Ultrasonic Attenuation of Pure Strong-Coupling Superconductors*

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The low-frequency longitudinal ultrasonic attenuation of strong-coupling superconductors is discussed. The starting point is a formula due to Kadanoff and Falko which expresses the attenuation in terms of various correlation functions. These functions were evaluated in the ladder approximation. The results are valid for all values of ql (l = electron mean free path, q = phonon wave number). In the limit $ql \gg 1$, the contribution of the density-density correlation function is dominant and we obtain Ambegaokar's result for the reduced attenuation (ratio of attenuation in superconducting to normal states). In the limit $ql \ll 1$, all correlation functions are equally important. The reduced attenuation is a function of q and differs from the results of the BCS isotropic model. Numerical calculations for the reduced attenuation in lead were performed for various values of q which satisfy $ql \ll 1$. The results are in rough agreement with those of Deaton's experiment.

I. INTRODUCTION

RECENTLY, there has been considerable interest in the ultrasonic attenuation of pure strong-coupling superconductors.[1–3] Although an expression for the longitudinal attenuation (α_L) in the $ql \gg 1$ limit (q, the impressed phonon wave number; l, the electronic mean free path) has been obtained,[3] no calculation valid for all values of ql has been done. In this paper, such a calculation is presented.

We consider a pure crystal with no defects. The only way the impressed sound wave can lose energy is then through the electron-phonon interaction, and the phonons will be assumed to be in equilibrium. The effects of anisotropy will not be considered. In the low-frequency limit, Kadanoff and Falko[4] showed that the attenuation can be expressed in the following form:

$$\alpha_L(\mathbf{q},\omega)=\mathrm{Re}\frac{q^2}{i\omega\rho_{\mathrm{ion}}v_s}\left\{\langle[\tau_{zz},\tau_{zz}]\rangle(\mathbf{q},\omega)-\frac{2p_F^2}{3m}\langle[\tau_{zz},n]\rangle(\mathbf{q},\omega)+\left(\frac{p_F^2}{3m}\right)^2\langle[n,n]\rangle(\mathbf{q},\omega)\right\}, \quad (1.1)$$

where v_s is the sound velocity, ρ_{ion} is the ionic mass, τ_{ij} is the stress tensor, and n is the electron density. In terms of the electronic wave function $\psi(r,t)$,

$$\tau_{zz}(\mathbf{r},t)=\sum_{\mathrm{spin}}\frac{\partial_z-\partial_{z'}}{2i}\frac{\partial_z-\partial_{z'}}{2mi}\psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r},t)\bigg|_{\mathbf{r}'=\mathbf{r}},$$

$$n(\mathbf{r},t)=\sum_{\mathrm{spin}}\psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t).$$

Brackets indicate equilibrium ensemble average. $\mathbf{q}$ is in the z direction.

The correlation functions that appear in Eq. (1.1) are given by

$$\langle[A,B]\rangle(\mathbf{q},z)=\frac{1}{i}\int_{-\infty}^{t}dt'\int d\mathbf{r}'\exp[iz(t-t')-i\mathbf{q}\cdot(r-r')]\times\langle[A(\mathbf{r},t),B(\mathbf{r}',t')]\rangle. \quad (1.2)$$

The last term on the right-hand side of Eq. (1.1) comes from the reciprocal of the lifetime of a phonon of wave number q. It is the term that would appear in a golden-rule calculation. The first term on the right-hand side represents the effects of collision drag. In the long-wavelength limit, it may be thought of as due to the viscosity of the electrons. The second term is then the result of interference between these two processes. For $ql \gg 1$, an electron sees only an averaged motion of the ions so that collision-drag effects are unimportant. Thus, the density-density correlation function should dominate in this limit. For $ql \ll 1$, the collision-drag terms are important. We will show that in this case, all terms of Eq. (1.1) are of the same order of magnitude.

In the following section, we will derive an integral equation for the stress-tensor–stress-tensor correlation function. Similar equations for the other functions will be obtained in Sec. III. In Sec. IV, we compare our results with experiment in the $ql \gg 1$ and $ql \ll 1$ limit.

II. THE STRESS-TENSOR–STRESS-TENSOR CORRELATION FUNCTION

We will use the techniques developed by Ambegaokar and Tewordt[5] in their solution of the thermal-conductivity problem.

It is convenient to define the function

$$P(\mathbf{q},t)\equiv\langle T\tau_{zz}(\mathbf{q},t)\tau_{zz}(-\mathbf{q},0)\rangle, \quad (2.1)$$

where $\tau_{zz}(\mathbf{q},t)$ is given by

$$\tau_{zz}(\mathbf{r},t)=\sum e^{i\mathbf{q}\cdot\mathbf{r}}\tau_{zz}(\mathbf{q},t)=\sum_{\mathbf{p},\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{r}}\left(\frac{p_z^2}{m}\right)c^\dagger_{\mathbf{p}-\mathbf{q}/2}(t)c_{\mathbf{p}+\mathbf{q}/2}(t), \quad (2.2)$$

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[3] V. Ambegaokar, Phys. Rev. Letters 16, 1047 (1966).
[4] L. P. Kadanoff and I. I. Falko, Phys. Rev. 136, A1170 (1964).
[5] V. Ambegaokar and L. Tewordt, Phys. Rev. 134, A805 (1964).

$c^\dagger$ and c are the electronic creation and annihilation operators. We will need the Fourier transform of $P(\mathbf{q},t)$.

$$P(\mathbf{q},t)=\frac{i}{\beta}\sum_{\nu_m} P(\mathbf{q},\nu_m)\exp[-i\nu_m t]\,, \tag{2.3}$$

where $\nu_m=2\pi mi/\beta$, m is an integer and $\beta=(k_BT)^{-1}$ where k_B is Boltzmann's constant and T is temperature. We will be working with Nambu's two-component field operators[6] defined by

$$\psi(1)=\begin{pmatrix}\psi_\uparrow(1)\\ \psi_\downarrow^\dagger(1)\end{pmatrix};$$

$\psi_\uparrow(1)$ destroys an electron of spin σ at the space-time point 1.

