi_n}{E-E_n}+\cdots\Big], \quad (2)$$

where ψ_0 may be expressed as a product of s-state orbitals belonging to a sum of single-particle harmonic oscillator Hamiltonians. It is Ψ_0 which forms the basis for our perturbative approach to the photonuclear cross sections.

In the special case of a Serber exchange potential $[\frac{1}{2}(1+P_{ij}^M)V_{ij}]$, the integrated $E1$ cross section can be written as[2]

$$\sigma_{\rm int}=\int_0^\infty\sigma(W)dW=\left(\frac{2\pi^2e^2\hbar}{mc}\right)$$
$$\times\left\{\frac{NZ}{A}-\frac{m}{6\hbar^2}\int\Psi_0^*\sum_{i,j}V_{ij}r_{ij}^2P_{ij}^M\Psi_0d\tau\right\}, \quad (3)$$

where $\mathbf{r}_{ij}$ is the separation vector between the ith proton and the jth neutron, and P_{ij}^M is the Majorana exchange operator. The other symbols have their usual meaning.

The use of the corresponding wave function, Eq. (2), in this expression gives the desired value of $\sigma_{\rm int}$. Note that this procedure does not reintroduce those center-of-mass excitation effects which were removed from the perturbation in Eq. (1), since Eq. (3) contains only relative coordinates.

The static nucleon-nucleon potentials employed are those suggested by Goldhammer[3] and by Hu and Massey.[4] The explicit expression for the former potential (hereafter referred to as potential A) is

$$V_{ij}=J_Re^{-r_{ij}^2/r^2}+J_C[\tfrac{1}{16}(1-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(3+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$$
$$+\tfrac{1}{16}(3+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(1-\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)]e^{-r_{ij}^2/r_0^2}+J_S[\tfrac{1}{4}(1-\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)]$$
$$\times(r_{ij}^2/r_0^2)[\boldsymbol{\sigma}_i\cdot\mathbf{n}_{ij}\boldsymbol{\sigma}_j\cdot\mathbf{n}_{ij}-\tfrac{1}{3}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j]e^{-r_{ij}^2/r_0^2}. \quad (4)$$

The parameters[5]

$$J_R=+189.75\text{ MeV},\quad J_C=-58.65\text{ MeV},$$
$$J_S=-107.29\text{ MeV},\quad r_0=1.55\times10^{-13}\text{ cm}, \quad (5)$$

and

$$r=r_0/\sqrt{8}$$

have been adjusted to give an adequate reproduction of the stationary properties of the $1s$ shell nuclei when used in conjunction with the second-order perturbation formalism of Eqs. (1) and (2). Potential A has the advantage of a repulsive-core term which is soft enough to exhibit the structural features without endangering the perturbation expansion.[6]

The Hu-Massey potential (hereafter referred to as potential B) is given by

$$V_{ij}=J[\tfrac{1}{16}(1-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(3+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)+\tfrac{1}{16}(3+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)$$
$$\times(1-\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)]e^{-r_{ij}^2/r_0^2}+J_S'[\tfrac{1}{4}(1-\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)]$$
$$\times[\boldsymbol{\sigma}_i\cdot\mathbf{n}_{ij}\boldsymbol{\sigma}_j\cdot\mathbf{n}_{ij}-\tfrac{1}{3}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j]e^{-r_{ij}^2/r_0^2}, \quad (6)$$

where the parameters

$$J=-29.49\text{ MeV},\quad J_S'=-52.20\text{ MeV}, \quad (7)$$

and

$$r_0=2.18\times10^{-13}\text{ cm}$$

were determined to fit the binding energy and quadrupole moment of the deuteron.

Following Bolsterli and Feenberg,[1] we express the matrix elements in terms of the following notation:

$$\begin{aligned}
-\hbar\omega\delta&=E-E_0, & a^2&=\hbar/m\omega,\\
\hbar\omega&=G(\eta-1), & r_0^2&=(2\hbar/m\omega)(\eta-1),\\
\mu&=\tfrac{1}{2}\hbar\omega\lambda, & u&=e^{-4\mu},\\
\alpha^2&=a^2/r_0^2, & \beta^2&=a^2/r^2,\\
r_{ij}^2/r_0^2&=\alpha^2q_{ij}^2, & r_{ij}^2/r_0^2&=\beta^2q_{ij}^2,\\
\eta&=1+1/2\alpha^2, & \zeta&=1+1/2\beta^2.
\end{aligned} \quad (8)$$

The explicit numerical values of G, $\hbar\omega$, and δ are listed in Appendix A for each potential. It will not be necessary to distinguish the parameters used in the two separate potentials since the appearance of β or ζ identifies potential A. Both potentials, with the exception of the core term in A, have a Serber exchange mixture. The repulsive-core part of A is assumed to have no exchange character and will not explicitly appear in Eq. (3). However, the core does make significant contributions to the perturbed ground-state wave function Ψ_0, and therefore will give rise to non-negligible cross terms in the final expressions.

Further simplification of the potentials can be achieved by recognizing that the space exchange

[2] M. L. Rustgi and J. S. Levinger, Phys. Rev. **106**, 530 (1957).
[3] P. Goldhammer, Phys. Rev. **116**, 676 (1959).
[4] T. Hu and H. S. W. Massey, Proc. Roy. Soc. (London) **A196**, 135 (1949).
[5] Note that this result differs slightly from the value of r_0 quoted in Ref. 3; we find, however, that the value above more nearly reproduces the binding energies listed in that reference.
[6] The usefulness of such core terms in structural calculations has been emphasized by Y. R. Waghmare, Phys. Rev. **136**, B1261 (1964).

operator P_{ij}^M, when applied to any pair of particles i, j in the same space state, simply yields the value $+1$ in the matrix element. Thus, for the $1s$ shell nuclei, assuming a completely space-symmetric wave function, $\frac{1}{2}(1+P_{ij}^M)=1$, and potentials A and B reduce to

$$V_{ij}^A=J_R e^{-\beta^2 q_{ij}^2}+J_C e^{-\alpha^2 q_{ij}^2}+J_S\alpha^2 q_{ij}^2 S_{ij} e^{-\alpha^2 q_{ij}^2}, \quad (9)$$

$$V_{ij}^B=J e^{-\alpha^2 q_{ij}^2}+J_S' S_{ij} e^{-\alpha^2 q_{ij}^2}. \quad (10)$$

An additional substitution,

$$J_C=\frac{K\hbar\omega}{\eta-1}, \quad J_R=\frac{K'\hbar\omega}{\zeta-1}, \quad J_S=\frac{K''\hbar\omega}{\eta-1}, \quad (11)$$

is convenient in order to avoid the use of the rather large numbers associated with the well depths (J's) in subsequent numerical evaluations. The numerical values of the K's for both potentials may be found in Appendix A.

III. $E1$ INTEGRATED CROSS SECTIONS

A. The Alpha Particle

Having defined our problem and indicated the methods to be used for its solution, we now consider the explicit evaluation of σ_{int}, defined by Eq. (3), for the alpha particle. In this case,

$$\sigma_{\text{int}}(\text{He}^4)=\sigma_0+\sigma_{\text{ex}}(\text{He}^4), \quad (12)$$

where $\sigma_0=2\pi^2e^2\hbar/mc=59.73$ MeV, and

$$\sigma_{\text{ex}}(\text{He}^4)=-\frac{m\sigma_0}{6\hbar^2}\int\Psi_0^*\sum_{i,j}^4 V_{ij}r_{ij}^2P_{ij}^M\Psi_0 d\tau. \quad (13)$$

The evaluation of (13) through second order leads to

$$\sigma_{\text{ex}}(\text{He}^4)=-\frac{m\sigma_0a^2}{6N^2\hbar^2}\times\left\{(\sum_{i,j}^4 V_{ij}q_{ij}^2)_{00}+2\sum_{n\neq0}^{\infty}\frac{(\sum_{i,j}{}^4 V_{ij}q_{ij}^2)_{0n}W_{n0}}{E-E_n}\right\}, \quad (14)$$

where

$$W=\sum_{i<j}^4 V_{ij}-M-U \quad \text{and} \quad M=\tfrac{1}{8}\hbar\omega\sum_{l<m}^4 q_{lm}^2.$$

Employing the techniques of Ref. 1, we arrive at

$$\sigma_{\text{ex}}(\text{He}^4)=-\frac{m\sigma_0a^2}{6N^2\hbar^2}\left\{(\sum_{i<j}^4 V_{ij}q_{ij}^2)_{00}-2\int e^{\lambda(E-U)}d\lambda((\sum_{i,j}^4 V_{ij}q_{ij}^2)e^{-\lambda H_0'}W)_{00}\right\}. \quad (15)$$

This expression may be simplified by noting that $\sum_{ij}{}^4$ may be replaced by $\frac{2}{3}\sum_{i<j}{}^4$ (all pairs) since only four of the six pairs of nucleons in the alpha particle are n-p pairs. However, since the operator $\frac{1}{4}(1-\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)S_{ij}$ has the effect of selecting only n-p pairs, it is necessary that we insert a compensatory factor of $\frac{3}{2}$ in the tensor terms. With these substitutions, and the use of potential A, we find that

$$\sigma_{\text{ex}}{}^A(\text{He}^4)=-\frac{2\sigma_0Ma^2}{3\hbar^2N^2}\left\{(C_{12}q_{12}{}^2)_{00}-2\int_0^{\infty}e^{\lambda(E-U)}[(C_{12}q_{12}{}^2eR_{12})_{00}+(C_{12}q_{12}{}^2eC_{12})_{00}+\tfrac{1}{3}((S_{12}'q_{12}{}^2eS_{12}'))_{00}\right.$$

$$+4(C_{12}q_{12}{}^2eR_{13})_{00}+4(C_{12}q_{12}{}^2eC_{13})_{00}-\tfrac{2}{3}((S_{12}'q_{12}{}^2eS_{13}'))_{00}+(C_{12}q_{12}{}^2eR_{34})_{00}+(C_{12}q_{12}{}^2eC_{34})_{00}$$

$$\left.-(C_{12}q_{12}{}^2eM)_{00}-U_{00}(C_{12}q_{12}{}^2e)_{00}]d\lambda\right\}. \quad (16)$$

Here,

$$C_{ij}=J_Ce^{-\alpha^2q_{ij}^2}, \quad R_{ij}=J_Re^{-\beta^2q_{ij}^2}, \quad S_{ij}'=J_S\alpha^2q_{ij}^2S_{ij}e^{-\alpha^2q_{ij}^2},$$

$$e=e^{-\lambda H_0'}=\exp[-\mu\sum_{i=1}^4(p_i^2+q_i^2)],$$

and the double parentheses on the tensor terms indicate that while the number of such terms has been counted and the spin and isospin averages have been determined, no angular or spatial integration has yet been performed.

The choice of

$$\psi_0=\pi^{-3}e^{-\frac{1}{2}(q_1^2+q_2^2+q_3^2+q_4^2)} \quad (17)$$

as the radial form of the alpha particle s-state zero-order function makes the calculation (16) straightforward. Here, it is convenient to make use of the integral transforms given in Appendix B.

The integrals in each case are most directly calculated by performing a binomial expansion and integrating termwise. Numerical results are tabulated in Appendix C. Substituting these into Eq. (16), we find that

$$\sigma_{\text{ex}}{}^A(\text{He}^4)=47.18 \text{ MeV mb}, \quad (18)$$

the first-order contribution being 27.69 MeV mb and

TABLE I. Contributions to $\sigma_{\text{int}}(\text{He}^4)$.

Type of contribution	Potential A [Eq. (4)] (MeV mb)	Potential B [Eq. (6)] (MeV mb)
Nonexchange	59.7	59.7
1st-order central	27.7	33.9
2nd-order central	−0.3	−0.7
Total central	27.4	33.2
Total tensor	19.8	13.0
Total	106.9	105.9

that from second-order 19.49 MeV mb. Thus

$$\sigma_{\text{int}}{}^{A}(\text{He}^4)=106.9 \text{ MeV mb}, \tag{19}$$

in agreement with the value reported in Ref. 7 and not inconsistent with the experimental result of Gorbunov and Spiridonov.[8]

The computation of the matrix elements for potential B is analogous to that for potential A, except that the repulsive-core contributions are absent.

For potential B, the corresponding results are found to be

$$\sigma_{\text{ex}}{}^{B}(\text{He}^4)=46.20 \text{ MeV mb}, \tag{20}$$

where the first- and second-order contributions are 33.93 and 12.27 MeV mb, respectively; and

$$\sigma_{\text{int}}{}^{B}(\text{He}^4)=105.9 \text{ MeV mb}. \tag{21}$$

As can be observed from Table I, showing individual component contributions to $\sigma_{\text{int}}{}^{A}(\text{He}^4)$ and $\sigma_{\text{int}}{}^{B}(\text{He}^4)$, the repulsive core contributes little directly in comparison to its role in enhancing the tensor contribution. That the addition of a soft repulsive core has such a relatively large effect on the tensor contribution results from the fact that the tensor interaction in potential A is long range in character. The region of attraction is thus pushed out to large radial distances where the $r_{ij}{}^2$ factor in the expression for the integrated cross section acts to further amplify its effect.

