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# Theory of the Gunn Effect

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This paper analyzes the possible forms of propagating field inhomogeneities in media with a negative differential resistance. A formula for the velocity is derived for both layers and domains. We discuss the stability of a pulse and calculate the impedance across a sample with a mode excited.

## I. INTRODUCTION

IN this paper, a mathematical framework for analyzing space-charge waves in a Gunn diode is presented. Some of the results have appeared in a prior publication.[1] Our intention here is to elaborate on the techniques, to further develop the physical implications, and to evaluate explicitly certain expressions which appear in the theory.

The approach is through the rigorous analysis of a model. We show that under steady operating conditions, field inhomogeneities persist only in specific propagating forms or "modes." These are: (1) isolated dipole domains (either high or low field, but selected by the current), (2) a family of periodic domains, and (3) charge accumulation or depletion layers. Salient features of the pulse are determined and the domain shape is reduced to quadratures. Velocity criteria are derived. Finally, we calculate the ac impedance for a diode with one of these modes excited.

What causes these waves to form and propagate is a negative differential resistance. Such media are unstable against charge bunching of the flowing carriers. These effects were first discussed by Shockley,[2] Reik,[3] and Ridley.[4] Ridley was able to argue the existence of dipole domains and charge layers. Indeed, his conclusions provided the support for Kroemer's[5] explanation of Gunn's observation of propagating electronic fronts in GaAs.

Subsequent numerical studies[6,7] have refined the model and revealed further properties. Other workers, Bonch-Bruevich[8] and Ridley[9] among them, have analyzed the linear-stability criterion in these systems. The present authors have solved the nonlinear problem of amplification of a fluctuation.[10] The first analytic work on steady-state domain structure is due to Butcher.[11] He derived a geometrical criterion for the peak field strength and proved that with field-independent diffusion the domain velocity was equal to the outside carrier velocity.

In Ref. 10 it was shown how a disturbance evolves, propagates, and develops into a domain bounded by shock discontinuities. The technique consisted of solving for the force on a bit of the compressible charge fluid assuming its trajectory was known. The trajectories were then evaluated and an exact solution resulted. When trajectories collide, shock fronts form. The present paper continues from that point, with particular emphasis on the effects of diffusion.

## II. FORMULATION

The model is defined through the law of internal current $J$,

$$J(X,T)=V(E)N(X,T)-D(E)N_X(X,T). \qquad (1)$$

$N$ is the total charge density of mobile carriers and $V$ is their *average* velocity. Subscripts denote partial differentiation.

We take $V$ to be a function of $E$ alone, the electrostatic field strength. This assumes that the intra- and intervalley relaxation times are short compared to the various times characterizing the dielectric relaxation. The important feature here is a negative differential resistance, i.e., $V(E)<0$ in some range of $E$ (see Fig. 2). In select semiconductors, such as GaAs and InP, the intervalley transfer mechanism of Ridley, Watkins,[12] and Hilsum[13] provides the microscopic basis for such an anomalous resistance. In their mechanism, fast electrons are converted to a slower species by shifting the population of carriers from a light- to a heavier-mass valley with increasing field. For other materials, resonance in the ionized impurity elastic-scattering cross section could lower the mobility at higher energy.[14,15] Virtual levels associated with subsidiary valleys could be responsible.

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In Eq. (1), $D(E)$ is the field-dependent diffusion coefficient. With species conversion, irreversible thermodynamics does not uniquely specify the form of the diffusion current. The term is sensitive to details of the scattering within and between valleys. Without a solution to the transport equation it is not possible to pass judgment on the most realistic model. The model chosen, however, is favored by analytic simplicity and by obeying the ordinary Einstein relation

$$eD(E)=kT\mu(E).$$

The analysis, however, does not depend on the commitment of functions $D(E)$ to this relation.

The equation of motion for the electrostatic field is simply derived from Eq. (1) with the aid of Poisson's equation and the law of continuity:

$$\text{Poisson's equation:}\quad E_X=(4\pi/\kappa)(N(X,T)-N_0), \quad (2)$$

$$\text{continuity:}\quad J_X=-N_T. \quad (3)$$

Here $N_0$ is the fixed, uniform neutralizing background charge density; $\kappa$ is the dielectric constant. These two equations combined result in

$$E_T(X,T)=(4\pi/\kappa)(-J(X,T)+F(T)). \quad (4)$$

[Take the partial time derivative of (3) and substitute for $N_T$ from (3). The subsequent equation admits (4) as a first integral.] This states that there are two sources for the displacement current: (1) due to the internal leakage current $J$ and (2) due to the charge flux supplied to the ends of the diode by the external circuit. Substituting for $J$ from Eq. (1) and eliminating from Poisson's equation $N$ and $N_X$ in favor of $E_X$ and $E_{XX}$, respectively, we derive the fundamental equation of the theory,

$$-D(E)E_{XX}+V(E)E_X+E_T =(4\pi/\kappa)(F(T)-V(E)N_0). \quad (5)$$

We render this to dimensionless form by the following scaling transformation:

$$T=\tau t,\ \tau=\left(\frac{4\pi}{\kappa}\frac{V(E_a)}{E_a}N_0\right)^{-1},$$

where $E_a$ is any characteristic field strength in $V$;

$$X=lx, \qquad l=V(E_a)\tau\,;$$

$$E=E_a\mathcal{E}, \qquad V=V(E_a)v,$$

$$D=(l^2/\tau)\mathfrak{D},$$

and

$$F=N_0V(E_a)f.$$

Thus Eq. (5) becomes

$$-\mathfrak{D}(\mathcal{E})\mathcal{E}_{xx}+v(\mathcal{E})\mathcal{E}_x+\mathcal{E}_t=f(t)-v(\mathcal{E}). \quad (6)$$

## III. ANALYSIS

The implications of Eq. (6) will be explored systematically. For the case of constant external current ($f=\text{const}$) the sequence of topics will be small departures from equilibrium, traveling waves of finite amplitude, and the stability of these waves. Finally, these results will be applied to the general situation of interaction between traveling waves and the current from the external circuit.