In terms of these

$$P(\mathbf{q},t)=\sum\frac{p_z^2}{m}\frac{p_{z'}^2}{m}\langle T\psi_i^\dagger(t,\,\mathbf{p}-\tfrac{1}{2}\mathbf{q})\psi_j(t^-,\,\mathbf{p}+\tfrac{1}{2}\mathbf{q})$$
$$\times\psi_k^\dagger(0^\dagger,\,\mathbf{p}'+\tfrac{1}{2}\mathbf{q})\psi_2(0,\,\mathbf{p}'-\tfrac{1}{2}\mathbf{q})\rangle(\tau_3)^{ij}(\tau_3)^{kl}, \tag{2.4}$$

where τ_3 is the Pauli spin matrix

$$\tau_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Using the functions defined above and the definition of $\langle[\tau_{zz},\tau_{zz}]\rangle(\mathbf{q},\omega)$, we obtain

$$\mathrm{Re}\frac{1}{i\omega}\langle[\tau_{zz},\tau_{zz}]\rangle(\mathbf{q},\omega)=\frac{\mathrm{Im}P(\mathbf{q},\,\omega+i0^\dagger)}{\omega}. \tag{2.5}$$

In the following, we will calculate $\mathrm{Im}P(\mathbf{q},\,\omega+i0^\dagger)$ in the ladder approximation and in the limit of small frequencies. But let us consider the Hartree-Fock approximation first. This is defined by

$$\langle T[\psi_i(1)\psi_k(2)\psi_l^\dagger(2')\psi_j^\dagger(1')]\rangle=G_{il}(1,2')G_{kj}(2,1')\,, \tag{2.6}$$

where $G_{ij}(1,2)$ is the electron Green's function

$$G_{ij}(1,2)=-i\langle T\psi_i(1)\psi_j^\dagger(2)\rangle; \tag{2.7}$$

using (2.6) in (2.4), we find

$$P(\mathbf{q},t)=\sum_{\mathbf{p}}\left(\frac{p_z^2}{m}\right)^2$$
$$\times\mathrm{tr}[\tau_3G(t,\,\mathbf{p}+\tfrac{1}{2}\mathbf{q})\tau_3G(0,\,\mathbf{p}-\tfrac{1}{2}\mathbf{q})]. \tag{2.8}$$

Thus

$$P(\mathbf{q},\nu_m)=\frac{i}{\beta}\sum_{\mathbf{p},\zeta_l}\left(\frac{p_z^2}{m}\right)^2$$
$$\times\mathrm{tr}[\tau_3G(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\,\zeta_l+\nu_m)\tau_3G(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\,\zeta_l)], \tag{2.9}$$

$\zeta_l=1(2l+1)\pi i/\beta$, l runs through all integers.

[6] Y. Nambu, Phys. Rev. **117**, 648 (1960).

The Hartree-Fock approximation, however, is inadequate. This approximation considers the relaxation of a particular electron when all the other electrons are in equilibrium. Thus, it does not consider properly the screening of the electrons. Furthermore, we know that the Ward identities are not satisfied in this approximation so that momentum conservation is violated. These defects are remedied in the ladder approximation.

In the ladder approximation, Eq. (2.9) becomes

$$P(\mathbf{q},\nu_m)=\frac{i}{\beta}\sum_{\mathbf{p},\zeta_l}\left(\frac{p_z^2}{m}\right)\mathrm{tr}\{\tau_3G(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\,\zeta_l+\nu_m)$$
$$\times\tau_3\Gamma(\mathbf{p},\mathbf{q},\zeta_l,\nu_m)\tau_3G(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\,\zeta_l)\}$$
$$\equiv\frac{i}{\beta}\sum_{\mathbf{p},\zeta_l}\left(\frac{p_z^2}{m}\right)\mathrm{tr}\tau_3\chi(\mathbf{p},\mathbf{q},\zeta_l,\nu_m)\,, \tag{2.10}$$

where

$$\Gamma(\mathbf{p},\mathbf{q},\zeta_l,\nu_m)=\frac{p_z^2}{m}\tau_3-\frac{1}{\beta}\sum_{\mu_n,\mathbf{k}}G(\mathbf{p}+\mathbf{k}+\tfrac{1}{2}\mathbf{q},\,\zeta_l+\mu_n+\nu_m)\tau_3$$
$$\times\Gamma(\mathbf{p}+\mathbf{k},\,\mathbf{q},\,\zeta_l+\mu_n,\,\nu_m)\tau_3G(\mathbf{p}+\mathbf{k}-\tfrac{1}{2}\mathbf{q},\,\zeta_l+\mu_n)D(\mathbf{k},\mu_n)$$
$$=\frac{p_z^2}{m}\tau_3-\frac{1}{\beta}\sum_{\mu_n,\mathbf{k}}\chi(\mathbf{p}+\mathbf{k},\,\mathbf{q},\,\zeta_l+\mu_n,\,\nu_m)D(\mathbf{k},\mu_n) \tag{2.11}$$

and $D(\mathbf{k},\mu_n)$ is the phonon propagator ($\mu_n=2\pi ni/\beta$, n is an integer) which has the spectral representation

$$D(\mathbf{k},\mu_n)=\int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\frac{d(\mathbf{k},\mu)}{\mu_n-\mu}. \tag{2.12}$$

The following double spectral representation for χ has been obtained in Ref. 5.

$$\chi(\mathbf{p},\mathbf{q},\zeta_l,\nu_m)=\int_{-\infty}^{\infty}\frac{d\omega_1d\omega_2}{(2\pi)^2}\left\{\frac{f_1(\mathbf{p},\mathbf{q},\omega_1,\omega_3)}{(\zeta_l-\omega_1)(\nu_m-\omega_2)}\right.$$
$$\left.+\frac{f_2(\mathbf{p},\mathbf{q},\omega_1,\omega_2)}{(\zeta_l-\omega_1)(\zeta_l+\nu_m-\omega_2)}\right\}. \tag{2.13}$$

Putting (2.13) into (2.10) and doing the frequency sum in the usual way, we obtain

$$P(\mathbf{q},\nu_m)=i\int\frac{d\omega_1d\omega_2}{(2\pi)^2}\sum_{\mathbf{p}}\left(\frac{p_z^2}{m}\right)\mathrm{tr}\left\{\tau_3\left[\frac{f(\omega_1)f_1(\mathbf{p},\mathbf{q},\omega_1,\omega_2)}{\nu_m-\omega_2}\right.\right.$$
$$\left.\left.+\frac{[f(\omega_1)-f(\omega_2)]f_2(\mathbf{p},\mathbf{q},\omega_1,\omega_2)}{(\omega_1+\nu_m-\omega_2)}\right]\right\}, \tag{2.14}$$

where $f(\omega)$ is the Fermi function. Thus,

$$2\,\mathrm{Im}(\mathbf{q},\,\omega+i0^\dagger)=\sum_{\mathbf{p}}\left(\frac{p_z^2}{m}\right)\int\frac{d\omega_1}{2\pi}$$
$$\times\mathrm{tr}\tau_3\{f(\omega_1)f_1(\mathbf{p},\mathbf{q},\omega_1,\omega)+[f(\omega_1)-f(\omega+\omega_1)]$$
$$\times f_2(\mathbf{p},\,\mathbf{q},\,\omega_1,\,\omega_1+\omega)\}. \tag{2.15}$$

Using (2.15) in (2.5) and taking the low-frequency limit, we have

$$\lim_{\omega\to 0}\mathrm{Re}\frac{1}{i\omega}\langle[\tau_{zz},\tau_{zz}]\rangle(\mathbf{q},\omega)=\frac{1}{2}\int\frac{d^3p\,d\omega_1}{(2\pi)^4}\left(\frac{p_z^2}{m}\right)\mathrm{tr}\tau_3$$
$$\times\left\{\lim_{\omega\to 0}\frac{f_1(\mathbf{p},\mathbf{q},\omega_1,\omega)}{\omega}f(\omega_1)-\frac{\partial f(\omega_1)}{\partial\omega_1}f_2(\mathbf{p},\mathbf{q},\omega_1,\omega_1)\right\}. \quad (2.16)$$