B. The Trinucleon

The procedure adopted in the previous section may be applied to the three-nucleon system.[9] The calculation differs only in that (1) the sum appearing in Eq. (3) now includes two n-p pairs and (2) the wave function is altered. In this case, the exchange contribution to the integrated cross section becomes

$$\sigma_{\text{ex}}(\text{T})=-\frac{\sigma_0 m}{6\hbar^2}\int \Psi_0{}^{*}\sum_{i,j}^{3}(V_{ij}r_{ij}{}^2)\Psi_0 d\tau, \tag{22}$$

where σ_0 has the constant value previously given and the i and j again refer to protons and neutrons, respectively. Expanding (22) in the manner indicated above, we find

$$\sigma_{\text{ex}}(\text{T})=-\frac{\sigma_0 m a^2}{6\hbar^2 N^2}\Big\{(\sum_{i,j}^{3}V_{ij}q_{ij}{}^2)_{00} -2\int e^{\lambda(E-U)}[(\sum_{i,j}^{3}V_{ij}q_{ij}{}^2)eW]_{00}d\lambda\Big\}, \tag{23}$$

where

$$W=\sum_{i<j}^{3}V_{ij}-M-U,\quad e=e^{-\lambda H_0'},\quad \text{and}\quad M=\tfrac{1}{6}\hbar\omega\sum_{l<m}^{3}q_{lm}{}^2.$$

The further expansion of (23) can again be considerably simplified by noting that $\sum_{i,j}{}^{3}$ may be replaced by $\frac{2}{3}\sum_{i<j}{}^{3}$ (all pairs). As before, the same argument does not apply to the tensor terms, since the corresponding operator already selects only n-p pairs.

With this simplification, Eq. (23) becomes

$$\sigma_{\text{ex}}{}^{A}(\text{T})=-\frac{\sigma_0 m a^2}{3\hbar^2 N^2}\Big\{(C_{12}q_{12}{}^2)_{00}-2\int_0^{\infty}e^{\lambda(E-U)}[(C_{12}q_{12}{}^2eR_{12})_{00}+(C_{12}q_{12}{}^2eC_{12})_{00}+\tfrac{1}{3}((S_{12}'q_{12}{}^2eS_{12}'))_{00} +2(C_{12}q_{12}{}^2eR_{13})_{00}+2(C_{12}q_{12}{}^2eC_{13})_{00}-\tfrac{1}{3}((S_{12}'q_{12}{}^2eS_{13}'))_{00}-(C_{12}q_{12}{}^2eM)_{00}-U_{00}(C_{12}q_{12}{}^2e)_{00}]d\lambda\Big\}, \tag{24}$$

where spin and isospin averages have been taken into account. A brief discussion of the evaluation of the spin and isospin sums is given in Appendix C.

The radial form for the trinucleon s-state zero-order function is given by

$$\psi_0=\pi^{-9/4}e^{-\frac{1}{2}(q_1{}^2+q_2{}^2+q_3{}^2)}. \tag{25}$$

The determination of the matrix elements then proceeds in the same fashion as that for the alpha particle, with the exception that $e^{\lambda(E-U)}d\lambda$ becomes $-(du/2\hbar\omega)u^{5/2-13/4}$ for the trinucleon, since H_0' now has the eigenvalue $\frac{9}{2}\hbar\omega$.

Substituting the results displayed in Appendices A and C into Eq. (24), we find that

$$\sigma_{\text{ex}}{}^{A}(\text{T})=25.20 \text{ MeV mb}, \tag{26}$$

where the first-order contribution is 12.10 MeV mb and second order is 13.10 MeV mb. Thus

$$\sigma_{\text{int}}{}^{A}(\text{T})=65.0 \text{ MeV mb}. \tag{27}$$

For potential B, the corresponding results are given by

$$\sigma_{\text{ex}}{}^{B}(\text{T})=24.58 \text{ MeV mb}, \tag{28}$$

[7] P. Goldhammer and H. S. Valk, Phys. Rev. **127**, 945 (1962).
[8] A. Gorbunov and V. Spiridonov, Zh. Eksperim. i Teor. Fiz. **33**, 21 (1957); **34**, 862, (1958); **34**, 866 (1958) [English transls.: Soviet Phys.—JETP **6**, 21 (1958); **7**, 596 (1958); **7**, 600 (1958)].
[9] P. O. Davey and H. S. Valk, Phys. Letters **7**, 155 (1963).

where the first- and second-order contributions are 16.68 and 7.90 MeV mb, respectively; and

$$\sigma_{\rm int}{}^{B}({\rm T})=64.4\ {\rm MeV\ mb}, \tag{29}$$

a result only slightly different from that predicted by potential A.

While it is perhaps somewhat early to assess the merit of these results, they do appear consistent with experimental data.[10,11] The most recent results of Fetisov *et al.*[11] yield the value of $\sigma_{\rm int}=62\pm6$ MeV mb, in quite good agreement with Eqs. (27) and (29).

A more detailed breakdown of the contributions to $\sigma_{\rm int}{}^{A}({\rm T})$ as in Table II again shows the indirect effect of the repulsive core in enhancing the exchange contribution to the cross section. Here the direct core terms contribute -1.3 MeV mb, whereas the central attractive and tensor contributions are 13.3 and 13.2 MeV mb, respectively. Furthermore it is apparent from the relative smallness of the total central contributions in second order (-0.1 MeV mb) that it is the tensor contribution (13.2 MeV mb) which is most responsive to the presence of the core.

TABLE II. Contributions to $\sigma_{\rm int}(H^3,He^3)$.

Type of contribution	Potential A [Eq. (4)] (MeV mb)	Potential B [Eq. (6)] (MeV mb)
Nonexchange	39.8	39.8
1st-order central	12.1	16.7
2nd-order central	−0.1	−0.5
Total central	12.0	16.2
Total tensor	13.2	8.4
Total	65.0	64.4

C. The Deuteron

We now consider the explicit evaluation of the integrated cross section defined by Eq. (3) for the deuteron. In this case, Eq. (3) becomes

$$\sigma_{\rm int}({\rm H}^2)=\sigma_0\{\tfrac{1}{2}-ma^2/6\hbar^2\int\Psi_0{}^*(V_{12}q_{12}{}^2)\Psi_0 d\tau\}. \tag{30}$$

Inserting the wave function given by Eq. (2) into Eq. (30), and including the $n=0$ term, we find for the exchange contribution

$$\sigma_{\rm ex}({\rm H}^2)=-\frac{\sigma_0 ma^2}{6\hbar^2N^2}\Bigg\{(V_{12}q_{12}{}^2)_{00}+2\Bigg[\sum_{n=0}^{\infty}\Bigg(\frac{(V_{12}q_{12}{}^2)_{0n}V_{n0}}{E-E_n}-\frac{(V_{12}q_{12}{}^2)_{0n}M_{n0}}{E-E_n}-\frac{(V_{12}q_{12}{}^2)_{0n}\delta_{0n}U_{n0}}{E-E_n}\Bigg)$$
$$-\frac{(V_{12}q_{12}{}^2)_{00}}{E-E_0}(V_{00}-M_{00}-U_{00})\Bigg]\Bigg\}, \tag{31}$$

which, after substitution of the components of potential A, becomes

$$\sigma_{\rm ex}{}^{A}({\rm H}^2)=-\frac{\sigma_0 ma^2}{6\hbar^2N^2}\Bigg\{(C_{12}q_{12}{}^2)_{00}-2\int_0^{\infty}e^{\lambda(E-U)}[(C_{12}q_{12}{}^2eR_{12})_{00}+(C_{12}q_{12}{}^2eC_{12})_{00}+(S_{12}{}'q_{12}{}^2eS_{12}{}')_{00}]d\lambda$$
$$+\frac{2}{\delta\hbar\omega}(C_{12}q_{12}{}^2)_{00}[(R_{12})_{00}+(C_{12})_{00}]+\frac{2}{(\delta+2)\hbar\omega}[(C_{12}q_{12}{}^2M)_{00}-(C_{12}q_{12}{}^2)_{00}M_{00}]\Bigg\}. \tag{32}$$

Here we have used the fact that M_{n0} vanishes unless $n=0$ or 2.

For the deuteron, the zero-order radial function is taken to be

$$\psi_0=\pi^{-3/2}e^{-\frac{1}{2}(q_1{}^2+q_2{}^2)}. \tag{33}$$

Again the evaluation of the individual matrix elements follows in the same manner as for the two-body alpha-particle terms and need not be pursued here; instead we refer to the list in Appendix C.

The result for potential A is

$$\sigma_{\rm ex}{}^{A}({\rm H}^2)=14.13\ {\rm MeV\ mb}. \tag{34}$$

Here the first-order exchange contribution amounts to 5.00 MeV mb, while the second order is 8.83 MeV mb. When combined with the nonexchange contribution, Eq. (34) leads to a total integrated cross section of

$$\sigma_{\rm int}{}^{A}({\rm H}^2)=44.0\ {\rm MeV\ mb}. \tag{35}$$

The calculation with potential B is quickly accomplished from Eq. (32) by setting $R_{12}=0$ and substituting the appropriate tensor operator. We find

$$\sigma_{\rm int}{}^{B}({\rm H}^2)=43.0\ {\rm MeV\ mb} \tag{36}$$

and

$$\sigma_{\rm ex}{}^{B}({\rm H}^2)=13.17\ {\rm MeV\ mb}, \tag{37}$$

where the first- and second-order exchange contributions are now 7.64 and 5.53 MeV mb, respectively.

The values of the integrated $E1$ cross section for both potentials turn out to be larger than the value (39.7 MeV mb) obtained by measuring the area under a plot of $\sigma(W)$ versus W using the available experimental data.[12,13] However, we should not expect exceptional

[10] J. R. Stewart, R. C. Morrison, and J. S. O'Connell, Phys. Rev. **138**, B372 (1965).

[11] V. N. Fetisov, A. N. Gorbunov, and A. T. Varfolomeev, Nucl. Phys. **71**, 305 (1965).

[12] J. S. Levinger, *Nuclear Photo-Disintegration* (Oxford University Press, London, 1960).

[13] M. J. Rediger and H. Neumann (private communication).

TABLE III. Contributions to $\sigma_{\rm int}(H^2)$.

Type of contribution	Potential A [Eq. (4)] (MeV mb)	Potential B [Eq. (6)] (MeV mb)
Nonexchange	29.9	29.9
1st-order central	5.3	7.6
2nd-order central	−0.3	−0.5
Total central	5.0	7.1
Total tensor	9.1	6.0
Total	44.0	43.0

agreement here, for, as Bolsterli and Feenberg[1] point out, the zero-order approximation to the wave function [Eq. (33)] is poorest for the deuteron. The oscillator potential is not a good description for a system where the constituent nucleons spend an appreciable amount of time far from each other. We therefore have little reason to expect better values than have been found with either potential using the present method. Moreover, as will be seen, other considerations seem to show that the third-order contributions to the binding energy, at least for the deuteron with potential A, cannot be disregarded and therefore the use of parameters determined by minimization only through second order appears not to be entirely justified.

It is interesting to note that for potential A, the direct repulsive-core terms help to decrease the exchange cross section, contributing −0.4 MeV mb, as compared to the total central attractive and tensor contributions of +5.4 and 9.1 MeV mb, respectively (see Table III). Because of the smallness of the second-order attractive term (+0.1 MeV mb), it seems that the tensor force again overcompensates for the presence of the core.

IV. THE BREMSSTRAHLUNG-WEIGHTED CROSS SECTION

It has been pointed out[14] that there exists a close connection between the root-mean-square (rms) charge radii of nuclei and the $E1$ bremsstrahlung-weighted cross section σ_b. The importance of this relation lies in the fact that the rms radii can be determined with quite reasonable precision by an analysis of the nuclear charge scattering of high-energy electrons, thus possibly providing an independent measure of σ_b.

Under the assumption that the nuclear ground-state wave function is fully spatially symmetric, Foldy[14] has shown that σ_b is simply related to the charge radius through the equation

$$\sigma_b=\frac{4\pi^2}{3}\left(\frac{e^2}{\hbar c}\right)\frac{NZ}{A-1}(R_c^2-R_p^2), \tag{38}$$

where R_c is the rms radius of the charge distribution for the nucleus and R_p is the rms charge radius of the proton; both R_c and R_p in Eq. (38) are measurable by electron-scattering experiments. Assuming that only the protons in a nucleus are responsible for the charge distribution, it is often convenient to express $(R_c^2-R_p^2)$ in (38) in terms of the equivalent quantity $\langle R^2\rangle$, where R^2 is the mean-square radius of a nucleus containing point nucleons. Thus

$$\sigma_b=\frac{4\pi^2}{3}\left(\frac{e^2}{\hbar c}\right)\frac{NZ}{A-1}\langle R^2\rangle. \tag{39}$$

To evaluate σ_b, then, we need only compute the mean-square radius for the nucleus under consideration and multiply the result by the constant given in Eq. (39).