### A. Small Departures From Equilibrium

Equation (6) evidently permits the equilibrium solutions ($v(\mathcal{E}_0)=f$). If there is a small departure from one of these solutions, then

$$\mathcal{E}(x,t)=\mathcal{E}_0+\mathcal{E}_1(x,t), \quad (7)$$

where $\mathcal{E}_1$ is small, and Eq. (6) yields the linearized equation

$$-\mathfrak{D}(\mathcal{E}_0)\mathcal{E}_{1xx}+v(\mathcal{E}_0)\mathcal{E}_{1x}+v_{\mathcal{E}}(\mathcal{E}_0)\mathcal{E}_1+\mathcal{E}_{1t}=0. \quad (8)$$

This equation has constant coefficients; the standard approach is to investigate solutions of the form

$$\mathcal{E}_1(x,t)=\mathcal{E}_1e^{\alpha x}e^{\lambda t}. \quad (9)$$

Substitution into (8) yields

$$-\mathfrak{D}\alpha^2+v\alpha+(v_{\mathcal{E}}+\lambda)=0, \quad (10)$$

which specifies the relationship between the "space constant" $\alpha$ and the "time constant" $\lambda$. To examine the implications of (10) it is convenient first to limit the investigation to small exponential traveling waves

$$\mathcal{E}_1(x,t)=\mathcal{E}_1(x-ct)=\mathcal{E}_1e^{\alpha(x-ct)} \quad (11)$$

moving at velocity $c$. This is equivalent to choosing $\lambda=-c\alpha$, and Eq. (10) becomes

$$\mathfrak{D}\alpha^2+(c-v)\alpha-v_{\mathcal{E}}=0. \quad (12)$$

If $v_{\mathcal{E}}$ is positive (the ohmic-resistance case, i.e., for $\mathcal{E}_0=\mathcal{E}_\alpha$ or $\mathcal{E}_\gamma$ in Fig. 2), Eq. (12) has real positive and negative solutions for $\alpha$. Either sign will lead the exponential (11) beyond the bounds of linearity in one end of the range $-\infty<x<\infty$. If $v_{\mathcal{E}}<0$ (the anomalous-resistance case, for $\mathcal{E}_0=\mathcal{E}_\beta$), $\alpha$ may be either complex or real depending on the magnitude of $(c-v)$. Unless $c-v=0$, $\alpha$ will have a real part, and $\mathcal{E}_1$ will again exceed linearity in one extreme range of $x$. The case $c=v$, $v_{\mathcal{E}}<0$ is special in that it yields a pure sinusoidal waveform with wavelength

$$L=2\pi(\mathfrak{D}/|v_{\mathcal{E}}|)^{1/2}, \quad (13)$$

which streams across the diode at the carrier velocity $v$.

Consider now more general streaming sinusoidal waves

$$\mathcal{E}_1(x,t)=\mathcal{E}_{10}e^{(2\pi i/L)(x-vt)}e^{\beta t}, \quad (14)$$

which stream at velocity $v$ but also grow or decay at a rate given by $\beta$. Here $L$ is not constrained by (13) but is freely chosen. Substitution into Eq. (8) yields

$$\beta=-v_{\mathcal{E}}-(2\pi/L)^2\mathfrak{D}. \quad (15)$$

From this result we draw the following conclusions: (1) The effect of diffusion is always dissipative and is emphasized more at shorter wavelengths; (2) the effect of the voltage-current characteristic is dissipative in the ohmic regions and antidissipative in the anomalous region; and (3) in the anomalous region short wavelengths will collapse but long wavelengths will explode; the critical wavelength is given by Eq. (13).

Note that any small initial perturbation of limited extent can be Fourier expanded as a superposition of the waves given by (9).

### B. Traveling Waves of Finite Amplitude

If one assumes in Eq. (6) that $\mathcal{E}$ has the form

$$\mathcal{E}(x,t)=\mathcal{E}(x-ct)\equiv\mathcal{E}(\xi) \tag{16}$$

of a traveling wave moving at velocity $c$, then Eq. (6) becomes

$$-\mathfrak{D}(\mathcal{E})\mathcal{E}_{\xi\xi}+(v(\mathcal{E})-c)\mathcal{E}_{\xi}=f-v(\mathcal{E})\,, \tag{17}$$

which is a *nonlinear ordinary* differential equation for $\mathcal{E}$. Solution of (17) by numerical means would be straightforward, but with no guarantee that the boundary conditions, as $\xi\rightarrow\pm\infty$, would make the solution acceptable. As in the linearized case, the wave velocity $c$ must be properly chosen.

The strategy against an equation such as (17) is to re-express it as a pair of first-order equations:

$$\frac{d\mathcal{E}}{d\xi}=\mathcal{E}_{\xi}\,, \tag{18}$$

$$\frac{d\mathcal{E}_{\xi}}{d\xi}=\frac{1}{\mathfrak{D}(\mathcal{E})}\{[v(\mathcal{E})-c](\mathcal{E}_{\xi}+1)-(f-c)\}\,. \tag{19}$$

Imagine a "phase-plane" whose points have coordinates $(\mathcal{E},\mathcal{E}_{\xi})$. Then as $\xi$ increases, the two components of motion of the "system point" on this phase plane will always be given by Eqs. (18) and (19). The coordinate $\mathcal{E}_{\xi}$ is the total charge density. Thus the Eqs. (18) and (19) express the rates of change for field and charge in terms of their local values. Evidently both derivatives in (18) and (19) vanish if $(\mathcal{E},\mathcal{E}_{\xi})$ satisfy both the relationships

$$v(\mathcal{E})=f\,,\quad \mathcal{E}_{\xi}=0\,. \tag{20}$$

These "singular points" (Fig. 1) are recognized as the equilibrium conditions already discussed.

If the phase-plane trajectory of the system point approaches close to a singular point, that part of its path can be calculated in detail from the linear analysis of Sec. III.A. It is easily shown that for the "ohmic" singular points, where the roots of (12) are real and opposite, there is a unique line of approach, and another unique line of departure (Fig. 1). At the "anomalous"

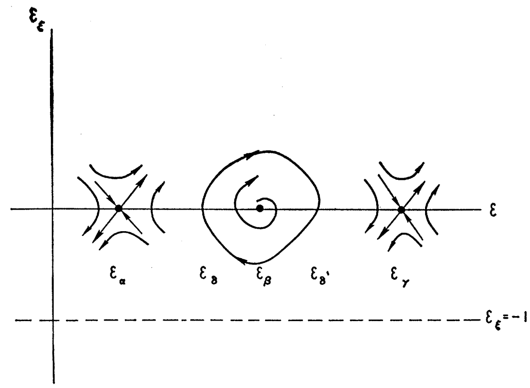


FIG. 1. Features of system-point trajectories.

singular point, complex roots of (12) lead to a spiral trajectory for the system point in that neighborhood.

Other interesting qualitative information may be deduced from the system-point equations (18) and (19). In particular, if the system point is on the line $\mathcal{E}_{\xi}=-1$ (Fig. 1), then by Eq. (19) it is crossing that line in a direction that is determined by the signature of $(f-c)$ alone; as a consequence, the system point can cross that line only once.