Thus, we must determine the function f_1 and f_2. To do this, we examine the equation for χ. From Eqs. (2.10) and (2.11), we have

$$\chi(\mathbf{p},\mathbf{q},\zeta_l,\nu_m)=(p_z^2/m)G(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta_l+\nu_m)\tau_3G(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta_l)-(1/\beta)\sum_{\mu_n,\mathbf{k}}G(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta_l+\nu_m)\tau_3$$
$$\times\chi(\mathbf{p}+\mathbf{k},\mathbf{q},\zeta_l+\mu_n,\nu_m)\tau_3G(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta_l)D(\mathbf{k},\mu_n). \quad (2.17)$$

Introducing the spectral representation for G

$$G(\mathbf{p},\zeta_l)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{a(\mathbf{p},\omega)}{\zeta_l-\omega} \quad (2.18)$$

into (2.17), using (2.12) and (2.13), and doing the sum over μ_n we obtain

$$\int\frac{d\omega_1d\omega_2}{(2\pi)^2}\left[\frac{f_1(\mathbf{p},\mathbf{q},\omega_1,\omega_2)}{(\zeta_l-\omega_1)(\nu_m-\omega_2)}+\frac{f_2(\mathbf{p},\mathbf{q},\omega_1,\omega_2)}{(\zeta_l-\omega_1)(\zeta_l+\nu_m-\omega_2)}\right]=\int\frac{d\omega_1d\omega_2}{(2\pi)^2}\left(\frac{p_z^2}{m}\right)\frac{a(\mathbf{p}+\frac{1}{2}\mathbf{q},\omega_2)\tau_3a(\mathbf{p}-\frac{1}{2}\mathbf{q},\omega_1)}{(\zeta_l+\nu_m-\omega_2)(\zeta_l-\omega_1)}$$
$$+\int\frac{d^3k\,d\omega_1d\omega_2d\xi_1d\xi_2d\mu}{(2\pi)^8}\frac{d(\mathbf{k},\mu)a(\mathbf{p}+\frac{1}{2}\mathbf{q},\omega_2)\tau_3}{\zeta_l+\nu_m-\omega_2}\left\{\left[\frac{n(\mu)+f(\xi_1)}{(\zeta_l+\mu-\xi_1)(\nu_m-\xi_2)}f_1(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)\right.\right.$$
$$+\left(\frac{n(\mu)}{(\zeta_l+\mu-\xi_1)(\zeta_l+\mu+\nu_m-\xi_2)}+\frac{f(\xi_1)}{(\zeta_l+\mu-\xi_1)(\xi_1+\nu_m-\xi_2)}\right.$$
$$\left.\left.\left.+\frac{f(\xi_2)}{(\zeta_l+\nu_m+\mu-\xi_2)(\xi_2-\xi_1-\nu_m)}\right)f_2(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)\right]\tau_3\frac{a(\mathbf{p}-\frac{1}{2}\mathbf{q},\omega_1)}{\zeta_l-\omega_1}\right\}, \quad (2.19)$$

where f is the Fermi function and n the Planck function. Equations for f_1 and f_2 are obtained by taking discontinuities across the real axis as we let $\zeta_l\to\zeta-i\delta$ and then $\nu_m\to\nu-i\delta$ and in the reversed order. The equations are

$$f_1(\mathbf{p},\mathbf{q},\zeta,\nu)=\int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega_1d\omega_2d\xi_1d\mu}{(2\pi)^4}d(\mathbf{k},\mu)a(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\omega_2)\tau_3\left\{2\,\mathrm{Im}\left[\frac{1}{(\zeta+\nu-\omega_2-i\delta)(\zeta+\mu-\xi_1-i\delta)(\zeta-\omega_1-i\delta)}\right]\right.$$
$$\times[n(\mu)+f(\xi_1)]f_1(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\nu)+2\,\mathrm{Im}\left[\frac{1}{(\zeta+\nu-\omega_2-i\delta)(\zeta+\mu-\xi_1-i\delta)(\zeta-\omega_1-i\delta)}\right]$$
$$\left.\times[f(\xi_1)-f(\xi_1+\nu)]f_2(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_1+\nu)\right\}\tau_3a(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\omega_1), \quad (2.20)$$

$$f_1(\mathbf{p},\mathbf{q},\zeta,\nu)+f_2(\mathbf{p},\mathbf{q},\zeta,\zeta+\nu)=\int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega_1d\omega_2d\xi_1d\xi_2d\mu}{(2\pi)^5}[d(\mathbf{k},\mu)a(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\omega_2)$$
$$\times\tau_3\{2\,\mathrm{Im}[(\zeta-\omega_1-i\delta)^{-1}(\zeta+\mu-\xi_1-i\delta)^{-1}]2\,\mathrm{Im}[(\nu-\xi_2-i\delta)^{-1}(\zeta+\nu-\omega_2-i\delta)^{-1}]$$
$$\times[n(\mu)+f(\xi_1)]f_1(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)+2\,\mathrm{Im}[(\zeta+\mu-\xi_1-i\delta)^{-1}(\zeta-\omega_1-i\delta)^{-1}]$$
$$\times 2\,\mathrm{Im}[(\zeta+\mu+\nu-\xi_2-i\delta)^{-1}(\zeta+\nu-\omega_2-i\delta)^{-1}]n(\mu)f_2(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)$$
$$+2\,\mathrm{Im}[(\zeta-\omega_1-i\delta)^{-1}(\zeta+\mu-\xi_1-i\delta)^{-1}]2\,\mathrm{Im}[(\zeta+\nu-\omega_2-i\delta)(\nu+\xi_1-\xi_2-i\delta)]$$
$$\times f(\xi_1)f_2(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)-2\pi\delta(\zeta-\omega_1)f(\xi_2)f_2(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)$$
$$\times 2\,\mathrm{Im}[(\zeta+\nu-\omega_2-i\delta)^{-1}(\zeta+\nu+\mu-\xi_2-i\delta)^{-1}(\nu+\xi_1-\xi_2-i\delta)^{-1}]\}$$
$$\times\tau_3a(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\omega_1)]+\left(\frac{p_z^2}{m}\right)a(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta+\nu)\tau_3a(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta). \quad (2.21)$$