The mean-square radius of a nucleus consisting of A point nucleons in terms of the expectation value of the square of the nucleon separation distance is given by

$$\langle R^2(A)\rangle=\frac{A-1}{2A}\langle r_{jk}^2\rangle. \tag{40}$$

The mean-square radius therefore becomes

$$\langle R^2(A)\rangle=\frac{A-1}{2AN^2}\left\{\int\psi_0^*r_{jk}^2\psi_0d\tau+2\int\psi_0^*r_{jk}^2\sum_{n\neq0}^{\infty}\frac{W_{0n}\psi_n}{E-E_n}d\tau+\int\sum_{n\neq0}^{\infty}\frac{W_{0n}^*\psi_n^*}{E-E_n}(r_{jk}^2)\sum_{m\neq0}^{\infty}\frac{W_{0m}\psi_m}{E-E_m}d\tau\right\}. \tag{41}$$

In terms of quantities previously employed, the mean-square radius of the deuteron is

$$\langle R^2(\mathrm{H}^2)\rangle=\frac{a^2}{4N^2}(q_{12}^2)_{00}-\frac{a^2}{2N^2}\left\{\int_0^\infty e^{\lambda(E-U)}(q_{12}^2eV)_{00}d\lambda+\frac{(q_{12}^2)_{00}V_{00}}{E-E_0}+\frac{1}{E-E_0-2\hbar\omega}[(q_{12}^2M)_{00}-(q_{12}^2)_{00}M_{00}]\right\}$$
$$+\frac{a^2}{4N^2}\int_0^\infty\int_0^\infty e^{\lambda(E-U)}e^{\lambda'(E-U)}\{(Ve'q_{12}^2eV)_{00}-2(Ve'q_{12}^2eM)_{00}+(Me'q_{12}^2eM)_{00}$$
$$-2(Ve'q_{12}^2eU)_{00}+(Ue'q_{12}^2eU)_{00}+2(Me'q_{12}^2U)_{00}\}d\lambda d\lambda'. \tag{42}$$

Since each matrix element required in Eq. (42) can be reduced to an integral of the type previously encountered in the determination of $\sigma_{\rm int}(H^2)$, we will not give them again. We find the mean-square radius using potential A and including *all* terms of (42) to be

$$\langle R_A^2(\mathrm{H}^2)\rangle=2.49\ \mathrm{F}^2. \tag{43}$$

Omitting the second-order (double integral) term in

[14] L. L. Foldy, Phys. Rev. **107**, 1303 (1957).

Eq. (42) for potential B, we find

$$\langle R_B^2(\mathrm{H}^2)\rangle = 2.51\ \mathrm{F}^2. \tag{44}$$

Substituting the values obtained in Eqs. (43) and (44) into Eq. (39) then yields

$$\sigma_b{}^A(\mathrm{H}^2) = 2.39\ \mathrm{mb}, \tag{45}$$

$$\sigma_b{}^B(\mathrm{H}^2) = 2.41\ \mathrm{mb}. \tag{46}$$

These values of the bremsstrahlung-weighted cross section are clearly in disagreement with the experimental result[12,13] of 3.8 mb obtained from the plots of $\sigma(W)$ versus W. This discrepancy, of course, reflects the small rms radii of the deuteron predicted by our procedure and potentials.

The critical role played here by the nuclear mean-square radius $\langle R^2\rangle$ requires that we also compare the above results with those derived from electron scattering.[15,16] Unfortunately, the quantity measured in the scattering experiments is the square of the charge radius R_c^2 and not the nuclear radius. The fact that the latter enters as the difference [compare Eqs. (38) and (39)] of the squares of two independently measured quantities permits considerable latitude in values of $\sigma_b(\mathrm{H}^2)$. In this regard it should be noted that very few elastic electron-scattering measurements with deuterium targets have been made in the low momentum transfer (or model-independent) region; therefore $R_c^2(\mathrm{H}^2)$ must be considered a derived quantity dependent on suitable deuteron wave functions.[15]

If we accept the deuteron rms charge radius of 2.17 ± 0.05 F, then the value 0.85 F for the rms proton charge radius seems most consistent with the experimental photonuclear calculation. Since there appears little *a priori* reason for the experimental photonuclear results to differ from those derived from the electron scattering, we shall adopt the 0.85 F value in the subsequent calculations.[17] The need for a precise low momentum transfer electron-scattering measurement for the deuteron is, however, apparent, and the model dependence cited above should be kept in mind when trying to make detailed comparisons.

Before proceeding further, it is desirable to call attention to an unresolved question concerning the use of the normalization constant N^2 in such expressions as Eqs. (14), (23), (31), and (42). Since the value of E entering into N^2 was determined by minimization only through second order in W, it may be argued that it is not strictly correct to retain the normalization N^2 which is of "second order" in the potentials if we do not retain terms of corresponding order in the expressions for the mean-square radius.

This situation is by no means clear, however, since it will be recalled that the Brillouin-Wigner energy series is able to achieve much of its convergence by "pulling down" or "mixing in" higher orders in the usual Rayleigh-Schrödinger expansion. Rather than attempt to resolve these questions here, we shall just adopt the prescription of neglecting N^2 when the "second-order" terms (in the potentials W and V) are not computed, a procedure which, although reasonable, is still open to question. Some measure of the effect of this rule of thumb can be determined for the deuteron with potential A, since all terms have been calculated in that case. As we noted above, $\langle R_A^2(\mathrm{H}^2)\rangle = 2.49\ \mathrm{F}^2$ when the double-integral terms in Eq. (44) and the normalization N^2 are retained. If both are dropped, $\langle R_A^2(\mathrm{H}^2)\rangle = 2.46$ fm^2, a change slightly greater than 1%. Since the convergence of the series in W is usually least rapid for such a loosely bound system, we would expect this to be the worst case.

Applying this prescription, we arrive at the revised values

$$\begin{aligned}\langle R_A^2(\mathrm{H}^2)\rangle &= 2.46\ \mathrm{F}^2,\\ \langle R_B^2(\mathrm{H}^2)\rangle &= 2.66\ \mathrm{F}^2,\end{aligned} \tag{47}$$

$$\begin{aligned}\sigma_b{}^A(\mathrm{H}^2) &= 2.36\ \mathrm{mb},\\ \sigma_b{}^B(\mathrm{H}^2) &= 2.55\ \mathrm{mb}.\end{aligned} \tag{48}$$

While these values reduce the discrepancy with experiment, they still fall short of good agreement. Although such a disagreement is disturbing, it is not unduly so, for, as alluded to earlier, the discrepancy seems to have its origin in the inapplicability of the second-order perturbation procedure when applied to a loosely bound system. Since both potentials A and B underbind, and it is reasonable to suppose that an underbound system should, if anything, yield too large an rms radius, the difficulty may be expected to lie primarily with the use of the restricted perturbation series. To further strengthen this point of view, it is instructive to employ the effective-range, model-independent estimate of σ_b due to Levinger,[18]

$$\sigma_b{}^{A,B}(\mathrm{H}^2) = \frac{\pi^2 e^2 \hbar}{6Mc\epsilon_{A,B}}\left[1-\left(\frac{m\epsilon_{A,B}}{\hbar^2}\right)^{1/2} r_{0t}{}^{A,B}\right]^{-1}, \tag{49}$$

in which m is the nucleon mass, ϵ_i is the deuteron binding energy predicted by the potential i, and $r_{0t}{}^i$ is the corresponding triplet effective range, and the other symbols retain their usual meaning. If in the approximation (49) we now insert the values of ϵ_i and $r_{0t}{}^i$ obtained by a numerical solution of the coupled differential equa-

[15] R. Herman and R. Hofstadter, *High-Energy Electron Scattering Tables* (Stanford University Press, Stanford, California, 1960).

[16] J. A. McIntyre and S. Dhar, Phys. Rev. **106**, 1074 (1957).

[17] This value is, in fact, the same as that used by J. N. Pappademos [Nucl. Phys. **42**, 122 (1963)] in his calculation of the radius and Coulomb energy of He^3, and is not too far from the result of L. N. Hand, D. G. Miller, and R. Wilson [Rev. Mod. Phys. **35**, 335 (1963)].

[18] J. S. Levinger, Phys. Rev. **97**, 970 (1955).

tions (see Sec. VI), we find

$$\sigma_b{}^A(\mathrm{H}^2)=4.10\ \mathrm{mb},$$
$$\sigma_b{}^B(\mathrm{H}^2)=3.77\ \mathrm{mb},$$

in fairly good agreement with experiment. The corresponding radii are

$$\langle R_A{}^2(\mathrm{H}^2)\rangle^{1/2}=2.07\ \mathrm{F},$$
$$\langle R_B{}^2(\mathrm{H}^2)\rangle^{1/2}=1.98\ \mathrm{F}.$$

Thus we anticipate that the deuteron systems A and B, if solved exactly, would be associated with rms radii which do not deviate appreciably from that of our actual physical system. The sources of this deviation and their relation to the binding energies calculated by the second-order-perturbation procedure will be discussed in more detail in Sec. VI.

Before going on with the calculation of the analogous cross sections for the trinucleon and alpha particle, it is desirable to devote a few remarks to the suitability of the second-order procedure as applied to more complex systems. In view of the comments in the preceding paragraph, we are led to expect that the results for H^3, He^3, and He^4 will turn out to be in much better agreement with experiment than they were for the deuteron. The correctness of this statement relies on the fact that the physical system is more compact and more tightly bound, so that the harmonic oscillator provides a more acceptable description of the potential seen by the individual nucleons. This, in turn, means that the higher orders of the perturbation expansion should become relatively less important. In other words, there will be a more rapid convergence of the series, and truncation at second order should not be too misleading as regards either the binding energy or the photonuclear cross sections. Insofar as the experimental data are available, we shall find that the above expectations are verified.

The problem of computing the $E1$ bremsstrahlung-weighted cross section for the three-body system consists in determining the value of $\langle R^2(\mathrm{T})\rangle$ as defined by Eq. (41). Writing out the expectation values of $r_{12}{}^2$ explicitly, we find

$$\langle R^2(\mathrm{T})\rangle=\frac{1}{3N^2}\int\psi_0{}^* r_{12}{}^2\psi_0 d\tau+\frac{2}{3N^2}\int\psi_0{}^* r_{12}{}^2 \times\sum_{n\neq 0}\frac{W_{0n}\psi_n}{E-E_n}d\tau+\frac{1}{3N^2}\int\sum_{n\neq 0}\frac{W_{0n}{}^*\psi_n{}^*}{E-E_n}(r_{12}{}^2) \times\sum_{m\neq 0}\frac{W_{0m}\psi_m}{E-E_m}d\tau. \quad (50)$$

Once again, we neglect the second-order contribution in Eq. (50), since this contribution was found to be small for the deuteron and, in line with our earlier comments, we therefore expect a negligible contribution to the trinculeon mean-square radius. With this modification and the suppression of N^2, Eq. (50) becomes

$$\langle R^2(\mathrm{T})\rangle=a^2(q_{12}{}^2)_{00}-\frac{2a^2}{3}\left\{\int_0^\infty e^{\lambda(E-U)}(q_{12}{}^2eV)_{00}d\lambda +\frac{(q_{12}{}^2)_{00}V_{00}}{E-E_0}+\frac{1}{E-E_0-2\hbar\omega} \times[(q_{12}{}^2M)_{00}-(q_{12}{}^2)_{00}M_{00}]\right\}. \quad (51)$$

The matrix elements needed in Eq. (51) are so similar to those previously evaluated for $\sigma_{\mathrm{int}}(\mathrm{T})$ that further discussion seems unnecessary. We find the following results for potentials A and B:

$$\langle R_A{}^2(\mathrm{T})\rangle=2.45\ \mathrm{F}^2, \quad (52)$$
$$\langle R_B{}^2(\mathrm{T})\rangle=2.49\ \mathrm{F}^2. \quad (53)$$

The $E1$ bremsstrahlung-weighted cross sections for the two potentials follow from Eq. (39). The values are

$$\sigma_b{}^A(\mathrm{T})=2.36\ \mathrm{mb}, \quad (54)$$
$$\sigma_b{}^B(\mathrm{T})=2.40\ \mathrm{mb}. \quad (55)$$

It will be observed that both of these numbers are within 7% of the experimental result

$$[\sigma_b(\mathrm{He}^3)=2.53\pm 0.19\ \mathrm{mb}]$$

reported by Fetisov, Gorbunov, and Varfolomeev.[11] Furthermore, they also lie between the range of results

$$\sigma_b(\mathrm{H}^3)=2.08\pm 0.16\ \mathrm{mb}, \quad (56)$$
$$\sigma_b(\mathrm{He}^3)=2.66\pm 0.17\ \mathrm{mb}, \quad (57)$$

obtained by substituting values of the charge radii from electron-scattering data[19] into Eq. (38). While the disparity in experimental values precludes any more sensitive test of the perturbative technique at this time, it is evident that the predicted values, Eqs. (54) and (55), are not inconsistent with the data and in relatively better agreement than was true for the deuteron. Unfortunately, the source of the difference in the charge radii for the mirror system is not well understood. Indeed, if charge symmetry is preserved, $\sigma_b(\mathrm{H}^3)$ should have the same value as $\sigma_b(\mathrm{He}^3)$.[20,21] This, combined with the fact that[22]

$$\sigma_b=\frac{4\pi^2}{3}\left(\frac{e^2}{\hbar c}\right)\frac{N^2Z^2}{A^2}\langle R_{pn}{}^2\rangle \quad (58)$$

is satisfied for the triton regardless of spatial-symmetry considerations,[20] leads to an apparent contradiction between the electron-scattering result, Eq. (56), and

[19] H. Collard, R. Hofstadter, E. B. Hughes, A. Johansson, M. R. Yearian, R. B. Day, and R. T. Wagner, Phys. Rev. **138**, B57 (1965).
[20] P. O. Davey and H. S. Valk, Phys. Letters **7**, 335 (1963).
[21] R. Bösch, J. Lang, R. Muller, and W. Wölfli, Helv. Phys. Acta **38**, 753 (1965).
[22] J. S. Levinger and H. A. Bethe, Phys. Rev. **78**, 115 (1950).

TABLE IV. Comparison of results derived in this paper with experiment. σ_b = bremsstrahlung-weighted cross section; $\langle R_c^2\rangle^{1/2}$ = rms charge radius; W_H = harmonic mean energy.