Another deduction from the phase plane: By the fixed-point theorem of topology, any closed trajectory must encircle a singular point.

The above two paragraphs lead to the observation that neither (a) a trajectory between singular points nor (b) a closed trajectory can cross the line $\mathcal{E}_{\xi}=-1$, whence

$$\mathcal{E}_{\xi}>-1 \tag{21}$$

for these categories. Now category (a) corresponds to traveling waves which approach equilibrium conditions as $\xi\rightarrow\pm\infty$, while category (b) includes periodic waves. These are the physically interesting solutions to Eq. (17).

Dividing (19) by (18) yields

$$\frac{\mathcal{E}_{\xi}}{\mathcal{E}_{\xi}+1}\frac{d\mathcal{E}_{\xi}}{d\mathcal{E}}=\frac{1}{\mathfrak{D}(\mathcal{E})}\left\{(v(\mathcal{E})-c)-\frac{f-c}{\mathcal{E}_{\xi}+1}\right\}\,, \tag{22}$$

and integration on $\mathcal{E}$ leads to the following results: Define

$$\Phi(\mathcal{E},\mathcal{E}_{\xi})\equiv\mathcal{E}_{\xi}-\ln(\mathcal{E}_{\xi}+1)-\int_{\mathcal{E}_\alpha}^{\mathcal{E}}d\mathcal{E}'\frac{v(\mathcal{E}')-c}{\mathfrak{D}(\mathcal{E}')}\,. \tag{23}$$

Then[11]

$$\Phi(\mathcal{E},\mathcal{E}_{\xi})=\Phi(\mathcal{E}_{\delta},0)+(c-f)\int_{\mathcal{E}_\delta}^{\mathcal{E}}\frac{d\mathcal{E}'}{\mathfrak{D}(\mathcal{E}')[\mathcal{E}_{\xi}(\mathcal{E}')+1]}\,, \tag{24}$$

where $\mathcal{E}_{\delta}$ is a value of $\mathcal{E}$ at which $\mathcal{E}_{\xi}=0$ (see Fig. 1), and $\mathcal{E}_{\xi}(\mathcal{E})$ is the solution to (22) which passes through this point. Expression (24) checks at the point $(\mathcal{E}=\mathcal{E}_{\delta}, \mathcal{E}_{\xi}=0)$ and also differentiates back to (22). Notice that $\Phi(\mathcal{E}_{\xi},\mathcal{E})$ has a unique value at any point on the phase plane.

If $c \neq f$, then the system point cannot follow a closed path on the phase plane. Proof is by contradiction. Assume a closed path as in Fig. 1. Extend the integration (24) all the way around the loop, so that $\mathcal{E}$ returns to $\mathcal{E}=\mathcal{E}_\delta$. Then Eq. (24) gives

$$\oint \frac{d\mathcal{E}'}{\mathfrak{D}(\mathcal{E}')[\mathcal{E}_\xi(\mathcal{E}')+1]}=0. \quad (25)$$

But the integrand of (25) is manifestly smaller on the outward arc than on the return, whence $c \neq f$ and a closed path are contradictory assumptions. Physically, this means that neither a periodic traveling wave nor a pulse bounded on each side by the same equilibrium condition (a domain) can propagate at a velocity other than the equilibrium drift velocity. However, a waveform bounded by two *different* equilibrium conditions (a depletion or accumulation layer) has no such simple constraint on its propagation velocity.

For $c=f$, Eq. (24) becomes

$$\Phi(\mathcal{E},\mathcal{E}_\xi)=-\int_{\mathcal{E}_\alpha}^{\mathcal{E}_\delta} d\mathcal{E}' \frac{v(\mathcal{E}')-c}{\mathfrak{D}(\mathcal{E}')}. \quad (26)$$

The right-hand expression is a constant. The curve on the phase plane, whose points yield this constant in (26), is the trajectory of the system point. Thus (26) implicitly specifies $\mathcal{E}_\xi(\mathcal{E})$, whereupon

$$\xi=\int^{\mathcal{E}} \frac{d\mathcal{E}'}{\mathcal{E}_\xi(\mathcal{E}')} \quad (27)$$

completes the integration of Eq. (17).

The various types of steady traveling waves now can be classified exhaustively. The various possibilities for $c=f$ are shown in Figs. 2 and 3. In each case the top diagram shows the graphical solution of $v(\mathcal{E})=f$. Directly below is the phase-plane trajectory, and on the bottom is the corresponding waveform.

Figure 2 shows the case where the closed locus of $\Phi(\mathcal{E},\mathcal{E}_\xi)=K$ [Eq. (26)] does not include any singular points. The solution is a periodic wave, and is in fact the finite-amplitude generalization of the wave discussed in Eq. (13). Notice that $\mathcal{E}_\xi=0$ for two values of $\mathcal{E}$, $\mathcal{E}=\mathcal{E}_\delta$ and $\mathcal{E}=\mathcal{E}_{\delta'}$. Substituting the latter value into Eq. (26) yields

$$\int_{\mathcal{E}_\delta}^{\mathcal{E}_{\delta'}} d\mathcal{E} \frac{v(\mathcal{E})-c}{\mathfrak{D}(\mathcal{E})}=0, \quad (28)$$

which is a relationship between the extreme field values of the wave. In the special case of $\mathfrak{D}$ not dependent on $\mathcal{E}$, the criterion (28) is equivalent to the statement that the two hatched areas in the figure are equal.[11] In the general case, the vertical elements of area must be weighted by $1/\mathfrak{D}(\mathcal{E})$.

Figure 3 shows three limiting cases of the previous mode, in which one or more singular points are included in the system-point trajectory. External current decreases in the sequence from left to right. For high external currents, the low-field singular point is included in the largest closed trajectory. This evidently corresponds to a high-field domain. As the current is reduced, a critical situation is achieved, in which both outside singular points are included in the trajectory, as shown in column 2 of Fig. 3. In this situation there is no longer a propagating domain, but instead two distinct solutions, $\Gamma_1$ and $\Gamma_2$, corresponding, respectively, to traveling accumulation and depletion layers. Further reduction of external current (column 3 of Fig. 3) leads to a critical closed trajectory including a singular point at maximum field. The field strength within the wave will be lower than that which bounds it: This is a propagating low-field domain.