Since the attenuation is expected to be finite, we expect $f_1(\mathbf{p},\mathbf{q},\zeta,0)$ to be zero. This conjecture is consistent with the above equations. Since we will only calculate in the limit of small frequencies, let us take the ν going to the zero limit. In this limit Eqs. (2.20) and (2.21) become

$$\frac{\partial f_1(\mathbf{p},\mathbf{q},\zeta,\nu)}{\partial\nu}\bigg|_{\nu=0}=\int\frac{d^3k d\omega_1 d\omega_2 d\xi_1 d\mu}{(2\pi)^7}d(\mathbf{k},\mu)a(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\omega_2)\tau_3$$
$$\times\bigg\{2\,\mathrm{Im}[(\zeta-\omega_2-i\delta)^{-1}(\zeta+\mu-\xi_1-i\delta)^{-1}(\zeta-\omega_1-i\delta)^{-1}][n(\mu)+f(\xi_1)]$$
$$\times\frac{\partial f_1(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\nu)}{\partial\nu}\bigg|_{\nu=0}-\frac{\partial f(\xi_1)}{\partial\xi_1}f_2(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_1)\bigg\}\tau_3 a(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\omega_1)\,. \quad (2.22)$$

$$f_2(\mathbf{p},\mathbf{q},\zeta,\zeta)=\left(\frac{p_z^2}{m}\right)a(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta)\tau_3 a(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta)+\int\frac{d^3k d\omega_1 d\omega_2 d\xi_1 d\xi_2 d\mu}{(2\pi)^8}d(\mathbf{k},\mu)a(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\omega_2)\tau_3$$
$$\times\{2\,\mathrm{Im}[(\zeta-\omega_1-i\delta)^{-1}(\zeta+\mu-\xi_1-i\delta)^{-1}]2\,\mathrm{Im}[(-\xi_2-i\delta)^{-1}(\zeta-\omega_2-i\delta)^{-1}]$$
$$\times[n(\mu)+f(\xi_1)]f_1(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)$$
$$+(n(\mu)2\,\mathrm{Im}[(\zeta+\mu-\xi_1-i\delta)^{-1}(\zeta-\omega_1-i\delta)^{-1}]2\,\mathrm{Im}[(\zeta+\mu-\xi_2-i\delta)^{-1}(\zeta-\omega_2-i\delta)^{-1}]$$
$$+f(\xi_1)2\,\mathrm{Im}[(\zeta+\mu-\xi_1-i\delta)^{-1}(\zeta-\omega_1-i\delta)^{-1}]2\,\mathrm{Im}[(\zeta-\omega_2-i\delta)^{-1}(\xi_1-\xi_2-i\delta)^{-1}]-f(\xi_2)2\pi\delta(\zeta-\omega_1)$$
$$\times 2\,\mathrm{Im}[(\zeta-\omega_2-i\delta)^{-1}(\zeta+\mu-\xi_2-i\delta)^{-1}(\xi_1-\xi_2-i\delta)^{-1}])f_2(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)\}\tau_3 a(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\omega_1)\,. \quad (2.23)$$

Since only the τ_3 components of $\partial f_1/\partial\nu$ and of f_2 appear in the expression for the attenuation constant, we consider only this component. We will show that the integral over $\mathbf{p}$ of the left-hand side of (2.22) multiplied by p_z^2/m is zero. The τ_3 component of $\partial f_1/\partial\nu$ is coupled to all components of $\partial f_1/\partial\nu$ and f_2. The coefficients of these components are even in q except for that of the τ_2 components. Let us forget about the τ_2 components for the present. The attenuation constant is an even function of q. Hence, we can assume the following form for the τ_0, τ_1, and τ_3 components:

$$f_2(\mathbf{p},\mathbf{q},\zeta,\zeta)=(p_z^2/m)\tilde{f}_2(\epsilon_p,\mathbf{q},\zeta)\,,$$
$$\frac{\partial f_1}{\partial\nu}(\mathbf{p},\mathbf{q},\zeta,\nu)=\frac{p_z^2}{m}\tilde{f}_1(\epsilon_p,\mathbf{q},\zeta)\,. \quad (2.24)$$

We put this into (2.23) and do the $\mathbf{k}$ integral. After integrating over the solid angle, the integrand becomes a function of k^2 and $\epsilon_{\mathbf{p}+\mathbf{k}}$. Now, we multiply by p_z^2 and integrate over $\mathbf{p}$. The following integrals occur:

$$I_1=\int d\epsilon_p\,\mathrm{tr}[\tau_3 G(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta-i\delta)\tau_3 G(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta-i\delta)]\,,$$
$$I_2=\int d\epsilon_p[G_3(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta-i\delta)G_{1(0)}(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta-i\delta)$$
$$+G_3(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta-i\delta)G_{1(0)}(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta-i\delta)]\,, \quad (2.25)$$

where G_i is the τ_ith component of G. In an Appendix, we show that $I_1=0$. I_2 is zero because the integrand is odd in ϵ_p. Now we come back to the τ_2 component. We integrate over ω_1 and ω_2 resulting in the term

$$G_0(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta-i\delta)G_1(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta-i\delta)$$
$$-G_1(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta-i\delta)G_0(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta-i\delta)\,. \quad (2.26)$$

Making the Eliashberg weak-momentum-dependence approximation, we find that (2.26) is zero. Thus, we find that only $f_2(\mathbf{p},\mathbf{q},\zeta,\zeta)$ contributes to the attenuation.

We now look at this function. It is convenient to add and subtract from Eq. (2.23) the terms

$$\frac{(2\pi)^4}{4}\{\delta(\zeta-\omega_1)\delta(\zeta-\omega_2)\delta(\zeta+\mu-\xi_1)\delta(\xi_2)[n(\mu)+f(\xi_1)]$$
$$\times f_1(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)+[n(\mu)\delta(\zeta-\omega_1)\delta(\zeta-\omega_2)$$
$$\times\delta(\zeta+\mu-\xi_1)\delta(\zeta+\mu-\xi_2)+f(\xi_1)\delta(\zeta-\omega_1)\delta(\zeta+\mu-\xi_1)$$
$$\times\delta(\zeta-\omega_2)\delta(\xi_1-\xi_2)]f_2(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)\}\,.$$

The equation for $f_2(\mathbf{p},\mathbf{q},\zeta,\zeta)$ may then be written in the form

$$f_2(\mathbf{p},\mathbf{q},\zeta,\zeta)=a(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\,\zeta)\tau_3\Big[\Gamma(\mathbf{p},\mathbf{q},\zeta,0)+\frac{1}{4}\int\frac{d\mu d^3k}{(2\pi)^4}\{d(\mathbf{k},\mu)[n(\mu)+f(\zeta+\mu)]f_2(\mathbf{p}+\mathbf{k},\,\mathbf{q},\,\zeta+\mu,\,\zeta+\mu)\}\Big]\tau_3a(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\,\zeta)$$

$$+\int\frac{d^3kd\omega_1d\omega_2d\xi_1d\xi_2d\mu}{(2\pi)^8}d(\mathbf{k},\mu)a(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\,\omega_2)\tau_3(2\pi)^2\Big\{\Big[\delta(\zeta-\omega_1)\delta(\xi_2)\,\mathrm{Re}\frac{1}{\zeta+\mu-\xi_1-i\delta}\,\mathrm{Re}\frac{1}{\zeta-\omega_2-i\delta}$$

$$+\delta(\zeta+\mu-\xi_1)\delta(\xi_2)\,\mathrm{Re}\frac{1}{\zeta-\omega_1-i\delta}\,\mathrm{Re}\frac{1}{\zeta-\omega_2-i\delta}+\delta(\zeta+\mu-\xi_1)\delta(\zeta-\omega_2)\,\mathrm{Re}\frac{1}{-\xi_2-i\delta}\,\mathrm{Re}\frac{1}{\zeta-\omega_1-i\delta}\Big]$$