Nucleus	Source	σ_b (mb)	$\langle R_c^2\rangle^{\frac{1}{2}}$ (fm)	W_H (MeV)
H^2	Potential A	2.36[a]	1.78	18.6
		2.39[b]	1.79	18.4
	Potential B	2.55	1.84	16.9
	e^- scattering expt.[c]	3.82±0.21	2.17±0.05	...
	Photo disintegration expt.[d]	3.8	2.11	10.4
H^3,He^3	Potential A	2.36	1.78	27.5
	Potential B	2.40	1.79	26.8
	e^- scattering expt.[e]	H^3−2.0±0.16	H^3−1.70±0.05	
		He^3−2.66±0.17	He^3−1.87±0.05	...
	Photodisintegration expt. (He^3)[f]	2.53±0.19	1.81±0.06	24.5
He^4	Potential A	2.61	1.66	41.0
	Potential B	2.70	1.68	39.2
	e^- scattering expt.[g]	2.48±0.17	1.63±0.04	...
	Photodisintegration expt.[h]	2.4 ±0.15	1.61±0.04	39.6

[a] Second-order terms and N^2 neglected. [b] Second-order terms and N^2 included. [c] References 15 and 16. [d] References 12, 13. [e] Reference 19. [f] Reference 11. [g] Reference 23. [h] Reference 2.

the photonuclear value of $\sigma_b(He^3)$. Whether this contradiction in fact remains, and, if so, whether it has its source in a breakdown of charge symmetry, or a failure of Siegert's theorem, awaits further experimental and theoretical study. In particular, a precise measurement of $\sigma_b(H^3)$ would be helpful in clarifying this problem.

Expanding Eq. (41) for the alpha particle and neglecting N^2 yields

$$\langle R^2(He^4)\rangle=\frac{3a^2}{8}\Big\{(q_{12}^2)_{00}-2\int_0^\infty e^{\lambda(E-U)}[(q_{12}^2eR_{12})_{00}+(q_{12}^2eC_{12})_{00}+4(q_{12}^2eR_{23})_{00}+4(q_{12}^2eC_{23})_{00}+(q_{12}^2eR_{34})_{00}+(q_{12}^2eC_{34})_{00}-(q_{12}^2eM)_{00}-U_{00}(q_{12}^2e)_{00}]d\lambda\Big\}. \quad (59)$$

Evaluating (59) we find

$$\langle R_A^2(He^4)\rangle=2.04\ F^2, \quad (60)$$

$$\langle R_B^2(He^4)\rangle=2.11\ F^2. \quad (61)$$

Hence

$$\sigma_b^A(He^4)=2.61\ mb, \quad (62)$$

$$\sigma_b^B(He^4)=2.70\ mb. \quad (63)$$

For potential A, these results differ by a small amount from those reported in Ref. 7 due to a slight change in the computed value of the mean-square radius.

These values of the bremsstrahlung-weighted cross section are in not unsatisfactory agreement with the experimental result of 2.4±0.15 mb found by Gorbunov and Spiridonov.[8] Moreover, acceptable agreement is obtained with the value $\sigma_b(He^4)=2.48\pm0.17$, derived from recent electron-scattering data.[23,24] The results derived in this section for the bremsstrahlung-weighted cross section σ_b, the rms charge radius $\langle R_c^2\rangle^{1/2}$, and the harmonic mean energy W_H, as well as those predicted from electron-scattering and photodisintegration experiments, are summarized in Table IV. Here it is seen that the experimental and theoretical values are not inconsistent with each other except in the case of the deuteron, where, for the reasons mentioned earlier, we should not expect agreement.

V. FURTHER ANALYSIS OF DEUTERON CALCULATIONS

A. Third-Order Binding-Energy Estimates

A measure of the effectiveness of any second-order perturbation scheme like the above can be obtained by examining the magnitude of the third-order term in the expansion. If the third-order term in the binding energy (say ϵ_3) is small compared to second order (ϵ_2), we can be satisfied that most of the contribution has been obtained [the remainder being of order of magnitude $\epsilon_3/(1-\epsilon_3/\epsilon_2)$].[25] Unfortunately, complete calculation of a third-order contribution is sufficiently extensive to warrant an examination of the possibility of using approximating procedures. Therefore, we first look at several rough estimates for the third-order contributions and then compare with the exact third-order term to ascertain their possible validity. In the present context, the third-order contribution is given by the term

$$\epsilon_3=\sum_{n,m\neq0}\frac{W_{0n}W_{nm}W_{m0}}{(E-E_n)(E-E_m)}, \quad (64)$$

in the energy expansion

$$E-E_0=-\delta\hbar\omega=\sum_{n\neq0}\frac{W_{0n}W_{0n}}{E-E_n}+\sum_{n,m\neq0}\frac{W_{0n}W_{nm}W_{m0}}{(E-E_n)(E-E_m)}. \quad (65)$$

[23] H. Frank, D. Haas, and H. Prange, Phys. Letters **19**, 391 (1965); **19**, 719 (1966).

[24] R. F. Frosch, R. E. Rand, J. S. McCarthy, and M. R. Yearian, Phys. Rev. (to be published).

[25] The estimate follows from the assumption that each successive term decreases in the constant ratio ϵ_3/ϵ_2.

Following a method adopted by Feingold,[26] we first determine an approximate mean energy ($\bar{\epsilon}$) for the spectrum of states E_n such that the second-order contribution ($-\delta\hbar\omega$) is given by the relation

$$-\delta\hbar\omega=\frac{(W^2)_{00}}{E-\bar{\epsilon}}=\epsilon_2. \tag{66}$$

An expansion of W^2, where $W=V-M-U$, leads to the result that

$$-\delta(\hbar\omega)^2\chi=(V^2)_{00}+(M^2)_{00} -2(VM)_{00}-(V_{00}-M_{00})^2, \tag{67}$$

where $\chi=(E-\epsilon)/\hbar\omega$. We then determine χ from Eq. (67) for the potential under consideration, and assume that

$$\epsilon_3\simeq(W^3)_{00}/\chi^2(\hbar\omega)^2. \tag{68}$$

Once again expanding, we find

$$\text{3rd order B.E. est.}=\frac{1}{\chi^2(\hbar\omega)^2}\{(V^3)_{00}-(M^3)_{00}+3(M^2V)_{00} -3(MV^2)_{00}+2(V_{00}-M_{00})^3-3(V_{00}-M_{00}) \times[(V^2)_{00}-2(VM)_{00}+(M^2)_{00}]\}. \tag{69}$$

The expansion, as well as the numerical values of the matrix elements, is given in Appendix D for both potentials A and B. A direct substitution of the required values into Eqs. (67) and (69) gives the following results:

potential A: $\chi=-7.077$; 3rd-order H^2 B.E. est. $=+0.90$ MeV. (70)

potential B: $\chi=-7.966$; 3rd-order H^2 B.E. est. $=+0.37$ MeV. (71)

To summarize, the use of an average energy denominator to evaluate third order indicates that for potential A the magnitude of this contribution is 15.8% of second order, while for potential B its magnitude is 10.5%. Although on the basis of these preliminary and quite crude approximations, it appears that the third-order contribution is larger than desirable, it is also clear that further verification of the results is required.

In addition to the above technique for making an estimate, an alternative simple procedure is suggested. The approximate evaluation of the mean energy $\bar{\epsilon}$ of the spectrum of states E_n may also be accomplished by noting that the normalization can be written as

$$N^2-1=(W^2)_{00}/(E-\bar{\epsilon})^2. \tag{72}$$

Substituting $y=(E-\bar{\epsilon})/\hbar\omega$ into Eq. (72), in the same manner as χ, we find

$$y^2=(W^2)_{00}/(N^2-1)(\hbar\omega)^2 \tag{73}$$

and

$$\epsilon_3\simeq(W^3)_{00}/y^2(\hbar\omega)^2. \tag{74}$$

Thus making use of the expansion of W^2 and W^3 given above and the results given in Appendix D, we now arrive at the following values:

potential A: $y^2=32.914$; 3rd-order H^2 B.E. est. $=+1.37$ MeV, (75)

potential B: $y^2=36.378$; 3rd-order H^2 B.E. est. $=+0.65$ MeV. (76)

We note that these estimates are essentially in agreement with those given in Eqs. (70) and (71). However, since the product of energy denominators in ϵ_3 is of a different character from those encountered in either N^2-1 or ϵ_2, there still exists a question about whether the magnitude of the mean energies used in these approximations is significant. In order to substantiate their validity, at least to some degree, it is necessary to compute ϵ_3 exactly. Before proceeding to this task, however, we first consider third-order estimates to σ_{int}.

B. σ_{int} Third-Order Estimates

An application of Eq. (3) to the deuteron yields the result

$$\sigma_{\text{int}}(H^2)=30\left\{1-\frac{M}{3\hbar^2}\int\Psi_0^*(V_{12}r_{12}^2)\Psi_0 d\tau\right\}, \tag{77}$$

where Ψ_0 is given by Eq. (2) and the numerical constant carries the units of MeV mb. Here we see that expanding the integrand gives the second-order contribution to σ_{int} as

$$-\frac{20M}{N^2\hbar^2}\sum_{n\neq0}\frac{(V_{12}r_{12}^2)_{0n}W_{n0}}{(E-E_n)}.$$

We now determine the average energy denominator in a manner analogous to that described earlier for the binding-energy estimates, by requiring that

$$\text{2nd-order cross section}=-\frac{20M}{N^2\hbar^2}\left\{\frac{[V_{12}r_{12}^2(V-M-U)]_{00}}{(E-\bar{\epsilon})}\right\}, \tag{78}$$

or

$$=-\frac{20}{zN^2(\hbar\omega)^2}[V_{12}q_1^{22}(V-M-U)]_{00},$$

where $z=(E-\bar{\epsilon})/\hbar\omega$ and $r_{12}^2=a^2q_{12}^2=(\hbar/M\omega)q_{12}^2$. Then the desired approximation to third order becomes

$$\text{3rd-order } \sigma_{\text{int}}(H^2)\text{ est.}=-\frac{10}{z^2(\hbar\omega)^3}[(V-M-U)V_{12}q_{12}^2(V-M-U)]_{00}. \tag{79}$$

[26] A. M. Feingold, Phys. Rev. **101**, 258 (1956).

The requisite matrix elements and numerical values for the calculation of Eq. (79) are supplied in Appendix D. Their use yields the following values:

potential A: $z=-5.824$; 3rd-order $\sigma_{\rm int}(H^2)$ est.
$=-0.41$ MeV mb, (80)

potential B: $z=-5.342$; 3rd-order $\sigma_{\rm int}(H^2)$ est.
$=-0.09$ MeV mb. (81)

It is obvious from these results that the appearance of the additional r_{12}^2 in the matrix elements considerably assists in the suppression of third order, and it would appear that truncation of the perturbation series at second order for the integrated cross section is at least qualitatively justified. In order to assess the correctness of this and the previous estimates, we turn to an exact calculation of the third-order binding-energy contribution.

C. Exact Third-Order Binding Energy

The determination of ϵ_3 as defined in Eq. (69) consists for the most part in the careful application of the methods of Ref. 1. Because of the algebraic complexity, it is well to introduce an abbreviated notation before beginning the calculation.

In the following we will use the symbols $e\equiv e^{-\lambda H_0'}$ and $e'\equiv e^{-\lambda' H_0'}$ and rewrite the potential A in the form

$$V=R+C+S, \tag{82}$$

where each term has an obvious meaning from Eq. (9).

Then applying Eq. (17) of Ref. 1 to ϵ_3, we find

$$\epsilon_3=\sum_{n,m\neq 0}\frac{W_{0n}W_{nm}W_{m0}}{(E-E_n)(E-E_m)}=\int_0^\infty\int_0^\infty e^{\lambda(E-U)}e^{\lambda'(E-U)}(We'WeW)_{00}d\lambda' d\lambda, \tag{83}$$

where

$$\begin{aligned}(We'WeW)_{00}=&(Ve'VeV)_{00}-(Ve'MeV)_{00}-(Ve'UeV)_{00}-2(Ve'VeM)_{00}-2(Ve'VeU)_{00}+2(Ve'MeM)_{00}\\&+2(Ve'MeU)_{00}+2(Ve'UeM)_{00}+2(Ve'UeU)_{00}+(Me'VeM)_{00}+2(Me'VeU)_{00}-2(Me'MeU)_{00}-(Me'MeM)_{00}\\&-(Me'UeM)_{00}-2(Me'UeU)_{00}+(Ue'VeU)_{00}-(Ue'MeU)_{00}-(Ue'UeU)_{00}.\end{aligned} \tag{84}$$

The reader will be spared details in the evaluation of this algebraic morass, but for the sake of completeness, a catalog of the term-by-term results appears in Appendix D.

Combining the results of Appendix D, one finds that

$$\epsilon_3{}^A=+1.24 \text{ MeV}. \tag{85}$$

The fact that this number lies between our two previous estimates, Eqs. (70) and (75), is most likely fortuitous and should not be taken too seriously; however, the proximity of the exact ϵ_3 to the approximate values is rather encouraging and permits us to perhaps place a little more confidence in the third-order estimates of $\sigma_{\rm int}$ for which exact calculations were not carried out.