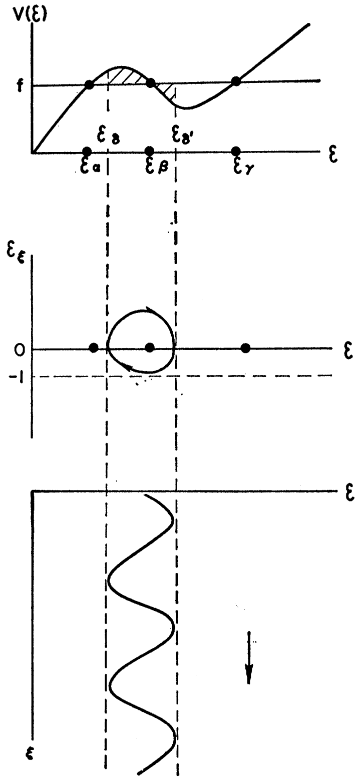


FIG. 2. Velocity versus field characteristic, and a periodic wave solution.

For $c \neq f$, only accumulation- and depletion-layer solutions, comparable to solutions $\Gamma_1$ and $\Gamma_2$ in Fig. 3, remain as possibilities. Section III.C examines the relation between external current and wave velocity for these cases.

### C. Propagation of Layers

Column 2 of Fig. 3 shows particular layer solutions which, because of the criterion of equal weighted areas, propagate at velocity $c=f$, which is the equilibrium drift velocity. If the current $f$ departs from the equal-weighted-area criterion, the propagation velocity will depart from the drift velocity. To develop a criterion

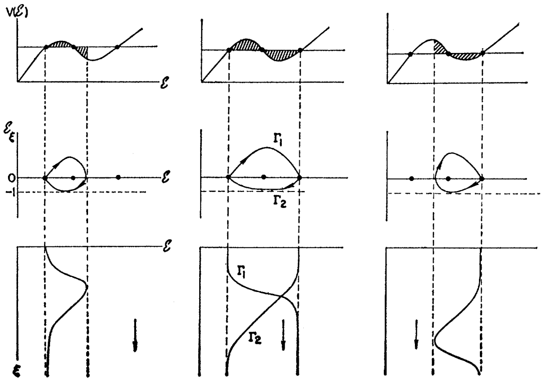

FIG. 3. High-field domain, accumulation and depletion layers, and low-field domain solutions.

for this departure, suppose a layer solution $\mathcal{E}_0(\xi)$ to Eq. (17) is known which propagates at velocity $c_0$ with external current $f_0$. Then a slight alteration in external current

$$f=f_0+f_1 \tag{29}$$

will yield a corresponding small change in waveform

$$\mathcal{E}(\xi)=\mathcal{E}_0(\xi)+\mathcal{E}_1(\xi), \tag{30}$$

and in propagation velocity

$$c=c_0+c_1. \tag{31}$$

Substitution into (17) yields the linearized equation

$$(v(\mathcal{E}_0)-c_0)\mathcal{E}_{1\xi}+(v_{\mathcal{E}}(\mathcal{E}_0)\mathcal{E}_1-c_1)\mathcal{E}_{0\xi}-\mathfrak{D}(\mathcal{E}_0)\mathcal{E}_{1\xi\xi} -\mathfrak{D}_{\mathcal{E}}(\mathcal{E}_0)\mathcal{E}_1\mathcal{E}_{0\xi\xi}=f_1-v_{\mathcal{E}}(\mathcal{E}_0)\mathcal{E}_1 \tag{32}$$

or

$$\left(\frac{d^2}{d\xi^2}-\frac{v-c_0}{\mathfrak{D}}\frac{d}{d\xi}-\frac{v_{\mathcal{E}}(\mathcal{E}_{0\xi}+1)-\mathfrak{D}_{\mathcal{E}}\mathcal{E}_{0\xi\xi}}{\mathfrak{D}}\right)\mathcal{E}_1 =-\frac{\mathcal{E}_{0\xi}c_1+f_1}{\mathfrak{D}}, \tag{33}$$

or even more briefly

$$A\mathcal{E}_1=-(\mathcal{E}_{0\xi}c_1+f_1)/\mathfrak{D}, \tag{34}$$

where $A$ is the linear operator in brackets in (33). If (33) can be utilized to obtain $c_1$ from $f_1$, then information concerning $c(f)$ can be obtained by noting that

$$dc/df=c_1/f_1. \tag{35}$$

Notice that a translated solution to (17) is also a solution. If $\epsilon$ is chosen small and constant, then

$$\mathcal{E}_0(\xi+\epsilon)=\mathcal{E}_0(\xi)+\epsilon\mathcal{E}_{0\xi}(\xi); \tag{36}$$

hence

$$\mathcal{E}_1=\epsilon\mathcal{E}_{0\xi},\ f_1=0,\ c_1=0 \tag{37}$$

solves (32) or (33). Equivalently, (34) gives

$$A\mathcal{E}_{0\xi}=0, \tag{38}$$

a result directly obtainable by differentiating Eq. (17), and which will be important for the remainder of this paper.

Determination of $c_1$ will be carried out in two pieces: First it will be shown that a self-adjoint (Hermitian) problem corresponding to (34) and (38) is solvable in a straightforward way. Then the actual problem will be reduced to a self-adjoint form.

Suppose a second-order differential operation $H$ has the self-adjoint property

$$\int d\xi r(\xi)Hs(\xi)=\int d\xi s(\xi)Hr(\xi) \tag{39}$$

for any well-behaved $r$ and $s$. Suppose further that the eigenfunctions $\psi_m(\xi)$, which satisfy

$$H\psi_m=\lambda_m\psi_m, \tag{40}$$

form a complete set in the sense that $s$ may be expressed as

$$s(\xi)=\sum_m a_m\psi_m(\xi). \tag{41}$$

It is familiar that (41) will hold under very general circumstances, with $a_m$ given by

$$a_m=\frac{\int d\xi\psi_m(\xi)s(\xi)}{\int d\xi(\psi_m(\xi))^2}. \tag{42}$$

The inhomogeneous equation

$$H\phi=s, \tag{43}$$

corresponding to (34) above, has the solution

$$\phi(\xi)=\sum_m\frac{1}{\lambda_m}a_m\psi_m(\xi) \tag{44}$$

as substitution into (43) and use of (40) shows at once. However, an evident difficulty arises in (44) if one of the eigenfunctions $\psi_M$ has a zero eigenvalue,

$$H\psi_M=0, \tag{45}$$

corresponding to (38) above. Then (43) can be solved only if $a_M=0$; hence

$$\int d\xi\,\psi_M(\xi)s(\xi)=0 \tag{46}$$

is a condition $s$ must satisfy, if $\phi$ in (43) exists.