$$\times[f(\xi_1)+n(\mu)]f_1(\mathbf{p}+\mathbf{k},\,\mathbf{q},\,\xi_1,\,\xi_2)+\Big(\Big[\delta(\zeta-\omega_1)\delta(\zeta+\mu-\xi_2)$$

$$\times\mathrm{Re}\frac{1}{\zeta+\mu-\xi_1-i\delta}\,\mathrm{Re}\frac{1}{\zeta-\omega_2-i\delta}+\delta(\zeta+\mu-\xi_1)\delta(\zeta+\mu-\xi_2)\,\mathrm{Re}\frac{1}{\zeta-\omega_1-i\delta}$$

$$\times\mathrm{Re}\frac{1}{\zeta-\omega_2-i\delta}+\delta(\zeta+\mu-\xi_1)\delta(\zeta-\omega_2)\,\mathrm{Re}\frac{1}{\zeta-\omega_1-i\delta}\,\mathrm{Re}\frac{1}{\zeta+\mu-\xi_2-i\delta}\Big]n(\mu)$$

$$+\Big[\delta(\zeta-\omega_1)\delta(\xi_1-\xi_2)\,\mathrm{Re}\frac{1}{\zeta+\mu-\xi_1-i\delta}\,\mathrm{Re}\frac{1}{\zeta-\omega_2-i\delta}+\delta(\zeta+\mu-\xi_1)\delta(\xi_1-\xi_2)$$

$$\times\mathrm{Re}\frac{1}{\zeta-\omega_1-i\delta}\,\mathrm{Re}\frac{1}{\zeta-\omega_2-i\delta}+\delta(\zeta+\mu-\xi_1)\delta(\zeta-\omega_2)\,\mathrm{Re}\frac{1}{\zeta-\omega_1-i\delta}$$

$$\times\mathrm{Re}\frac{1}{\xi_1-\xi_2-i\delta}\Big]f(\xi_1)-\Big[\delta(\zeta+\mu-\xi_1)\,\mathrm{Re}\frac{1}{\zeta-\omega_2-i\delta}\,\mathrm{Re}\frac{1}{\xi_1-\xi_2-i\delta}+\delta(\xi_1-\xi_2)$$

$$\times\mathrm{Re}\frac{1}{\zeta-\omega_2-i\delta}\,\mathrm{Re}\frac{1}{\zeta+\mu-\xi_2-i\delta}\Big]\delta(\zeta-\omega_1)f(\xi_2)\Big)f_2(\mathbf{p}+\mathbf{k},\,\mathbf{q},\,\xi_1,\,\xi_2)\Big\}\tau_3a(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\,\omega_1).\quad(2.27)$$

In (2.27) $\Gamma(\mathbf{p},\mathbf{q},\zeta,0)$ describes the vertex associated with the ladder approximation. It is given by

$$\Gamma(\mathbf{p},\mathbf{q},\zeta,\nu)=\frac{p_z^2}{m}\tau_3+\sum_k\int\frac{d\nu_1d\omega_1d\omega_2}{(2\pi)^3}d(\mathbf{k},\nu_1)\Big\{[n(\nu_1)+f(\omega_1)]f_1(\mathbf{p}+\mathbf{k},\,\mathbf{q},\,\omega_1,\,\omega_2)\,\mathrm{Re}\frac{1}{(\zeta+\nu_1-\omega_1-i\delta)(\nu-\omega_2-i\delta)}$$

$$+\Big[u(\nu_1)\,\mathrm{Re}\frac{1}{(\zeta+\nu_1-\omega_1-i\delta)(\zeta+\nu_1+\nu-\omega_2-i\delta)}+f(\omega_1)\,\mathrm{Re}\frac{1}{(\zeta+\nu_1-\omega_1-i\delta)(\omega_1-\omega_2+\nu-i\delta)}$$

$$-f(\omega_2)\,\mathrm{Re}\frac{1}{(\zeta+\nu+\nu_1-\omega_2-i\delta)(\omega_1+\nu-\omega_2-i\delta)}\Big]f_2(\mathbf{p}+\mathbf{k},\,\mathbf{q},\,\omega_1,\,\omega_2)\Big\}.\quad(2.28)$$

This function is related to the analytic continuation of Eq. (2.11) by

$$\Gamma(\mathbf{p},\mathbf{q},\zeta,\nu)=\tfrac{1}{2}[\Gamma(\mathbf{p},\mathbf{q},\zeta_l,\nu_m)|_{\zeta_l=\zeta-i\delta,\ \nu_m=\nu-i\delta}+\Gamma(\mathbf{p},\mathbf{q},\zeta_l,\nu_m)|_{\zeta_l=\zeta+i\delta,\ \nu_m=\nu+i\delta}].\quad(2.29)$$

The Ward identity, reflecting momentum conservation, satisfied by this function in the weak momentum dependence approximation is

$$\Gamma(\mathbf{p},\mathbf{q},\zeta,\nu)=(p_z^2/m+p_zq_z/m+q_z^2/4m)\tau_3.\quad(2.30)$$

Now, the wave number of the impressed sound wave is generally much smaller than that of a thermal phonon. If we restrict ourselves to this approximation, we can neglect q_z^2/m compared to p_z^2/m. The p_zq_z/m term will vanish upon integration over p since it is odd in p_z. That is,

$$\Gamma(\mathbf{p},\mathbf{q},\zeta,0)=(p_z^2/m)\tau_3.\quad(2.31)$$

Equation (2.27) can be simplified further. The first and second terms in the integrand of the second integral are zero because of $\delta(\xi_2)f_1(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)$. After using Eq. (2.24) and integrating Eq. (2.27) over $\mathbf{p}$, we find that to order (kT/cp_F) (c=sound velocity), the τ_3 component of f_2 satisfies the following equation

$$\int \mathrm{tr}\tau_3 f_2(\mathbf{p},\mathbf{q},\zeta,\zeta)\frac{p_z^2}{m}\frac{d^3p}{(2\pi)^3}=\int\frac{d^3p}{(2\pi)^3}\left(\frac{p_z^2}{m}\right)^2\mathrm{tr}[a(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta)\tau_3 a(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta)\tau_3]$$
$$+\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d\mu}{2\pi}\int\frac{d^3k}{(2\pi)^3}\bigg\{d(\mathbf{k},\mu)[u(\mu)+f(\zeta+\mu)]\left(\frac{p_z^2}{m}\right)\mathrm{tr}[a(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta)\tau_3 a(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta)\tau_3]$$
$$+\mathrm{tr}\tau_3 f_2(\mathbf{p}+\mathbf{k},\mathbf{q},\zeta+\mu,\zeta+\mu)\bigg\}+f'(\mathbf{q},\zeta)\,, \quad (2.32)$$