Taking into account these approximate third-order estimates, we find for the resultant values of

$$\begin{aligned}&\sigma_{\rm int}{}^A(H^2)\simeq 43.6 \text{ MeV mb},\\&\sigma_{\rm int}{}^A(H^3)\simeq 42.9 \text{ MeV mb}.\end{aligned} \tag{86}$$

Thus we see that while the corrected values are lower than second-order perturbation theory would predict, they are still greater than the experimental result. This, of course, is not unexpected, since both potentials A and B somewhat underbind, and a more loosely bound system in turn should yield a somewhat larger integrated cross section.

A less encouraging consequence of Eq. (85) is that the magnitude of ϵ_3 would seem to throw some doubt on the extensive use of parameters obtained by minimization only through second order and to point up the need for their redetermination through third order. Unhappily, the latter prospect deprives the perturbative approach of much of its simplicity and elegance, and suggests that we consider the practicability either of minimization with the use of approximate third-order terms such as might be obtained by modifications of the techniques employed in this section, or of the inclusion of refinements in the perturbation series.[27]

VI. SUMMARY AND CONCLUSIONS

We have in the preceding sections utilized what we believe to be a simple and consistent perturbative prescription for determining $E1$ photonuclear cross sections in the lightest nuclei. This procedure was first introduced by Goldhammer and Valk[7] in their study of the photodisintegration of He^4, and its present application forms a logical extension of their work, completing a program of calculations in the $1s$ shell. With a full set of calculations at our disposal, we are now in a position to discuss not only the merits of the perturbation approach, but also the defects and limitations, limitations which would not always be evident in isolated calculations.

It is the intent of the present section to analyze in more detail the various contributions both to the $E1$ cross sections and to the binding energies and to ascer-

[27] Preliminary results of such a study have been reported by H. Neumann and H. S. Valk, Bull. Am. Phys. Soc. **9**, 545 (1964); H. S. Valk and H. Neumann, *ibid.* **10**, 448 (1965). Some of this work is summarized in Ref. 29.

TABLE V. Contributions to the binding energy of the deuteron.

	Perturbation theory					Exact[a]				
Potential	ϵ (MeV)	ϵ_3 Est. 1 (MeV)	ϵ_3 Est. 2 (MeV)	ϵ_3 (MeV)	$\%D$	ϵ (MeV)	Quadrupole moment (cm^2)	$\%D$	a_{0t}(fm)	r_{0t} (fm)
A	−2.16	+0.90	+1.37	+1.24	6.7	−1.91	2.99×10^{-27}	3.99	6.06	1.7
B	−2.08	+0.37	+0.65	···	4.7	−2.16	2.75×10^{-27}	2.93	Very Good	Very Good
					Accepted	−2.22[b]	2.82×10^{-27}[c]	···	5.396[d]	1.726[d]

[a] References 28, 29.
[b] R. Wilson, *The Nucleon-Nucleon Interaction* (Interscience Publishers, Inc. New York, 1963).
[c] J. P. Auffray, Phys. Rev. Letters **6**, 120 (1961).
[d] H. P. Noyes, Phys. Rev. **130**, 2025 (1963).

tain on this basis to what extent the second-order perturbation procedure might be expected to yield meaningful values. In the process of this analysis, we hope to clarify, at least to some degree, the role played in such calculations by the repulsive core and tensor component of the nucleon-nucleon interaction.

As a starting point for the discussion, it is desirable to consider the contributions in different orders (order here refers to the number of times the interaction operator appears in a given contribution) to the integrated $E1$ cross sections for the $1s$ shell nuclei. These are shown in Tables I–III.

Here it is seen that for both potentials A and B, the ratio $R^{A,B}$ of the total second-order contribution to that in first order is a decreasing function of the atomic weight corresponding to the following sequence of values:

$$\begin{aligned} &R^A(\mathrm{H}^2)=1.67\,,\quad R^A(\mathrm{T})=1.08\,,\quad R^A(\mathrm{He}^4)=0.70\,;\\ &R^B(\mathrm{H}^2)=0.72\,,\quad R^B(\mathrm{T})=0.47\,,\quad R^B(\mathrm{He}^4)=0.36\,. \end{aligned} \tag{87}$$

Although it is not possible to assign significance to the individual numbers, their relative magnitudes for a given potential do give some indication of the rapidity of convergence of the corresponding perturbation expansions, and hence of the validity of truncating the series at second order. Since these series are generated from a sum of harmonic oscillator Hamiltonians, this convergence rate is related to the fact that for the $1s$ shell, the higher the atomic weight, the more appropriate is an oscillator description. From such a statement one is led to conclude, as has in fact already been observed, that the deuteron is not as suitable for a second-order perturbation calculation as is the trinucleon or the alpha particle.

The occurrence of ratios greater than unity in (87) has its source in the presence of the repulsive-core term in potential A. To see this, we note from Tables I–III that the over-all contribution to σ_{int} or σ_b from either potential is almost the same. [This result is not too surprising in view of the fact that the parameters characterizing both potentials were constrained to fit approximately the same set of deuteron ground-state properties; compare the remarks following Eqs. (5) and (7).] Since potential A is rather different from potential B_8 there must be compensatory changes occurring within the individual components. Indeed, reference to Tables I–III will show, for example, in the case of the trinucleon that the introduction of a repulsive core, as in potential A, brings about a 26% decrease in the central and a 57% *increase* in the tensor contribution from those components in potential B. Because of the fact that this large tensor component of potential A can only enter initially in second order, we are led immediately to a distinction between the greater ratios on the top line of (87) and the smaller ones below. It is worth noting that the above arguments regarding the relative behavior of components within the different potentials would be expected to retain a qualitative validity even though the perturbation procedure might be inappropriate.

Since we are using a semivariational-perturbation approach, the occurrence of a relatively large second-order contribution is not necessarily in itself a cause for concern. However, it should serve to indicate possible limitations imposed by a second-order minimization process. In order to understand some of these limitations, we refer to the deuteron binding-energy contributions listed in Table V. There we find, as derived in Sec. V (see Refs. 28–29), that the third-order contribution ϵ_3 is non-negligible, and has the effect of reducing the computed binding energy ϵ in second order from 2.16 MeV for potential A down to 0.92 MeV. It should be noted that this value lies well above the result of 1.91 MeV computed by Signell[28] and Neumann[29] directly from potential A by a numerical solution to the coupled differential equations for the S- and D-state radial functions. It is thus consistent with the fact that the first-order wave function, Eq. (2), represents a trial function for the Hamiltonian, Eq. (1), when the energy is computed through third order. The disparity between the upper limit of −0.92 MeV and the exact solution of −1.91 MeV is again an indication of the poor quality of Eq. (2) (with $\hbar\omega=14.49$ MeV) when applied to a loosely bound system: It has already been observed in Sec. IV that the true rms radius corresponding to potential A probably should be around

[28] P. Signell (private communication).
[29] H. Neumann, Ph.D. thesis, University of Nebraska, 1965 (unpublished).

2 F,[30] whereas the use of Eq. (2), led, via Eq. (43), to a value of 1.58 F. In other words, the first-order wave function, Eq. (2), is to be associated with a much more localized system and derives its contribution to the binding energy from a much smaller region about the origin than is demanded by potential A. Although a complete evaluation was not carried out for ϵ_3 with potential B, the estimates in Table V would indicate that a similar set of arguments applies. The fact that the disparity between the estimated binding energy and the numerical solution of Signell and Neumann is less in this case may be attributed to the role of the core in "pushing out" the long-range tensor force in potential A: Since the two trial functions derive the majority of their binding over essentially the same region of space [compare Eqs. (43) and (44)], the absence of the core in potential B corresponds to greater binding with its trial function.

On the basis of the above remarks it is possible to assign most of the lack of agreement between the theoretical and experimental photonuclear cross sections noted in Secs. III C and IV to the inappropriateness of Eq. (2) with an $\hbar\omega$ determined by means of a second-order minimization process. The question as to whether this deficiency arises from the first-order perturbative form of Eq. (2) or from the procedure for determining $\hbar\omega$ is not clear from the present analysis. However, other calculations indicate that a more suitable $\hbar\omega$ can be found for the loosely bound system by higher-order minimization.[27]

The evidence cited in the preceding paragraphs for the localized character of Eq. (2) (with an $\hbar\omega$ determined in second order) seems to augur well for its usefulness when applied to more localized systems. Here, not only should the form of Eq. (2) be more appropriate, but the ϵ_3 should be so small as to throw less doubt on the second-order minimization procedure for finding $\hbar\omega$. The favorable comparison of experiment with the theoretical cross-section predictions for the three- and four-body nuclei seems to bear this out. Unfortunately, the uncertainties associated with the experimental data are as yet too great to yield a critical quantitative test of the technique. We can, however, from our experience with the deuteron, conjecture that the calculated values of $\sigma_{\text{int}}{}^A(\text{T})$ and $\sigma_{\text{int}}{}^A(\text{He}^4)$ should be slightly overestimated [recall that even the poor deuteron function gave a 10% overestimate for $\sigma_{\text{int}}{}^A(\text{H}^2)$]. Similarly, the values of $\sigma_b{}^A(\text{T})$ and $\sigma_b{}^A(\text{He}^4)$ should be underestimates.

Although we must acknowledge certain defects associated with the potentials and the truncation of perturbation series, the approach outlined here has several significant advantages: One lies in its ability to provide a simple and systematic prescription for calculating the photonuclear cross sections with relatively realistic forms for the interaction operators; another in its ability to provide a complete set of $1s$ shell calculations with the same potential and the same type of wave-function properties. The latter is particularly desirable if one is to make valid comparisons among the different contributions to the cross sections and their behavior as functions of atomic weight. The fact that fair agreement could be achieved with both interaction operators for both the trinucleon and the alpha particle without additional adjustable parameters is most encouraging, and gives a strong indication that it is possible to fit all of the photonuclear cross sections in the shell with one form of the nucleon-nucleon interaction.

APPENDIX A: THE PARAMETERS

After performing various consistency checks, the values determined by Goldhammer for potential A and by Bolsterli and Feenberg[1] for potential B were accepted. For potential A,

$$G=34.5\text{ MeV},\quad K=-1.70,\quad K'=0.6875,\quad K''=-3.1099;$$

and for potential B,

$$G=17.4\text{ MeV},\quad K=-1.695,\quad \text{and}\quad K''=-3.0002.$$

The remaining parameters required for the $1s$-shell nuclei are listed in Table VI.

TABLE VI. Parameters for $1s$-shell nuclei.

	Potential A					Potential B			
Nucleus	δ	$\hbar\omega$ (MeV)	η	ζ	N^2	δ	$\hbar\omega$ (MeV)	η	N^2
H^2	0.3934	14.49	1.42	1.0525	1.08	0.274	13.2	1.75	1.06
H^3,He^3	0.489	17.94	1.52	1.065	1.11	0.342	17.4	1.99	1.07
He^4	0.65	24.15	1.70	1.0875	1.14	0.485	22.2	2.27	1.10

[30] In fact an exact numerical calculation (Ref. 29) shows it to be 2.16 F.

APPENDIX B: INTEGRAL TRANSORMS

The following results reduce the evaluation of the matrix elements to a drill. Each can be demonstrated by the careful application of Eq. (17) of Ref. 1.

$$e^{-\lambda H_0'}e^{-\gamma Q_{12}^2}=\left(\frac{k}{2g\gamma+1}\right)^{3/2}\times\exp\left\{-\left[\frac{1}{2g}-\frac{k^2}{2g(2g\gamma+1)}\right]Q_{12}^2\right\}.$$

Thus,

$$e^{-\lambda H_0'}e^{-\frac{1}{2}Q_{12}^2}=u^{3/4}e^{-\frac{1}{2}Q_{12}^2},$$

$$e^{-\lambda H_0'}e^{-(2\alpha^2+\frac{1}{2})Q_{12}^2}=\frac{(\eta-1)^{3/2}u^{3/4}}{(\eta-u)^{3/2}}\exp\left[-\frac{1}{2}\left(\frac{\eta+u}{\eta-u}\right)Q_{12}^2\right].$$

By taking derivatives with respect to γ, we find that

$$e^{-\lambda H_0'}Q_{12}^2e^{-\frac{1}{2}Q_{12}^2}=\{\tfrac{3}{2}(1-u)+uQ_{12}^2\}u^{3/4}e^{-\frac{1}{2}Q_{12}^2},$$

and

$$e^{-\lambda H_0'}Q_{12}^4e^{-\frac{1}{2}Q_{12}^2}=\tfrac{1}{4}\{15(1-u)^2+20u(1-u)Q_{12}^2+4u^2Q_{12}^4\}u^{3/4}e^{-\frac{1}{2}Q_{12}^2}.$$

The evaluation of the angular part of the tensor terms is more complicated but straightforward and gives the result

$$e^{-\lambda H_0'}Q_{12}^2S_{12}\exp(-\tfrac{1}{2}Q^2-(2\alpha^2+\tfrac{1}{2})Q_{12}^2]=\frac{(\eta-1)^{7/2}u^{5/2}}{(\eta-u)^{7/2}}Q_{12}^2S_{12}e^{-\frac{1}{2}Q^2}\exp\left[-\frac{1}{2}\left(\frac{\eta+u}{\eta-u}\right)Q_{12}^2\right].$$

APPENDIX C: CROSS SECTION MATRIX ELEMENTS

The following list of matrix elements are those needed in the determination of the integrated cross sections. Where the algebraic form of the integrals is the same as one previously recorded it will not be repeated.

1. The Alpha Particle

The matrix elements required by Eq. (16) are given below.