In Eq. (33) it is the $d/d\xi$ term which prevents the operator $A$ from being self-adjoint. This annoyance can be remedied in the following way: Construct the function

$$T(\xi)=\exp\left\{-\frac{1}{2}\int^{\xi}d\xi'\frac{v(\mathcal{E}_0(\xi'))-c}{\mathfrak{D}(\mathcal{E}_0(\xi'))}\right\}; \tag{47}$$

then the transformations

$$\phi = T\mathcal{E}_1, \tag{48}$$

$$s = T(\mathcal{E}_{0\xi}c_1 + f_1)/\mathfrak{D}, \tag{49}$$

and

$$H = TAT^{-1} \tag{50}$$

convert Eq. (34) (upon multiplication by $T$) into Eq. (43). Explicit evaluation of $H\phi = TAT^{-1}\phi$ demonstrates that $H$ is indeed self-adjoint.

The transformation

$$\psi_M = T\mathcal{E}_{0\xi} \tag{51}$$

converts (38) to (45); hence by (46) a solution $\mathcal{E}_1$ exists only if

$$c_1\int_{-\infty}^{\infty} d\xi T^2\frac{(\mathcal{E}_{0\xi})^2}{\mathfrak{D}} + f_1\int_{-\infty}^{\infty} d\xi T^2\frac{\mathcal{E}_{0\xi}}{\mathfrak{D}} = 0. \tag{52}$$

With (35) this gives

$$\frac{dc}{df} = -\frac{\displaystyle\int_{-\infty}^{\infty} d\xi T^2\mathcal{E}_{0\xi}/\mathfrak{D}}{\displaystyle\int_{-\infty}^{\infty} d\xi T^2(\mathcal{E}_{0\xi})^2/\mathfrak{D}}. \tag{53}$$

Now for the *depletion layer* ($\Gamma_2$ in Fig. 2) the slope $\mathcal{E}_{0\xi}$ is negative but less negative than $-1$ [Eq. (21) or Fig. 2]. This gives $[-\mathcal{E}_{0\xi} > (\mathcal{E}_{0\xi})^2]$ everywhere in the integrands of (53); therefore

$$dc/df > 1 \tag{54}$$

for a depletion layer. This says that if the external current is higher than $f_c$, then for the equal-weighted-area (or $c = f_c$) condition, the wave velocity exceeds the equilibrium drift velocity. Similarly, for an *accumulation layer,*

$$dc/df < 0, \tag{55}$$

so that increasing the external current makes the wave velocity of the accumulation layer *decrease.*

### D. Stability of Traveling Waves

If one makes the Galilean transformation to the moving space coordinate

$$\xi = x - ct, \tag{56}$$

which was introduced in Eq. (16), a waveform previously dependent on space only will subsequently depend on time; the elementary differential operators undergo the transformations

$$\partial/\partial x \to \partial/\partial\xi, \tag{57}$$

$$\partial/\partial t \to \partial/\partial t - c(\partial/\partial\xi), \tag{58}$$

which brings Eq. (6) to the form

$$-\mathfrak{D}\mathcal{E}_{\xi\xi} + \mathcal{E}_t + (v-c)\mathcal{E}_\xi = f - v. \tag{59}$$

For a steady waveform traveling at velocity $c$, $\mathcal{E}_t = 0$ above and the equation reduces to (17). This makes the form (59) convenient for studying the evolution of small initial departures from a traveling wave-form. Let

$$\mathcal{E}(\xi,t) = \mathcal{E}_0(\xi) + \mathcal{E}_1(\xi,t), \tag{60}$$

$$f(t) = f_0 + f_1(t). \tag{61}$$

Linearizing, as in Eq. (32), gives

$$\mathcal{E}_{1t} + (v-c)\mathcal{E}_{1\xi} + v_{\mathcal{E}}\mathcal{E}_1\mathcal{E}_{0\xi} - \mathfrak{D}\mathcal{E}_{1\xi\xi} - \mathfrak{D}_{\mathcal{E}}\mathcal{E}_1\mathcal{E}_{0\xi\xi} = f_1 - v_{\mathcal{E}}\mathcal{E}_1, \tag{62}$$

or briefly

$$A\mathcal{E}_1 = -\frac{1}{\mathfrak{D}}\left(f_1 - \frac{\partial}{\partial t}\mathcal{E}_1\right), \tag{63}$$

where $A$ is the operator which first appeared in Eqs. (33) and (34).

The term $f_1(t)$ has been introduced to include external current changes brought about by changes in voltage across the diode. This term will be considered later, but first we investigate the case of fixed external current (or infinite driving impedance) for which $f_1 = 0$, giving

$$A\mathcal{E}_1 = \frac{1}{\mathfrak{D}}\frac{\partial}{\partial t}\mathcal{E}_1. \tag{64}$$

The solution of (64) can be expressed as a superposition of separated functions of the form

$$\mathcal{E}_1(\xi,t) = \mathcal{E}_{1m}(\xi)e^{\lambda_m t}. \tag{65}$$

Substitution into (64) yields

$$A\mathcal{E}_{1m} = (1/\mathfrak{D})\lambda_m\mathcal{E}_{1m}, \tag{66}$$

an eigenvalue equation jointly for the spatial function $\mathcal{E}_{1m}(\xi)$ and for the $m$th time-rate constant $\lambda_m$.

The traveling waveform $\mathcal{E}_0(\xi)$ is stable only if all the eigenvalues of Eq. (66) are nonpositive. This question of signature will now be considered. Again utilizing Eq. (51), and defining

$$\phi_m = T\mathcal{E}_{1m} \tag{67}$$

brings Eq. (66) to the form

$$H\phi_m = \lambda_m(1/\mathfrak{D})\phi_m. \tag{68}$$

Now $H$ can be written

$$H = \partial^2/\partial\xi^2 + h(\xi), \tag{69}$$

and the function $1/\mathfrak{D}(\xi)$ is always positive. These features of Eq. (68) justify the following assertion: The eigenfunctions may be listed in such an order that $\phi_m(\xi)$ has $m$ zeros, starting with $m=0$; and $\lambda_m$ will be increasingly negative with increasing $m$.[16] Notice finally that if

$$\phi_M(\xi)=T(\xi)\mathcal{E}_{0\xi}(\xi), \tag{70}$$

then

$$\lambda_M=0 \tag{71}$$

in Eq. (68) follows easily from the earlier result (38). From the above observations we conclude: (1) If $\mathcal{E}_0(\xi)$ is an accumulation or depletion layer, then $\mathcal{E}_{0\xi}(\xi)$ has no zeros. By (70) $\phi_M(\xi)$ has no zeros, giving $M=0$, and (71) becomes $\lambda_0=0$. Thus all eigenvalues are zero or negative, and a layer solution is stable if the external current is fixed.[17] (2) For a propagating domain, $\mathcal{E}_0(\xi)$ has an extreme electric field value at which $\mathcal{E}_{0\xi}(\xi)=0$. Equation (70) gives a zero to $\phi_M(\xi)$ at that point; thus $M=1$ and $\lambda_1=0$. There must be one eigenfunction $\phi_0(\xi)$ with no zeros and an eigenvalue $\lambda_0>0$. Thus for fixed external current a propagating domain will be unstable.