$f'(\mathbf{q},\zeta)$ represents the contributions of the τ_2 components of f_1 and f_2. Its explicit form is

$$f'(\mathbf{q},\zeta)=i\int\frac{d^3p}{(2\pi)^3}\frac{p_z^2}{m}\int\frac{d^3kd\xi_1d\xi_2d\mu}{(2\pi)^6}(2\pi)^2d(\mathbf{k},\mu)\bigg\{(f(\xi_1)+n(\mu))\delta(\zeta+\mu-\xi_1)\,\mathrm{Re}\frac{1}{-\xi_2-i\delta}$$
$$\times\mathrm{tr}\tau_2 f_1(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)+[f(\xi_1)+f(\xi_2)+2n(\mu)]\delta(\zeta+\mu-\xi_1)\,\mathrm{Re}\frac{1}{\zeta+\mu-\xi_2-i\delta}\,\mathrm{tr}\tau_2 f_2(\mathbf{p}+\mathbf{k},\mathbf{q},\xi_1,\xi_2)\bigg\}$$
$$\times\zeta|Z(\zeta)|^2\Delta_2(\zeta)\left[\mathrm{Re}\frac{1}{^2Z^2(\zeta)-\epsilon_+^2-\phi^2(\zeta)}\,\mathrm{Re}\frac{1}{\zeta^2Z^2(\zeta)-\epsilon_-^2-\phi^2(\zeta)}+\mathrm{Im}\frac{1}{\zeta^2Z^2(\zeta)-\epsilon_+^2-\phi^2(\zeta)}\,\mathrm{Im}\frac{1}{\zeta^2Z^2(\zeta)-\epsilon_-^2-\phi^2(\zeta)}\right],$$

where we have used

$$G(p,\zeta)=\frac{\zeta Z(\zeta)+\epsilon_p\tau_3+\phi(\zeta)\tau_1}{\zeta^2Z^2(\zeta)-\epsilon_p^2-\phi^2(\zeta)}\,,$$
$$\epsilon_\pm=\frac{(\mathbf{p}\pm\frac{1}{2}\mathbf{q})^2}{2m}-\mu_c;\quad \mu_c=\text{chemical potential}$$
$$\Delta_2(\zeta)=\mathrm{Im}(\phi(\zeta)/Z(\zeta))\,.$$

This term is of negligible importance for our problem because of the smallness of Δ_2.[5,7] After explicitly doing the p integral Eq. (2.32) becomes

$$\int\frac{d^3p}{(2\pi)^3}\left(\frac{p_z^2}{m}\right)^2\mathrm{tr}\tau_3\tilde{f}_2(\mathbf{p},\mathbf{q},\zeta)=\left(\frac{p_F^2}{m}\right)^2\frac{\pi N(0)}{(p_Fq/2m)}$$
$$\times\left[1+\frac{\zeta^2-|\Delta|^2}{|\zeta^2-\Delta^2|}\right]\frac{1}{[ql(\zeta)]^4}\left[\frac{(ql(\zeta))^3}{3}-ql(\zeta)+\tan^{-1}ql(\zeta)\right]$$
$$\times\bigg[1+\frac{1}{2}\int\frac{d\mu}{2\pi}\sum_\lambda\int d\omega_\lambda F_\lambda(\omega_\lambda)\int d\epsilon_{p+k}d(\omega_\lambda,\mu)$$
$$\times[n(\mu)+f(\zeta+\mu)]\,\mathrm{tr}\tau_3\tilde{f}_2(\mathbf{p}+\mathbf{k},\mathbf{q},\zeta+\mu)\bigg]\,, \quad (2.33)$$

where we have neglected the contribution of the τ_2 terms. $N(0)$ is the density of states at the Fermi surface and $l(\zeta)$, the mean free path, is defined by

$$l(\zeta)=\frac{p_F}{2m|\mathrm{Im}Z(\zeta)(\zeta^2-\Delta^2(\zeta))^{1/2}|}\,.$$

From Eq. (2.16) we have

$$\lim_{\omega\to0}\mathrm{Re}\frac{1}{i\omega}\langle[\tau_{zz},\tau_{zz}]\rangle(\mathbf{q},\omega)=-\frac{1}{2}\int\frac{d\omega_1}{2\pi}\frac{\partial f(\omega_1)}{\partial\omega_1}$$
$$\times\int\frac{d^3p}{(2\pi)^3}\mathrm{tr}\tau_3\tilde{f}_2(\mathbf{p},\mathbf{q},\omega_1)\left(\frac{p_z^2}{m}\right)^2. \quad (2.34)$$

In Eq. (2.33), λ is the phonon branch index and F_λ is the phonon density of states.

III. THE OTHER FUNCTIONS

The evaluation of $\mathrm{Re}(1/i\omega)\langle[\tau_{zz},n]\rangle(\mathbf{q},\omega)$ and $\mathrm{Re}(1/i\omega)\langle[n,n]\rangle(\mathbf{q},\omega)$ proceeds in exactly the same way. The only differences are in the number of p_z^2/m that arise in the equations for χ and p. For the stress-tensor–density correlation function, the analog of Eq. (2.10) is

$$P(\mathbf{q},\nu_m)=(i/\beta)\sum_{\mathbf{p},\zeta_l}\mathrm{tr}\tau_3\chi(\mathbf{p},\mathbf{q},\zeta_l,\nu_m)\,, \quad (3.1)$$

while Eq. (2.17) remains unchanged. Making the same assumptions and approximations as in Sec. II, we ob-

[7] J. C. Swihart (private communication); J. W. F. Woo, thesis, Cornell University, 1966 (unpublished). See also V. Ambegaokar and J. Woo, Phys. Rev. 139, A1818 (1965). The model of the electron-phonon coupling constant and the frequency distribution of phonons used in the present calculation is identical to that used in the paper of Ambegaokar and Woo.

tain the equation corresponding to (2.33).

$$\int \frac{d^3p}{(2\pi)^3}\left(\frac{p_z^2}{m}\right) \mathrm{tr}\tau_3\tilde{f}_2^{(\tau)}(\mathbf{p},\mathbf{q},\zeta) = \left(\frac{p_F^2}{m}\right)\frac{\pi N(0)}{[p_F q/2m]} \times\left[1+\frac{\zeta^2-|\Delta|^2}{|\zeta^2-\Delta^2|}\right]\frac{1}{[ql(\zeta)]^2}[\tan^{-1}ql(\zeta)-ql(\zeta)] \times\left[1+\frac{1}{2}\int\frac{d\mu}{2\pi}\sum_\lambda\int d\omega_\lambda F_\lambda(\omega_\lambda)\int d\epsilon_{p+k}d(\omega_\lambda,\mu) \times[n(\mu)+f(\zeta+\mu)]\,\mathrm{tr}\tau_3\tilde{f}_2^{(\tau)}(\mathbf{p}+\mathbf{k},\,\mathbf{q},\,\zeta+\mu)\right], \quad (3.2)$$

where $\tilde{f}_2^{(\tau)}$ is the function which corresponds to $\tilde{f}_2$. The function which corresponds to f_1 is also zero. For the density-density correlation function, Eq. (3.1) is correct while Eq. (2.17) becomes

$$\chi(\mathbf{p},\mathbf{q},\zeta_l,\nu_m) = G(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\,\zeta_l+\nu_m)\tau_3 G(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\,\zeta_l) -\frac{1}{\beta}\sum_{k,\mu_n} G(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\,\zeta_l+\nu_m)\tau_3\chi(\mathbf{p}+\mathbf{k},\,\mathbf{q},\,\zeta_l+\mu_n,\,\nu_m) \times\tau_3 G(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\,\zeta_l)D(\mathbf{k},\mu_n)\,. \quad (3.3)$$