Potential A:

$$(C_{12}q_{12}^2)_{00}=\frac{3K(\eta-1)^{3/2}}{\eta^{7/2}}\hbar\omega=-0.792669\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}^2eR_{12})_{00}d\lambda=\tfrac{3}{2}KK'(\hbar\omega)(\zeta-1)^{1/2}(\eta-1)^{3/2}\int_0^1\frac{(\zeta-u)u^{\delta/2-1}}{(\eta\zeta-u)^{5/2}}du=-0.230999\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}^2eC_{12})_{00}d\lambda=\tfrac{3}{2}K^2(\hbar\omega)(\eta-1)^2\int_0^1\frac{(\eta-u)u^{\delta/2-1}}{(\eta^2-u)^{5/2}}du=+0.831280\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}((S_{12}'q_{12}^2eS_{12}))_{00}d\lambda=\frac{105}{4}K''^2(\hbar\omega)(\eta-1)^4\int_0^1\frac{(\eta-u)u^{\delta/2}}{(\eta^2-u)^{9/2}}du=+1.225876\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}^2eR_{12})_{00}d\lambda=12KK'(\hbar\omega)(\eta-1)^{3/2}(\zeta-1)^{1/2}\int_0^1\frac{(4\zeta-u)u^{\delta/2-1}}{(4\eta\zeta-u)^{5/2}}du=-0.224495\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}^2eC_{13})_{00}d\lambda=12K^2(\hbar\omega)(\eta-1)^2\int_0^1\frac{(4\eta-u)u^{\delta/2-1}}{(4\eta^2-u)^{5/2}}du=+0.795874\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}((S_{12}'q_{12}^2eS_{13}))_{00}d\lambda=2^6\left(\frac{105}{4}\right)K''^2(\eta-1)^4(\hbar\omega)\int_0^1\frac{(4\eta-u)u^{\delta/2}}{(4\eta^2-u)^{9/2}}du=+0.188748\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}^2eR_{34})_{00}d\lambda=\tfrac{3}{2}KK'(\hbar\omega)\frac{(\zeta-1)^{1/2}(\eta-1)^{3/2}}{\eta^{5/2}\zeta^{3/2}}\int_0^1u^{\delta/2-1}du=-0.218683\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}^2eC_{34})_{00}d\lambda=\tfrac{3}{2}K^2(\hbar\omega)\frac{(\eta-1)^2}{\eta^4}\int_0^1u^{\delta/2-1}du=+0.782539\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}^2eM)_{00}d\lambda=\tfrac{3}{8}K(\hbar\omega)\frac{(\eta-1)^{3/2}}{\eta^{7/2}}\int_0^1[9\eta+(2\eta-5)u]u^{\delta/2-1}du=-2.67349\hbar\omega,$$

$$U_{00}\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}{}^2e)_{00}d\lambda=\tfrac{3}{2}KU_{00}\frac{(\eta-1)^{3/2}}{\eta^{5/2}}\int_0^1 u^{\delta/2-1}du=+6.12701\hbar\omega,$$

$$U_{00}=\hbar\omega\left[\frac{6K'(\zeta-1)^{1/2}}{\zeta^{3/2}}+\frac{6K(\eta-1)^{1/2}}{\eta^{3/2}}-\frac{9}{4}\right]=-5.024204\hbar\omega.$$

Potential B:

$$(C_{12}q_{12}{}^2)_{00}=-0.937415\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}{}^2eC_{12})_{00}d\lambda=+1.08053\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}((S_{12}{}'q_{12}{}^2eS_{12}{}'))_{00}d\lambda=3K''{}^2(\hbar\omega)(\eta-1)^4\int_0^1\frac{u^{\delta/2}du}{(\eta-u)(\eta^2-u)^{5/2}}=+0.775410\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}{}^2eC_{13})_{00}d\lambda=+1.08146\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}((S_{12}{}'q_{12}{}^2eS_{13}{}'))_{00}d\lambda=2^6(3)K''{}^2(\hbar\omega)(\eta-1)^4\int_0^1\frac{u^{\delta/2}du}{(4\eta-u)(4\eta^2-u)^{5/2}}=+0.236244\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}{}^2eC_{34})_{00}d\lambda=+1.079501\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}{}^2eM)_{00}d\lambda=-4.329722\hbar\omega,$$

$$U_{00}\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}{}^2e)_{00}d\lambda=+10.82584\hbar\omega,$$

$$U_{00}=-5.601074\hbar\omega.$$

2. The Trinucleon Matrix Elements

$$(C_{12}q_{12}{}^2)_{00}=\frac{3K(\hbar\omega)(\eta-1)^{3/2}}{\eta^{5/2}}=-0.671377\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}{}^2eR_{12})_{00}d\lambda=\tfrac{3}{2}KK'(\hbar\omega)(\zeta-1)^{1/2}(\eta-1)^{3/2}\int_0^1\frac{(\zeta-u)u^{\delta/2-1}}{(\eta\zeta-u)^{5/2}}du=-0.241126\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}{}^2eC_{12})_{00}d\lambda=\tfrac{3}{2}K^2(\hbar\omega)(\eta-1)^2\int_0^1\frac{(\eta-u)u^{\delta/2-1}}{(\eta^2-u)^{5/2}}du=+0.978563\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}((S_{12}{}'q_{12}{}^2eS_{12}{}'))_{00}d\lambda=\frac{105}{4}K''{}^2(\hbar\omega)(\eta-1)^4\int_0^1\frac{(\eta-u)u^{\delta/2}}{(\eta^2-u)^{9/2}}du=+1.258979\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}{}^2eR_{13})_{00}d\lambda=12KK'(\hbar\omega)(\eta-1)^{3/2}(\zeta-1)^{1/2}\int_0^1\frac{(4\zeta-u)u^{\delta/2-1}}{(4\eta\zeta-u)^{5/2}}du=-0.225782\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}{}^2eC_{13})_{00}d\lambda=12K^2(\hbar\omega)(\eta-1)^2\int_0^1\frac{(4\eta-u)u^{\delta/2-1}}{(4\eta^2-u)^{5/2}}du=+0.917482\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}((S_{12}{}'q_{12}{}^2eS_{13}))_{00}d\lambda=2^5\left(\frac{105}{4}\right)K''{}^2(\hbar\omega)(\eta-1)^4\int_0^1\frac{(4\eta-u)u^{\delta/2}}{(4\eta^2-u)^{9/2}}du=+0.157936\hbar\omega.$$

The spin and isospin averages in the latter matrix element are best computed by expressing each tensor operator as the scalar product of two second-rank irreducible tensors[31] as outlined by Bolsterli[32]; the result is simply $-\frac{1}{9}$.

Potential B:

$$(C_{12}q_{12}^2)_{00}=-0.896625\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}^2eC_{12})_{00}d\lambda=+1.595456\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}((S_{12}'q_{12}^2eS_{12}'))_{00}d\lambda=3K''^2(\hbar\omega)(\eta-1)^4\int_0^1\frac{u^{\delta/2}du}{(\eta-u)(\eta^2-u)^{5/2}}=+0.784858\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}^2eC_{13})_{00}d\lambda=+1.582099\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}((S_{12}'q_{12}^2eS_{13}'))_{00}=+2^5(3)K''^2(\hbar\omega)(\eta-1)^4\int_0^1\frac{u^{\delta/2}du}{(4\eta-u)(4\eta^2-u)^{5/2}}=+0.104783\hbar\omega.$$

3. The Deuteron Matrix Elements

The following list of matrix elements are those needed in the determination of $\sigma_{ex}(H^2)$ as defined by Eq. (32).

Potential A:

$$(C_{12}q_{12}^2)_{00}=\frac{3K(\hbar\omega)(\eta-1)^{3/2}}{\eta^{5/2}}=-0.577728\hbar\omega,$$

$$V_{00}=\frac{K(\hbar\omega)(\eta-1)^{1/2}}{\eta^{3/2}}+\frac{K'(\hbar\omega)(\zeta-1)^{1/2}}{\zeta^{3/2}}=-0.505202\hbar\omega,$$

$$(C_{12}q_{12}^2M)_{00}=\frac{15}{4}(\hbar\omega)^2\frac{K(\eta-1)^{5/2}}{\eta^{7/2}}=-0.213597(\hbar\omega)^2,$$

$$M_{00}=\tfrac{3}{4}\hbar\omega.$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}^2eR_{12})_{00}d\lambda=+\tfrac{3}{2}KK'(\hbar\omega)(\zeta-1)^{1/2}(\eta-1)^{3/2}\int_0^1\frac{(\zeta-u)u^{\delta/2-1}}{(\eta\zeta-u)^{5/2}}du=-0.238103\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}^2eC_{12})_{00}d\lambda=+\tfrac{3}{2}K^2(\hbar\omega)(\eta-1)^2\int_0^1\frac{(\eta-u)u^{\delta/2-1}}{(\eta^2-u)^{5/2}}du=+1.055976\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(S_{12}'q_{12}^2eS_{12}')_{00}d\lambda=+\frac{35}{3}K''^2(\hbar\omega)(\eta-1)^4\int_0^1\frac{(\eta-u)u^{\delta/2}}{(\eta^2-u)^{9/2}}du=+0.530912\hbar\omega.$$

Potential B: Here only those matrix elements which differ in form from those given for potential A are listed.

$$(C_{12}q_{12}^2)_{00}=-0.815244\hbar\omega,$$

$$V_{00}=\frac{K(\hbar\omega)(\eta-1)^{1/2}}{\eta^{3/2}}=-0.634079\hbar\omega,$$

$$(C_{12}q_{12}^2M)_{00}=-0.436738(\hbar\omega)^2,$$

$$\int_0^\infty e^{\lambda(E-U)}(C_{12}q_{12}^2eC_{12})_{00}d\lambda=+1.936150\hbar\omega,$$

$$\int_0^\infty e^{\lambda(E-U)}(S_{12}'q_{12}^2eS_{12}')_{00}d\lambda=\tfrac{4}{3}K''^2(\hbar\omega)(\eta-1)^4\int_0^1\frac{u^{\delta/2}}{(\eta-u)(\eta^2-u)^{5/2}}du=+0.320830\hbar\omega.$$

[31] D. M. Brink and G. R. Satchler, *Angular Momentum* (Oxford University Press, London, 1962).
[32] A. M. Bolsterli, Ph.D. thesis, Washington University, 1955 (unpublished).

APPENDIX D: MATRIX ELEMENTS OF THIRD ORDER

1. Third-Order Estimates of Binding Energy

The matrix elements of the binding-energy estimates are listed below in algebraic form together with the corresponding numerical value obtained by substituting the appropriate parameters listed in Appendix A.

Potential A:

$$(W^2)_{00}=(V^2)_{00}+(M^2)_{00}-2(VM)_{00}-(V_{00}-M_{00})^2=+2.78417(\hbar\omega)^2,$$

$$(V^2)_{00}=(\hbar\omega)^2\left\{\frac{K'^2}{(\zeta-1)^{1/2}(\zeta+1)^{3/2}}+\frac{K^2}{(\eta-1)^{1/2}(\eta+1)^{3/2}}+\frac{2KK'(\eta-1)^{1/2}(\zeta-1)^{1/2}}{(\eta\zeta-1)^{3/2}}+\frac{10}{3}\frac{K''^2}{(\eta-1)^{1/2}(\eta+1)^{7/2}}\right\}$$

$$=+3.144255(\hbar\omega)^2,$$

$$M^2)_{00}=\tfrac{15}{16}(\hbar\omega)^2,$$

$$(VM)_{00}=\tfrac{3}{4}(\hbar\omega)^2\left\{\frac{K'(\zeta-1)^{3/2}}{\zeta^{5/2}}+\frac{K(\eta-1)^{3/2}}{\eta^{5/2}}\right\}=-0.138974(\hbar\omega)^2,$$

$$(V_{00}-M_{00})=U_{00}=\hbar\omega\left\{\frac{K'(\zeta-1)^{1/2}}{\zeta^{3/2}}+\frac{K(\eta-1)^{1/2}}{\eta^{3/2}}-\frac{3}{4}\right\}=-1.255202\hbar\omega,$$

$$(W^3)_{00}=(V^3)_{00}-(M^3)_{00}+3(M^2V)_{00}-3(MV^2)_{00}+2(V_{00}-M_{00})^3$$

$$-3(V_{00}-M_{00})[(V^2)_{00}-2(VM)_{00}+(M^2)_{00}]=+3.11270(\hbar\omega)^3,$$

$$(V^3)_{00}=(\hbar\omega)^3\left\{\frac{K'^3}{(\zeta-1)^{3/2}(\zeta+2)^{3/2}}+\frac{K^3}{(\eta-1)^{3/2}(\eta+2)^{3/2}}-\frac{70}{9}\frac{K''^3}{(\eta-1)^{3/2}(\eta+2)^{9/2}}+\frac{3KK'^2(\eta-1)^{1/2}}{(\zeta-1)^{1/2}(\eta\zeta+\eta-2)^{3/2}}\right.$$

$$\left.+\frac{3K^2K'(\zeta-1)^{1/2}}{(\eta-1)^{1/2}(\eta\zeta+\zeta-2)^{3/2}}+\frac{10K'K''^2(\zeta-1)^{3/2}}{(\eta-1)^{1/2}(\eta\zeta+\zeta-2)^{7/2}}+\frac{10KK''^2}{(\eta-1)^{3/2}(\eta+2)^{7/2}}\right\}=-4.608986(\hbar\omega)^3,$$