The positive eigenvalue $\lambda_0$ for a propagating domain will be of subsequent interest, and is evaluated for certain limiting cases in Appendix C. That Appendix also reinforces the following qualitative argument concerning the nature of the instability: If the external current is fixed near (but not at) the critical value $f_c$ of column 2 in Fig. 3, then the front and back faces of the domain will be separated by a long "plateau" (or "valley" for a low-field domain). If the two edges, where $\mathcal{E}_{0\xi}$ is large, are well separated, then in these regions the solution $\mathcal{E}_{10}$ will not differ significantly from the solution $\mathcal{E}_{11}=\mathcal{E}_{0\xi}$, and we write approximately

$$\mathcal{E}_{10}\cong\alpha\mathcal{E}_{0\xi} \quad \text{for the trailing edge}, \tag{72}$$

$$\mathcal{E}_{10}\cong-\beta\mathcal{E}_{0\xi} \quad \text{for front edge}. \tag{73}$$

Here $\alpha$ and $\beta$ have the same sign, and the minus sign appears because $\mathcal{E}_{0\xi}$ crosses zero, while $\mathcal{E}_{10}$ does not. The full field [Eq. (60)] may then be written

$$\mathcal{E}(\xi,t)\cong\mathcal{E}_0(\xi)+\alpha e^{\lambda_0 t}\mathcal{E}_{0\xi}(\xi)\cong\mathcal{E}_0(\xi+\alpha e^{\lambda_0 t}) \quad \text{for the trailing edge}, \tag{74}$$

$$\mathcal{E}(\xi,t)\cong\mathcal{E}_0(\xi-\beta e^{\lambda_0 t}) \quad \text{for front edge}. \tag{75}$$

Thus the effect of the instability is to make the two faces of the domain slide in opposite directions at an exponentiating rate.

### E. Domain Impedance and Interaction with the External Circuit

A space-independent perturbation from equilibrium reduces Eq. (62) to the simple form

$$\mathcal{E}_{1t}=f_1-v_g\mathcal{E}_1. \tag{76}$$

[16] E. L. Ince, *Ordinary Differential Equation* (Longmans-Green and Company, Inc., London, 1927), p. 231.

[17] In another context, the stability of a monotonic front has been analyzed by Y. B. Zeldovich and G. I. Barenblatt, Combust. Flame 3, 61 (1959).

If the ends of the diode in which this occurs are at $a$ and $b$, then the voltage change across the diode is

$$V_1=(b-a)\mathcal{E}_1. \tag{77}$$

If, further, $V_1$ has the time dependence

$$V_1(t)=V_1(0)e^{i\omega t}, \tag{78}$$

then $\mathcal{E}_1$ and $f_1$ will respond sympathetically, and the above equations yield the equilibrium impedance

$$Z_e=V_1/f_1=(b-a)/(v_g+i\omega). \tag{79}$$

This may be written in terms of equivalent parallel resistance and capacitance circuit components as

$$Z_e=\frac{1/C_e}{(1/R_eC_e)+i\omega}, \tag{80}$$

where in our dimensionless units $C_e=(b-a)^{-1}$ is the distributed capacitance, $R_e=(b-a)/v_g$ is the dc resistance, and $(1/R_eC_e)$ is the dielectric relaxation time.

Now a similar analysis may be performed on Eq. (62) in the less trivial case where a propagating domain is present, and departures from (79) can be ascribed to the impedance of the domain itself. The voltage perturbation across the diode is

$$V_1=\int_a^b d\xi\mathcal{E}_1(\xi). \tag{81}$$

We assume that the domain is well removed from both ends of the diode, so that fixing the limits of (81) in the moving coordinate system will do no violence to the physics of the situation. Assume again the oscillating voltage (78); the impedance is still given by

$$Z(\omega)=V_1/f_1, \tag{82}$$

which brings Eq. (63) to the form

$$A\mathcal{E}_1-\frac{i\omega}{\mathfrak{D}}\mathcal{E}_1=-\frac{V_1}{Z}\frac{1}{\mathfrak{D}} \tag{83}$$

or

$$A\mathcal{E}_1(\xi)-\frac{i\omega}{\mathfrak{D}(\xi)}\mathcal{E}_1(\xi)=-\frac{1}{\mathfrak{D}(\xi)}\frac{1}{Z(\omega)}\int_a^b d\xi'\mathcal{E}_1(\xi'). \tag{84}$$

What follows is the determination of $Z(\omega)$ from (84).

Multiplying (83) by $T(\xi)$ brings it to the form

$$H\chi-(i\omega/\mathfrak{D})\chi=(1/\mathfrak{D})s, \tag{85}$$

where

$$\chi=T\mathcal{E}_1, \tag{86}$$

and

$$s(\xi)=-(V_1/Z)T(\xi). \tag{87}$$

Now a familiar generalization of Eqs. (40), (41), and (42) is that if $\phi_m(\xi)$ satisfies (68), then any well-behaved function $s(\xi)$ can be represented as

$$s(\xi)=\sum_m b_m\phi_m(\xi)\,, \tag{88}$$

where

$$b_m=\frac{\int d\xi\frac{1}{\mathfrak{D}(\xi)}\phi_m(\xi)s(\xi)}{\int d\xi\frac{1}{\mathfrak{D}(\xi)}(\phi_m(\xi))^2}\,. \tag{89}$$

Thus a solution to (85) is

$$\chi(\xi)=\sum_m\frac{b_m}{\lambda_m-i\omega}\phi_m(\xi)\,. \tag{90}$$

Multiplying by $T^{-1}$ gives

$$\mathcal{E}_1(\xi)=\sum_m\frac{b_m}{\lambda_m-i\omega}\mathcal{E}_{1m}(\xi); \tag{91}$$

hence

$$V_1=\sum_m\frac{b_m}{\lambda_m-i\omega}\int d\xi\mathcal{E}_{1m}(\xi)\,. \tag{92}$$