We see that $\sum_{\mathbf{p},\zeta_l}\mathrm{tr}\tau_3\chi(\mathbf{p},\mathbf{q},\zeta_l,\nu_m)$ where χ is given by Eq. (3.3) is just the phonon self-energy. In the ladder approximation, it satisfies the following integral equation.

$$\int\frac{d^3p}{(2\pi)^3}\mathrm{tr}\tau_3\tilde{f}_2^{(n)}(\mathbf{p},\mathbf{q},\zeta) = \frac{\pi N(0)}{[p_F q/2m]}\left[1+\frac{\zeta^2-|\Delta|^2}{|\zeta^2-\Delta^2|}\right] \times\tan^{-1}(ql(\zeta))\left[1+\frac{1}{2}\int\frac{d\mu}{2\pi}\sum_\lambda\int d\omega_\lambda F_\lambda(\omega_\lambda) \times\int d\epsilon_{p+k}d(\omega_\lambda,\mu)[n(\mu)+f(\zeta+\mu)] \times\mathrm{tr}\tau_3\tilde{f}_2^{(n)}(\mathbf{p}+\mathbf{k},\,\mathbf{q},\,\zeta+\mu)\right]. \quad (3.4)$$

The density-density correlation function which appears in the formula for α_L is then given by multiplying Eq. (3.4) by $-\frac{1}{2}[\partial f(\zeta)/\partial\zeta]$ and integrating the result over ζ.

IV. $ql\gg1$ AND $ql\ll1$

In this section, we will consider the reduced attenuation (ratio of the attenuations in the superconducting (α_{Ls}) and the normal (α_{Ln}) states) in the $ql\gg1$ and $ql\ll1$ limits and compare our results with Deaton's experiment.[1] Since the mean free path is a function of frequency, by $ql\ll1$ we mean that this inequality holds over most of the frequencies important in our problem. The attenuation in the normal state is obtained by letting Δ go to zero in the formula for α_{Ls}.

Perhaps it is appropriate to point out one difference between our calculation and that for the impurity-scattering case. In our problem, the mean free path depends on frequency and differs in the normal and superconducting states. For impurity scattering, however, the mean free path does not depend on frequency nor does it differ in the two phases. This accounts for the different results in the two cases.

(a) $ql\gg1$

In this limit,

$$\lim_{\omega\to0}\mathrm{Re}(1/i\omega)\langle[\tau_{zz},\tau_{zz}]\rangle(\mathbf{q},\omega)\propto 1/q\bar{l}\,,$$
$$\lim_{\omega\to0}\mathrm{Re}(1/i\omega)\langle[\tau_{zz},n]\rangle(\mathbf{q},\omega)\propto 1/q\bar{l}\,, \quad (4.1)$$
$$\lim_{\omega\to0}\mathrm{Re}(1/i\omega)\langle[n,n]\rangle(\mathbf{q},\omega)\propto 1\,,$$

where $\bar{l}$ is some averaged mean free path. That is, the density-density correlation function dominates and we recover Ambegaokar's result[3] that for $ql\gg1$,

$$\alpha_{Ls}/\alpha_{Ln} = 2/(e^{\beta\Delta}+1)\,. \quad (4.2)$$

This result is not in agreement with Deaton's experiment.[1,3] He finds that the attenuation is anomalously small in this limit. (For $T/T_c\sim0.95$, $(\alpha_{Ls}/\alpha_{Ln})_{\mathrm{expt}}\sim\frac{1}{3}(2f(\Delta))$ for $ql\gg1$, $2\Delta(0)/kT_c=4.3$.)

(b) $ql\ll1$

There is now no real difference between Eqs. (2.33), (3.2), and (3.4). The only differences are in the factors of p_F^2/m. Since the vertex corrections for the electron-phonon interaction is small, the vertex corrections in Eqs. (2.33) and (3.2) are also small. The inhomogeneous terms are all proportional to $[1+(\omega^2-|\Delta^2|/|\omega^2-\Delta^2|)]\times ql_s(\omega)$, except for a very small region around Δ where $ql_s\sim1$. Thus, to a good approximation, ($\sim10\%$) the reduced attenuation is

$$\frac{\alpha_{Ls}}{\alpha_{Ln}} = \frac{\displaystyle\int_0^\infty d\omega[1+(\omega^2-|\Delta^2|)/(|\omega^2-\Delta^2|)]\,\mathrm{sech}^2\tfrac{1}{2}\beta\omega\,\tan^{-1}ql_s(\omega)}{\displaystyle 2\int_0^\infty d\omega\,\mathrm{sech}^2\tfrac{1}{2}\beta\omega\,\tan^{-1}ql_n(\omega)}\,. \quad (4.3)$$

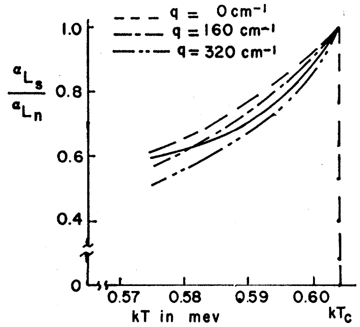

FIG. 1. The reduced attenuation for $q=320$, 160, 0. V_F is assumed to be 10^8 cm/sec. The critical temperature is $kT_c=0.6040$ meV. For $q=320$, $ql_n\sim 0.4$. The solid line represents $2f(\Delta)$ with $2\Delta(0)/kT_c=4.3$.

In particular, our result is a function of q while that for impurity scattering is not.[4] Using the data of Swihart[7] for the values of Z and Δ in lead, we did the integrals in Eq. (4.3) for various values of q for which $ql\ll 1$ over most of the important frequencies. The results are shown in Fig. 1. For comparison, we have plotted $2f(\Delta)$ with $2\Delta(0)/kT_c=4.3$ on the same graph. We notice that over the temperature range for which we have theoretical data, the curve for $q=320$ cm^{-1} differs only slightly from $2f(\Delta)$. This result is in approximate agreement with that of Deaton. He finds that for $q\sim 290$, the experimental value differs very little from the predictions of the BCS model with impurity scattering and $2\Delta(0)/kT_c=3.5$.

Finally we remark that for very low temperatures ($T/T_c\ll 1$, T_c=critical temperature) the mean free path becomes very long, so that $ql\gg 1$. Thus for low temperatures, the reduced attenuation should behave like $2f(\Delta)$.