$$(M^3)_{00}=\frac{105}{64}(\hbar\omega)^3,$$

$$(MV^2)_{00}=(\hbar\omega)^3\left\{\frac{3}{4}\frac{K'^2(\zeta-1)^{1/2}}{(\zeta+1)^{5/2}}+\frac{3}{4}\frac{K^2(\eta-1)^{1/2}}{(\eta+1)^{5/2}}+\frac{35}{6}\frac{K''^2(\eta-1)^{1/2}}{(\eta+1)^{9/2}}+\frac{3}{2}\frac{KK'(\eta-1)^{3/2}(\zeta-1)^{3/2}}{(\eta\zeta-1)^{5/2}}\right\}$$

$$=+0.980072(\hbar\omega)^3,$$

$$(M^2V)_{00}=\tfrac{15}{16}(\hbar\omega)^3\left\{\frac{K'(\zeta-1)^{5/2}}{\zeta^{7/2}}+\frac{K(\eta-1)^{5/2}}{\eta^{7/2}}\right\}=-0.0530588(\hbar\omega)^3,$$

$$(V_{00}-M_{00})^3=-1.977612(\hbar\omega)^3,$$

$$(V_{00}-M_{00})[(V^2)_{00}-2(VM)_{00}+(M^2)_{00}]=-5.472309(\hbar\omega)^3.$$

Potential B:

$$(W^2)_{00}=+2.18274(\hbar\omega)^2,$$

$$(V^2)_{00}=(K^2+(8/9)K''^2)\frac{(\hbar\omega)^2}{(\eta-1)^{1/2}(\eta+1)^{3/2}}=+2.753295(\hbar\omega)^2,$$

$$(VM)_{00}=\tfrac{3}{4}(\hbar\omega)^2\frac{K(\eta-1)^{3/2}}{\eta^{5/2}}=-0.203811(\hbar\omega)^2,$$

$$(V_{00}-M_{00})=\hbar\omega\left\{\frac{K(\eta-1)^{1/2}}{\eta^{3/2}}-\frac{3}{4}\right\}=-1.384079(\hbar\omega),$$

$$(W^3)_{00}=+1.79173(\hbar\omega)^3,$$

$$(V^3)_{00}=(\hbar\omega)^3\frac{(K^3+(8/3)KK''^2-(16/27)K''^3)}{(\eta-1)^{3/2}(\eta+2)^{3/2}}=-6.265272(\hbar\omega)^3,$$

$$(M^2V)_{00}=\tfrac{15}{16}(\hbar\omega)^3\frac{K(\eta-1)^{5/2}}{\eta^{7/2}}=-0.109184(\hbar\omega)^3,$$

$$(MV^2)_{00}=(\hbar\omega)^3(\tfrac{3}{4}K^2+\tfrac{2}{3}K''^2)\frac{(\eta-1)^{1/2}}{(\eta+1)^{5/2}}=+0.563174(\hbar\omega)^3,$$

$$(V_{00}-M_{00})^3=-2.651444(\hbar\omega)^3,$$

$$(V_{00}-M_{00})[(V^2)_{00}-2(VM)_{00}+(M^2)_{00}]=-5.672532(\hbar\omega)^3.$$

2. Third-Order Estimates of $\sigma_{\rm int}(H^2)$

Potential A:

$$\langle 0|\,V_{12}q_{12}^2(V-M-U)\,|0\rangle=(V_{12}q_{12}^2V)_{00}-(V_{12}q_{12}^2M)_{00}-(V_{12}q_{12}^2)_{00}(V_{00}-M_{00})=+2.77945(\hbar\omega)^2,$$

$$(V_{12}q_{12}^2V)_{00}=(Cq_{12}^2R)_{00}+(Cq_{12}^2C)_{00}+(Sq_{12}^2S)_{00}$$

$$=\left\{\frac{3KK'(\eta-1)^{3/2}(\zeta-1)^{3/2}}{(\eta\zeta-1)^{5/2}}+\frac{3K^2(\eta-1)^{1/2}}{(\eta+1)^{5/2}}+\frac{70}{3}\frac{K''^2(\eta-1)^{1/2}}{(\eta+1)^{9/2}}\right\}(\hbar\omega)^2$$

$$=+3.291020(\hbar\omega)^2,$$

$$(V_{12}q_{12}^2M)_{00}=\frac{15}{4}(\hbar\omega)^2\frac{K(\eta-1)^{5/2}}{\eta^{7/2}}=-0.213597(\hbar\omega)^2,$$

$$(V_{12}q_{12}^2)_{00}=\frac{3K(\eta-1)^{3/2}}{\eta^{5/2}}(\hbar\omega)=-0.577728(\hbar\omega),$$

$$(V_{00}-M_{00})=-1.255202\hbar\omega,$$

$$\langle 0|\,(V-M-U)(V_{12}q_{12}^2)(V-M-U)\,|0\rangle=(R^2Cq_{12}^2)_{00}+2(RC^2q_{12}^2)+2(RS^2q_{12}^2)_{00}-2(RMCq_{12}^2)_{00}-2(RUCq_{12}^2)_{00}$$
$$+(C^3q_{12}^2)_{00}+3(CS^2q_{12}^2)_{00}-2(MC^2q_{12}^2)_{00}-2(C^2Uq_{12}^2)_{00}+(S^3q_{12}^2)_{00}-2(MS^2q_{12}^2)_{00}-2(US^2q_{12}^2)_{00}$$
$$+2(MUCq_{12}^2)_{00}+(M^2Cq_{12}^2)_{00}+(UCq_{12}^2)_{00},$$

$$(R^2Cq_{12}^2)_{00}=\frac{3KK'^2(\hbar\omega)^3(\eta-1)^{3/2}(\zeta-1)^{1/2}}{(\eta\zeta+\eta-2)^{5/2}}=-0.187954(\hbar\omega)^3,$$

$$(RC^2q_{12}^2)_{00}=\frac{3K^2K'(\hbar\omega)^3(\eta-1)^{1/2}(\zeta-1)^{3/2}}{(\eta\zeta+\zeta-2)^{5/2}}=+0.209936(\hbar\omega)^3,$$

$$(RS^2q_{12}^2)_{00}=\frac{70}{3}\frac{K'K''^2(\hbar\omega)^3(\zeta-1)^{7/2}(\eta-1)^{1/2}}{(\eta\zeta+\zeta-2)^{9/2}}=+0.0503258(\hbar\omega)^3,$$

$$(RMCq_{12}^2)_{00}=\frac{15}{4}\frac{KK'(\hbar\omega)^3(\eta-1)^{5/2}(\zeta-1)^{5/2}}{(\eta\zeta-1)^{7/2}}=-0.00371997(\hbar\omega)^3,$$

$$(RUCq_{12}^2)_{00}=\frac{3KK'(\hbar\omega)^2(\eta-1)^{3/2}(\zeta-1)^{3/2}}{(\eta\zeta-1)^{5/2}}(V_{00}-M_{00})=+0.0837807(\hbar\omega)^3$$

$$(C^3q_{12}^2)_{00}=\frac{3K^3(\hbar\omega)^3}{(\eta-1)^{1/2}(\eta+2)^{5/2}}=-1.051425(\hbar\omega)^3,$$

$$(CS^2q_{12}^2)_{00}=\frac{70}{3}\frac{KK''^2(\hbar\omega)^3}{(\eta-1)^{1/2}(\eta+2)^{9/2}}=-2.339720(\hbar\omega)^3,$$

$$(MC^2q_{12}^2)_{00}=\frac{15}{4}\frac{K^2(\hbar\omega)^3(\eta-1)^{3/2}}{(\eta+1)^{7/2}}=+0.133798(\hbar\omega)^3,$$

$$(C^2Uq_{12}^2)_{00}=\frac{3K^2(\hbar\omega)^2(\eta-1)^{1/2}}{(\eta+1)^{5/2}}(V_{00}-M_{00})=-0.774139(\hbar\omega)^3,$$

$$(S^3q_{12}^2)_{00}=-\frac{70K''^3(\hbar\omega)^3}{(\eta-1)^{1/2}(\eta+2)^{11/2}}=+3.754484(\hbar\omega)^3,$$

$$(MS^2q_{12}^2)_{00}=\frac{105}{2}\frac{K''^2(\hbar\omega)^3(\eta-1)^{3/2}}{(\eta+1)^{11/2}}=+1.070358(\hbar\omega)^3,$$

$$(US^2q_{12}^2)_{00}=\frac{70}{3}\frac{K''^2(\hbar\omega)^2(\eta-1)^{1/2}}{(\eta+1)^{9/2}}(V_{00}-M_{00})=-3.440537(\hbar\omega)^3,$$

$$(MUCq_{12}^2)_{00}=\frac{15}{4}\frac{K(\hbar\omega)^2(\eta-1)^{5/2}}{\eta^{7/2}}(V_{00}-M_{00})=+0.268107(\hbar\omega)^3,$$

$$(M^2Cq_{12}^2)_{00}=\frac{105}{16}\frac{K(\hbar\omega)^3(\eta-1)^{7/2}}{\eta^{9/2}}=-0.110559(\hbar\omega)^3,$$

$$(U^2Cq_{12}^2)_{00}=\frac{3K(\hbar\omega)(\eta-1)^{3/2}}{\eta^{5/2}}(V_{00}-M_{00})^2=-0.910229(\hbar\omega)^3,$$

$$\langle 0|(V-M-U)(V_{12}q_{12}^2)(V-M-U)|0\rangle=+1.39281(\hbar\omega)^3.$$

Potential B:

$$\langle 0|V_{12}q_{12}^2(V-M-U)|0\rangle=(C^2q_{12}^2)_{00}+(S^2q_{12}^2)_{00}-(MCq_{12}^2)_{00}-(UCq_{12}^2)_{00},$$

$$\langle 0|V_{12}q_{12}^2(V-M-U)|0\rangle=+1.56107(\hbar\omega)^2,$$

$$(C^2q_{12}^2)_{00}=\frac{3K^2(\hbar\omega)^2(\eta-1)^{1/2}}{(\eta+1)^{5/2}}=+0.595195(\hbar\omega)^2,$$

$$(S^2q_{12}^2)_{00}=\frac{8}{3}\frac{K''^2(\hbar\omega)^2(\eta-1)^{1/2}}{(\eta+1)^{5/2}}=+1.657500(\hbar\omega)^2,$$

$$(MCq_{12}^2)_{00}=\frac{15}{4}\frac{K(\hbar\omega)^2(\eta-1)^{5/2}}{\eta^{7/2}}=-0.436738(\hbar\omega)^2,$$

$$(UCq_{12}^2)_{00}=\frac{3K(\hbar\omega)(\eta-1)^{3/2}}{\eta^{5/2}}(V_{00}-M_{00})=+1.128362(\hbar\omega)^2,$$

$$(V_{00}-M_{00})=-1.384079\hbar\omega,$$

$$\langle 0|(V-M-U)(V_{12}q_{12}^2)(V-M-U)|0\rangle=(C^3q_{12}^2)_{00}-2(MC^2q_{12}^2)_{00}+3(CS^2q_{12}^2)_{00}+(S^3q_{12}^2)_{00}-2(MS^2q_{12}^2)_{00}$$
$$-2(UC^2q_{12}^2)_{00}+2(MUCq_{12}^2)_{00}-2(US^2q_{12}^2)_{00}+(CU^2q_{12}^2)_{00}+(M^2Cq_{12}^2)_{00},$$

$$\langle 0|(V-M-U)(V_{12}q_{12}^2)(V-M-U)|0\rangle=+0.260388(\hbar\omega)^3,$$

$$(C^3q_{12}^2)_{00}=-0.619472(\hbar\omega)^3,$$

$$(MC^2q_{12}^2)_{00}=+0.202908(\hbar\omega)^3,$$

$$(CS^2q_{12}^2)_{00}=-\frac{8}{3}\frac{KK''^2(\hbar\omega)^3}{(\eta-1)^{1/2}(\eta+2)^{5/2}}=-1.725105(\hbar\omega)^3,$$

$$(S^3q_{12}^2)_{00}=-\frac{16}{9}\frac{K''^3(\hbar\omega)^3}{(\eta-1)^{1/2}(\eta+2)^{5/2}}=+2.035624(\hbar\omega)^3,$$

$$(MS^2q_{12}^2)_{00}=\frac{10}{3}\frac{K''^2(\hbar\omega)^3(\eta-1)^{3/2}}{(\eta+1)^{7/2}}=+0.565057(\hbar\omega)^3,$$

$$(UC^2q_{12}^2)_{00}=-0.823797(\hbar\omega)^3,$$

$$(MUCq_{12}^2)_{00}=+0.604480(\hbar\omega)^3,$$

$$(US^2q_{12}^2)_{00}=-\frac{8}{3}\frac{K''^2(\hbar\omega)^2(\eta-1)^{1/2}}{(\eta+1)^{5/2}}(V_{00}-M_{00})=-2.294111(\hbar\omega)^3,$$

$$(CU^2q_{12}^2)_{00}=-1.561742(\hbar\omega)^3,$$

$$(M^2Cq_{12}^2)_{00}=-0.327553(\hbar\omega)^3.$$

3. Exact Third-Order Binding Energy

The evaluation of the matrix elements occurring in the exact third-order calculation is facilitated by the use of the integral transforms given in Appendix B. The numerical determination of the values of $\int\int e^{\lambda(E-U)}e^{\lambda'(E-U)}\times(\ )_{00}d\lambda' d\lambda$, in those cases where it was not possible to perform analytically the double integration over u and u', was carried out by expanding the integral and then integrating the series on a Burroughs 205 electronic computer. It will be convenient in the following extensive list of matrix elements to use the abbreviated form

$$\int$$

to mean

$$\int_0^\infty\int_0^\infty e^{\lambda(E-U)}e^{\lambda'(E-U)}(\ )_{00}d\lambda' d\lambda.$$