Now by (87) and (89),

$$b_m=-\frac{V_1}{Z}\frac{\int d\xi\frac{T^2}{\mathfrak{D}}\mathcal{E}_{1m}}{\int d\xi\frac{T^2}{\mathfrak{D}}(\mathcal{E}_{1m})^2}\,. \tag{93}$$

Upon substitution into (92), $V_1$ cancels, giving finally

$$Z(\omega)=-\sum_m\frac{1}{\lambda_m-i\omega}\frac{\left(\int d\xi\mathcal{E}_{1m}\right)\left(\int d\xi\frac{T^2}{\mathfrak{D}}\mathcal{E}_{1m}\right)}{\int d\xi\frac{T^2}{\mathfrak{D}}(\mathcal{E}_{1m})^2}$$

$$\equiv-\sum_m\frac{u_m}{\lambda_m-i\omega}\,. \tag{94}$$

The physical distortion of a domain in response to a sinusoidal fluctuation of external current can be seen by inspection of (91), which can be written

$$\mathcal{E}_1(\xi)=\frac{b_0}{\lambda_0-i\omega}\mathcal{E}_{10}(\xi)-\frac{b_1}{i\omega}\mathcal{E}_{0\xi}(\xi)+\cdots\,. \tag{95}$$

The same arguments that led to the results (74) and (75) may again be applied, leading to the conclusion that the $m=0$ term above leads to a periodic "breathing" of the domain, while the $m=1$ term leads to a rigid "back-and-forth" motion of the entire domain. This rigid motion is just sufficient to prevent the external current oscillations from building up a charge difference on the ends of the domain, for the $m=1$ contribution to the resulting voltage in (92) vanishes with

$$\int_a^b d\xi\mathcal{E}_{0\xi}=\mathcal{E}_0(\mathrm{b})-\mathcal{E}_0(a)=0\,. \tag{96}$$

As $m$ increases in (91), the higher modes must account for the fluctuations in the field outside the domain, in the quiescent regions of the diode. The sinusoidal form of the eigenfunctions in these regions has already been indicated in Eq. (14), and the eigenvalues have been given by $\beta$ in Eq. (15). If $(b-a)$ is large, there will be many eigenfunctions for which $L$ is large in Eq. (15), so that the corresponding eigenvalues will be well approximated by

$$\lambda_m\cong-v_{\mathcal{E}}\,. \tag{97}$$

The denominators of these terms in (92) and (94) will thus be in agreement with that of (79), and no violence is done if we make the approximation

$$Z(\omega)\cong-u_0/(\lambda_0-i\omega)+Z_e(\omega)\,. \tag{98}$$

Therefore the contribution specific to the domain is

$$Z_d(\omega)=-u_0/(\lambda_0-i\omega)\,. \tag{99}$$

Equation (99) is in the form that one anticipates for the impedance of a variable capacitor whose plate separation at equilibrium is determined by the impressed voltage, and which seeks that equilibrium with a relaxation constant $-\lambda_0$. This corresponds to the physical picture of the dipole domain under conditions of external voltage forcing. However, the lowest eigenvalue $\lambda_0$ is *positive*, and this gives an equivalent picture of a voltage-forced capacitor whose equilibrium is unstable. Another way of viewing Eq. (99) is to re-express it as a parallel resistance-capacitance combination, as was done for Eq. (79). It is seen that the numerator $u_0$ is positive, whereupon the positive eigenvalue $\lambda_0$ leads to an $RC$ combination in which $R$ is *negative*.

The evaluation of the diode impedance equation (94) leads immediately to an understanding of the behavior of a diode containing a domain plus the external circuit [with known impedance $Z_c(\omega)$] to which the diode is connected. Since the ac voltage drop around the entire circuit must sum to zero, the impedance satisfies

$$Z(\omega)+Z_c(\omega)=0\,, \tag{100}$$

where $Z(\omega)$ is given by (94). Equation (100) is to be solved for $\omega$, which is the free-running frequency (generally complex) of the entire system. A simple example illustrates: Use the approximation of Eq. (98) and ignore the capacitative part of the response in $Z_e$. (This is realistic if the time scale is slow compared to that of dielectric relaxation in the diode). Assume further that the external load is purely resistive. The quiescent diode impedance and that of the external load may be lumped together as a single resistance $R$, and Eq. (100) becomes

$$-u_0/(\lambda_0-i\omega)+R=0, \tag{101}$$

giving

$$i\omega=\lambda_0-u_0/R. \tag{102}$$

From (102) we conclude that the propagating domain is stable for small external loads $R<u_0/\lambda_0$ and unstable if this critical load value is exceeded.

Note that the previous result for stability of a domain against fluctuations which do not alter the external current (see Sec. III.D) is recovered for the case of an infinite external impedance.

## IV. CONCLUDING REMARKS

This investigation should be regarded as exploratory; however, its results are encouraging. Since our analysis was completed, Copeland[18] has made available a set of numerical calculations. Though his model differs somewhat from ours, his results show gratifying agreement. We call attention particularly to Fig. 3 of that reference.

We suggest that the methods of this paper might be applied to such questions as (a) more general current models, (b) the input boundary condition, (c) analysis of doping inhomogeneities, and (d) interaction of the external circuit with accumulation and depletion layers, and with periodic waves. In closing we mention that we found in the course of this work a multitude of tantalizing relationships which we were unable to exploit; this suggests that the analytic theory of the Gunn effect is susceptible to extensive further development.

*Note added in proof.* An early paper containing analytic work has recently appeared in translation: V. Bonch-Bruevich, Fiz. Tverd. Tela **8**, 1753 (1966) [English transl.: Soviet Phys.—Solid State **8**, 1397 (1966)].

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## APPENDIX A: HAMILTONIAN, LAGRANGIAN, AND NATURAL SELF-ADJOINT FORMULATIONS

In the case where $c=f$, which leads to the propagating-domain solutions, the traveling-wave equation can be derived from a Lagrangian. This feature is of more than aesthetic interest because a Lagrange equation linearizes to a self-adjoint operator. Thus a natural self-adjoint form of the linearized laws is easily derived.