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APPENDIX A

In this Appendix, we evaluate the integrals

$$B_n=\int\frac{d^3p}{(2\pi)^3}\left(\frac{p_z^2}{m}\right)^n \mathrm{tr}[\tau_3 a(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta)\tau_3 a(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta)],\quad n=0,1,2.$$

From

$$a(\mathbf{p},\zeta)=i[G(\mathbf{p},\zeta+i0^+)-G(\mathbf{p},\zeta-i0^+)]\equiv i[G(\mathbf{p},\zeta^+)-G(\mathbf{p},\zeta^-)],$$

we find that

$$\mathrm{tr}[\tau_3 a(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\zeta)\tau_3 a(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\zeta)]=-\mathrm{tr}[G(\mathbf{p}_+,\zeta^+)\tau_3 G(\mathbf{p}_-,\zeta^+)\tau_3 +G(\mathbf{p}_+,\zeta^-)\tau_3 G(\mathbf{p}_-,\zeta^-)\tau_3-G(\mathbf{p}_+,\zeta^+)\tau_3 G(\mathbf{p}_-,\zeta^-)\tau_3-G(\mathbf{p}_+,\zeta^-)\tau_3 G(\mathbf{p}_-,\zeta^+)\tau_3],$$

where $p_\pm=\mathbf{p}\pm\frac{1}{2}\mathbf{q}$. We first show that

$$\int\frac{d^3p}{(2\pi)^3}\left(\frac{p_z^2}{m}\right)^n \mathrm{tr}[G(p_+,\zeta^{+(-)})\tau_3 G(p_-,\zeta^{+(-)})\tau_3]=0.$$

Using

$$G(p,\zeta^+)=\frac{\zeta Z(\zeta^+)+\epsilon_p\tau_3+\phi(\zeta^+)\tau_1}{\zeta^2Z^2(\zeta^+)-\epsilon_p^2-\phi^2(\zeta^+)}$$

and

$$\mathrm{Im}G(\mathbf{p},\zeta^+)=-\mathrm{Im}G(\mathbf{p},\zeta^-),$$

we find

$$\int\frac{d^3p}{(2\pi)^3}\left(\frac{p_z^2}{m}\right)^n \mathrm{tr}[G(\mathbf{p}_+,\zeta^{+(-)})\tau_3 G(\mathbf{p}_-,\zeta^{+(-)})\tau_3]$$

$$=\left(\frac{p_F^2}{m}\right)^n\int\frac{d\Omega}{4\pi}N(0)x^{2n}\int d\epsilon\frac{\zeta^2Z^2(\zeta^{+(-)})+\epsilon_+\epsilon_--\phi^2(\zeta^{+(-)})}{(\epsilon-E_+^{+(-)})(\epsilon-E_-^{+(-)})(\epsilon+E_+^{+(-)})(\epsilon+E_-^{+(-)})},$$

where

$$\epsilon_\pm=\frac{p^2}{2m}+\frac{q^2}{8m}-\mu\pm\frac{pqx}{2m}\equiv\epsilon\pm\frac{pq}{2m}x,$$

$$E_\pm^{+(-)}=[\zeta^2Z^2(\zeta^{+(-)})-\phi^2(\zeta^{+(-)})]^{1/2}\mp\frac{pq}{2m}x,\quad \mathrm{Im}E_\pm^{+(-)}>0,$$

$$x=\cos\theta.$$

Consider

$$\int d\epsilon \frac{\zeta^2 Z^2(\zeta^{+(-)})+\epsilon^2-(pqx/2m)^2-\phi^2(\zeta^{+(-)})}{(\epsilon-E_+{}^{+(-)})(\epsilon-E_-{}^{+(-)})(\epsilon+E_+{}^{+(-)})(\epsilon+E_-{}^{+(-)})}=\frac{\zeta^2 Z^2(\zeta^{+(-)})+(E_+{}^{+(-)})^2-(pqx/2m)^2-\phi^2(\zeta^{+(-)})}{2E_+{}^{+(-)}([E_+{}^{+(-)}]^2-[E_-{}^{+(-)}]^2)}-\frac{\zeta^2 Z^2(\zeta^{+(-)})+[E_-{}^{+(-)}]^2-(pqx/2m)^2-\phi^2(\zeta^{+(-)})}{2E_-{}^{+(-)}([E_+{}^{+(-)}]^2-[E_-{}^{+(-)}]^2)}=0.$$

Next, we evaluate

$$\int \frac{d^3p}{(2\pi)^3}\left(\frac{p_z{}^2}{m}\right)^n \mathrm{tr}[\tau_3 G(p_+,\zeta^{+(-)})\tau_3 G(p_-,\zeta^{-(+)})]=\left(\frac{p_F{}^2}{m}\right)^n N(0)\int_{-1}^{1}\frac{dx}{2}x^{2n}\int d\epsilon\frac{\zeta^2|Z(\zeta^\pm)|^2+\epsilon^2-\bar{x}^2-|\phi(\zeta^\pm)|^2}{(\epsilon-\epsilon_1+\bar{x})(\epsilon+\epsilon_1+\bar{x})(\epsilon-\epsilon_1{}^*-\bar{x})(\epsilon+\epsilon_1{}^*-\bar{x})},$$

where $\epsilon_1{}^2+\phi^2(\zeta^+)-\zeta^2Z^2(\zeta^+)=0$, $\mathrm{Im}\epsilon_1>0$; $\bar{x}=p_F qx/2m$. The ϵ integral can be done by contour integration with the result

$$\begin{aligned}\frac{B_n}{2}&=-\tfrac{1}{4}(N(0)\pi i)\int_{-1}^{1}\frac{dx}{2}\left[1+\frac{\zeta^2-|\Delta(\zeta^+)|^2}{|\zeta^2-\Delta^2(\zeta^+)|}\right]\frac{2i\,\mathrm{Im}\epsilon_1}{[(\mathrm{Im}\epsilon_1)^2+\bar{x}^2]}\left(\frac{p_F{}^2x^2}{m}\right)^n\\&=\frac{1}{2}\frac{\pi N(0)}{[p_Fq/2m]}\left(\frac{p_F{}^2}{m}\right)^n\frac{1}{[ql(\zeta)]^{2n}}\left(1+\frac{\zeta^2-|\Delta(\zeta^+)|^2}{|\zeta^2-\Delta^2(\zeta^+)|^2}\right)\int_{-ql(\zeta)}^{ql(\zeta)}\frac{dy}{2}\frac{y^{2n}}{1+y^2}\\&\equiv A_n(p,q,\zeta)\int_{-ql(\zeta)}^{ql(\zeta)}\frac{dy}{2}\frac{y^{2n}}{1+y^2}.\end{aligned}$$

Therefore,

$$\frac{B_0}{2}=A_0(p,q,\zeta)\tan^{-1}(ql(\zeta)),\quad \frac{B_1}{2}=A_1(p,q,\zeta)[ql(\zeta)-\tan^{-1}[ql(\zeta)]],\quad \frac{B_2}{2}=A_2(p,q,\zeta)\left[\frac{[ql(\zeta)]^3}{3}-ql(\zeta)+\tan^{-1}ql(\zeta)\right].$$