The algebraic expressions and numerical values (in units of $\hbar\omega$) of the matrix elements appearing in $(We'WeW)_{00}$ and ϵ_3 are

$$(Re'ReR)_{00}=\frac{K'^3(\zeta-1)^{3/2}(uu')^{3/2}}{[(\zeta-1)(\zeta^2-uu')+(\zeta-u')(\zeta-u)]^{3/2}},\qquad \int=+0.078163\ ;$$

$$2(Re'ReC)_{00}=\frac{2KK'^2(\eta-1)^{1/2}(\zeta-1)(uu')^{3/2}}{[(\zeta-1)(\eta\zeta-uu')+(\zeta-u')(\eta-u)]^{3/2}},\qquad \int=-0.470424\ ;$$

$$(Re'CeR)_{00}=\frac{KK'^2(\zeta-1)(\eta-1)^{1/2}(uu')^{3/2}}{[(\eta-1)(\zeta^2-uu')+(\zeta-u')(\zeta-u)]^{3/2}},\qquad \int=-0.170240\ ;$$

$$2(Re'CeC)_{00}=\frac{2K^2K'(\eta-1)(\zeta-1)^{1/2}(uu')^{3/2}}{[(\eta-1)(\eta\zeta-uu')+(\zeta-u')(\eta-u)]^{3/2}},\qquad \int=+1.303454\ ;$$

$$2(Re'SeS)_{00}=\frac{20}{3}\frac{K'K''^2(\eta-1)^3(\zeta-1)^{1/2}u^{5/2}u'^{3/2}}{[(\eta-1)(\eta\zeta-uu')+(\zeta-u')(\eta-u)]^{7/2}},\qquad \int=+0.277060\ ;$$

$$(Ce'ReC)_{00}=\frac{K^2K'(\eta-1)(\zeta-1)^{1/2}(uu')^{3/2}}{[(\zeta-1)(\eta^2-uu')+(\eta-u')(\eta-u)]^{3/2}},\qquad \int=+0.717435\ ;$$

$$(Ce'CeC)_{00}=\frac{K^3(\eta-1)^{3/2}(uu')^{3/2}}{[(\eta-1)(\eta^2-uu')+(\eta-u')(\eta-u)]^{3/2}}, \qquad \int=-2.558910\ ;$$

$$2(Ce'SeS)_{00}=\frac{20}{3}\frac{KK''^2(\eta-1)^{7/2}(\eta-u')^2u^{5/2}u'^{3/2}}{[(\eta-1)(\eta^2-uu')+(\eta-u)(\eta-u')]^{7/2}}, \qquad \int=-1.237100\ ;$$

$$(Se'ReS)_{00}=\frac{10}{3}\frac{K'K''^2(\eta-1)^3(\zeta-1)^{5/2}(uu')^{5/2}}{[(\zeta-1)(\eta^2-uu')+(\eta-u')(\eta-u)]^{7/2}}, \qquad \int=+0.001330\ ;$$

$$(Se'CeS)_{00}=\frac{10}{3}\frac{KK''^2(\eta-1)^{11/2}(uu')^{5/2}}{[(\eta-1)(\eta^2-uu')+(\eta-u')(\eta-u)]^{7/2}}, \qquad \int=-0.041607\ ;$$

$$(Se'SeS)_{00}=-\frac{70}{9}\frac{K''^3(\eta-1)^{11/2}(\eta-u)(\eta-u')(uu')^{5/2}}{[(\eta-1)(\eta^2-uu')+(\eta-u')(\eta-u)]^{9/2}}, \qquad \int=+0.072547;$$

$$-(Re'MeR)_{00}=\frac{3}{4}\frac{K'^2(\zeta-1)(\zeta-u)(\zeta-u')(uu')^{3/2}}{(\zeta^2-uu')^{5/2}}, \qquad \int=-0.075625\ ;$$

$$-(Ce'MeC)_{00}=\frac{3}{4}\frac{K^2(\eta-1)(\eta-u)(\eta-u')(uu')^{3/2}}{(\eta^2-uu')^{5/2}}, \qquad \int=-1.639238\ ;$$

$$-2(Re'MeC)_{00}=\frac{-3}{2}\frac{KK'(\zeta-1)^{1/2}(\eta-1)^{1/2}(\zeta-u')(\eta-u)(uu')^{3/2}}{(\eta\zeta-uu')^{5/2}}, \qquad \int=+0.703500\ ;$$

$$-(Se'MeS)_{00}=\frac{-35}{6}\frac{K''^2(\eta-1)^3(\eta-u)(\eta-u')(uu')^{5/2}}{(\eta^2-uu')^{9/2}}, \qquad \int=-0.047447$$

$$-(Re'UeR)_{00}=\frac{-K'^2U_{00}(\zeta-1)(uu')^{3/2}}{(\zeta^2-uu')^{3/2}}, \qquad \int=+0.184357\ ;$$

$$-(Ce'UeC)_{00}=\frac{-K^2U_{00}(\eta-1)(uu')^{3/2}}{(\eta^2-uu')^{3/2}}, \qquad \int=+3.525231\ ;$$

$$-2(Re'UeC)_{00}=\frac{KK'U_{00}(\zeta-1)^{1/2}(\eta-1)^{-/2}(uu')^{3/2}}{(\eta\zeta-uu')^{3/2}}, \qquad \int=-1.600106\ ;$$

$$-(Se'UeS)_{00}=\frac{-10}{3}\frac{K''^2U_{00}(\eta-1)^3(uu')^{5/2}}{(\eta^2-uu')^{7/2}}, \qquad \int=+0.093814\ ;$$

$$-2(Re'ReM)_{00}=-\tfrac{3}{2}K'^2(\zeta-1)(uu')^{3/2}\left\{\frac{1-u}{(\zeta^2-u')^{3/2}}+\frac{(\zeta-1)(\zeta-u')u}{(\zeta^2-u')^{5/2}}\right\}, \qquad \int=-0.340272\ ;$$

$$-2(Re'CeM)_{00}=-\tfrac{3}{2}KK'(\zeta-1)^{1/2}(\eta-1)^{1/2}(uu')^{3/2}\left\{\frac{1-u}{(\eta\zeta-u')^{3/2}}+\frac{(\eta-1)(\zeta-u')u}{(\eta\zeta-u')^{5/2}}\right\}, \qquad \int=+1.080900\ ;$$

$$-2(Ce'ReM)_{00}=-\tfrac{3}{2}KK'(\zeta-1)^{1/2}(\eta-1)^{1/2}(uu')^{3/2}\left\{\frac{1-u}{(\eta\zeta-u')^{3/2}}+\frac{(\zeta-1)(\eta-u')u}{(\eta\zeta-u')^{5/2}}\right\}, \qquad \int=+1.040498\ ;$$

$$-2(Se'SeM)_{00}=\frac{-5}{3}K''^2(\eta-1)^3(uu')^{3/2}\left\{\frac{3u'(1-u)}{(\eta^2-u')^{7/2}}+\frac{7(\eta-1)(\eta-u')(uu')}{(\eta^2-u')^{9/2}}\right\}, \qquad \int=-1.202480\ ;$$

$$-2(Ce'CeM)_{00}=-\tfrac{3}{2}K^2(\eta-1)(uu')^{3/2}\left\{\frac{1-u}{(\eta^2-u')^{3/2}}+\frac{(\eta-1)(\eta-u')u}{(\eta^2-u')^{5/2}}\right\}, \qquad \int=-4.315252;$$

$$-2(Re'ReU)_{00}=-2K'^2U_{00}\frac{(\zeta-1)(uu')^{3/2}}{(\zeta^2-u')^{3/2}}, \qquad \int=+0.675590;$$

$$-4(Re'CeU)_{00}=-4KK'U_{00}(\zeta-1)^{1/2}(\eta-1)^{1/2}\frac{(uu')^{3/2}}{(\eta\zeta-uu')^{3/2}}, \qquad \int=-4.128132;$$

$$-2(Ce'CeU)_{00}=-2K^2U_{00}(\eta-1)\frac{(uu')^{3/2}}{(\eta^2-u')^{3/2}}, \qquad \int=+8.200766;$$

$$-2(Se'SeU)_{00}=\frac{-20}{3}K''^2U_{00}(\eta-1)^3\frac{u'^{5/2}u^{3/2}}{(\eta^2-u')^{7/2}}, \qquad \int=+2.186198;$$

$$2(Re'MeM)_{00}=\tfrac{3}{8}K'\frac{(\zeta-1)^{1/2}(uu')^{3/2}}{\zeta^{7/2}}\{3\zeta^2-3\zeta u'+2\zeta^2u-7\zeta uu'+5uu'^2\}, \qquad \int=+0.971038;$$

$$2(Ce'MeM)_{00}=\tfrac{3}{8}K\frac{(\eta-1)^{1/2}(uu')^{3/2}}{\eta^{7/2}}\{3\eta^2-3\eta u'+2\eta^2u-7\eta uu'+5uu'^2\}, \qquad \int=-4.551120;$$

$$2(Re'MeU)_{00}=\tfrac{3}{2}K'U_{00}\frac{(\zeta-1)^{1/2}(\zeta-u')(uu')^{3/2}}{\zeta^{5/2}}, \qquad \int=-1.497650;$$

$$2(Ce'MeU)_{00}=\tfrac{3}{2}KU_{00}\frac{(\eta-1)^{1/2}(\eta-u')(uu')^{3/2}}{\eta^{5/2}}, \qquad \int=+7.004080;$$

$$2(Re'UeM)_{00}=\tfrac{3}{2}K'U_{00}\frac{(\zeta-1)^{1/2}(\zeta-uu')(uu')^{3/2}}{\zeta^{5/2}}, \qquad \int=-1.729266;$$

$$2(Ce'UeM)_{00}=\tfrac{3}{2}KU_{00}\frac{(\eta-1)^{1/2}(\eta-uu')(uu')^{3/2}}{\eta^{5/2}}, \qquad \int=+7.770246,$$

$$2(Re'UeU)_{00}=2K'U_{00}^2\frac{(\zeta-1)^{1/2}(uu')^{3/2}}{\zeta^{3/2}}, \qquad \int=+2.970352;$$

$$2(Ce'UeU)_{00}=2KU_{00}^2\frac{(\eta-1)^{1/2}(uu')^{3/2}}{\eta^{3/2}}, \qquad \int=-13.256524;$$

$$(Me'ReM)_{00}=\tfrac{3}{16}K'\frac{(\zeta-1)^{1/2}(uu')^{3/2}}{\zeta^{7/2}}\{3\zeta(\zeta-u-u')+(2\zeta^2-4\zeta+5)uu'\}, \qquad \int=+0.377581;$$

$$(Me'CeM)_{00}=\tfrac{3}{16}K\frac{(\eta-1)^{1/2}(uu')^{3/2}}{\eta^{7/2}}\{3\eta(\eta-u-u')+(2\eta^2-4\eta+5)uu'\}, \qquad \int=-1.854030;$$

$$2(Me'ReU)_{00}=\tfrac{3}{2}K'U_{00}\frac{(\zeta-1)^{1/2}(\zeta-u')(uu')^{3/2}}{\zeta^{5/2}}, \qquad \int=-1.497650;$$

$$2(Me'CeU)_{00}=\tfrac{3}{2}KU_{00}\frac{(\eta-1)^{1/2}(\eta-u')(uu')^{3/2}}{\eta^{5/2}}, \qquad \int=+7.004080;$$

$$-(Me'MeM)_{00}=\frac{-3}{64}(uu')^{3/2}\{9+6u+6u'+14uu'\}, \qquad \int=-3.437904;$$

$$-2(Me'MeU)_{00}=-\tfrac{3}{8}U_{00}(uu')^{3/2}(3+2u), \qquad \int=+10.124084;$$

$$-(Me'UeM)_{00}=-\tfrac{3}{16}U_{00}(uu')^{3/2}(3+2uu'), \qquad \int=+4.644298;$$

$$-2(Me'UeU)_{00}=-\tfrac{3}{2}U_{00}^{2}(uu')^{3/2}, \qquad \int=-15.270382;$$

$$(Ue'ReU)_{00}=K'U_{00}^{2}\frac{(\zeta-1)^{1/2}(uu')^{3/2}}{\zeta^{3/2}}, \qquad \int=+1.485176;$$

$$(Ue'CeU)_{00}=KU_{00}^{2}\frac{(\eta-1)^{1/2}(uu')^{3/2}}{\eta^{3/2}}, \qquad \int=-6.628262;$$

$$-(Ue'MeU)_{00}=-\tfrac{3}{4}U_{00}^{2}(uu')^{3/2}, \qquad \int=-7.635191;$$

$$-(Ue'UeU)_{00}=-U_{00}^{3}(uu')^{3/2}, \qquad \int=+12.778278.$$

The third-order contribution to the energy ϵ_3 is the sum of the above terms, 1.24 MeV.

ACKNOWLEDGMENTS

It is indeed a pleasure to thank Professor Paul Goldhammer for providing us with his unpublished variational parameters and to express our appreciation to Professor Peter Signell for performing the analysis of the potentials employed. In addition, particular thanks are due Dr. Herschel Neumann for many fruitful discussions and supporting computations. The assistance provided by Dr. William A. Fraser and Dr. Bruce Anspaugh is also gratefully acknowledged. Finally, the authors wish to thank the National Science Foundation for financial support.