If $c=f$, Eqs. (18) and (19) become

$$\frac{d\mathcal{E}}{d\xi}=\mathcal{E}_\xi,$$

$$\frac{d\mathcal{E}_\xi}{d\xi}=\left(\frac{v(\mathcal{E})-c}{\mathfrak{D}(\mathcal{E})}\right)(\mathcal{E}_\xi+1), \tag{A1}$$

which possesses the integral invariant

$$\Phi(\mathcal{E},\mathcal{E}_\xi)=\mathcal{E}_\xi-\ln(\mathcal{E}_\xi+1)-\int_{\mathcal{E}_\alpha}^{\mathcal{E}} d\mathcal{E}'\frac{v(\mathcal{E}')-c}{\mathfrak{D}(\mathcal{E}')}. \tag{A2}$$

The mobile charge density

$$n=\mathcal{E}_\xi+1$$

is an alternative variable which brings these relations to the form

$$\Phi=n-1-\ln n-\int_{\mathcal{E}_\alpha}^{\mathcal{E}} d\mathcal{E}'\frac{v-c}{\mathfrak{D}}, \tag{A3}$$

$$d\mathcal{E}/d\xi=n-1, \tag{A4}$$

$$dn/d\xi=[(v-c)/\mathfrak{D}]n. \tag{A5}$$

The further transformation

$$n=e^s \tag{A6}$$

yields

$$\Phi=e^s-1-s-\int_{\mathcal{E}_\alpha}^{\mathcal{E}} d\mathcal{E}'\frac{v-c}{\mathfrak{D}}, \tag{A7}$$

$$d\mathcal{E}/d\xi=e^s-1=\partial\Phi/\partial s, \tag{A8}$$

$$ds/d\xi=(v-c)/\mathfrak{D}=-\partial\Phi/\partial\mathcal{E}. \tag{A9}$$

These equations are in Hamiltonian form, with $\Phi$ playing the role of the Hamiltonian. The corresponding Lagrangian is

$$\begin{aligned}\mathfrak{L}&=s\mathcal{E}_\xi-\Phi\\ &=\mathcal{E}_\xi\ln(\mathcal{E}_\xi+1)-\mathcal{E}_\xi+\ln(\mathcal{E}_\xi+1)-\int_{\mathcal{E}_\alpha}^{\mathcal{E}} d\mathcal{E}'\frac{v-c}{\mathfrak{D}},\end{aligned} \tag{A10}$$

and the Lagrange equation of the system becomes

$$\begin{aligned}0&=\frac{d}{d\xi}\frac{\partial\mathfrak{L}}{\partial\mathcal{E}_\xi}-\frac{\partial\mathfrak{L}}{\partial\mathcal{E}}\\ &=\frac{d}{d\xi}(\ln(\mathcal{E}_\xi+1))-\frac{v-c}{\mathfrak{D}}.\end{aligned} \tag{A11}$$

Performing the differentiation retrieves the original equation for a propagating domain. The useful feature

[18] J. A. Copeland, J. Appl. Phys. **37**, 3602 (1966).

of the above expression is that the nonlinear operator

$$Q(\mathcal{E})=\frac{d}{d\xi}\ln(\mathcal{E}_\xi+1)-\frac{v(\mathcal{E})-c}{\mathfrak{D}(\mathcal{E})} \tag{A12}$$

linearizes at once to

$$Q(\mathcal{E}_0+\mathcal{E}_1)-Q(\mathcal{E}_0)=\left[\frac{d}{d\xi}\frac{1}{\mathcal{E}_{0\xi}+1}\frac{d}{d\xi}-\frac{d}{d\mathcal{E}}\left(\frac{v-c}{\mathfrak{D}}\right)\bigg|_{\mathcal{E}_0}\right]\mathcal{E}_1$$

$$\equiv L\mathcal{E}_1\,, \tag{A13}$$

which gives the linearized operator in manifestly self-adjoint form.

## APPENDIX B: EVALUATION OF $T(\xi)$ AND $dc/df$ FOR DOMAINS

Equation (A11) can be integrated with respect to $\xi$, yielding at once

$$\frac{1}{\mathcal{E}_{0\xi}+1}=\exp\left\{-\int^{\xi}d\xi'\frac{v-c}{\mathfrak{D}}\right\}=T^2, \tag{B1}$$

which gives a simpler expression for $T$ than that in the text.

For a domain, $c=f$ implies $dc/df=1$ which must agree with the general expression (53). For a domain we can prove

$$1=-\int_{-\infty}^{\infty}d\xi\frac{T^2}{\mathfrak{D}}\mathcal{E}_{0\xi}\bigg/\int_{-\infty}^{\infty}d\xi\frac{T^2}{\mathfrak{D}}(\mathcal{E}_{0\xi})^2\,. \tag{B2}$$

Since this is equivalent to

$$\int_{-\infty}^{\infty}d\xi\mathcal{E}_{0\xi}(\mathcal{E}_{0\xi}+1)\frac{T^2}{\mathfrak{D}}=0\,, \tag{B3}$$

Eq. (B2) is seen to be true by using (B1) which reduces the left-hand side above to

$$\int_{-\infty}^{\infty}\frac{\mathcal{E}_{0\xi}d\xi}{\mathfrak{D}}=\oint\frac{d\mathcal{E}}{\mathfrak{D}(\mathcal{E})}\,, \tag{B4}$$

which is manifestly zero.

## APPENDIX C: DOMAINS; EVALUATION OF $\lambda_0$

An expression for $\lambda_0$ valid as $f\to f_c$, $\mathfrak{D}\to 0$ is derived below. Because of the ineffable nature of the coefficients in Eq. (62), a precise evaluation appears difficult. These coefficients have a complicated spatial variation through $\mathcal{E}_0(\xi)$, which in turn is a solution to the nonlinear system (17). Also, the usual maneuver of exploiting a variational principle was unsuccessful, inasmuch as we were unable to construct a usable trial function falling below $\mathcal{E}_{0\xi}$.

The stratagem taken here is to make maximal use

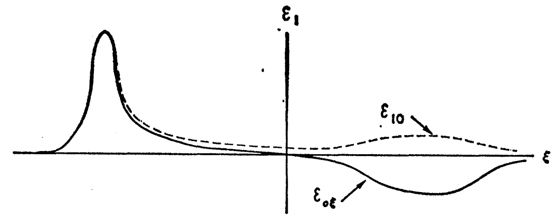


FIG. 4. The eigenfunctions $\mathcal{E}_{10}$ and $\mathcal{E}_{0\xi}$.

of the one known eigenfunction $\mathcal{E}_{0\xi}$ and its eigenvalue, zero. Starting with (A13),

$$L\mathcal{E}_{10}=\lambda_0[1/(\mathcal{E}_{0\xi}+1)\mathfrak{D}]\mathcal{E}_{10}\,, \tag{C1}$$

and

$$L\mathcal{E}_{0\xi}=0\,. \tag{C2}$$

Multiplying the first equation by $\mathcal{E}_{0\xi}$, the second by $\mathcal{E}_{10}$, subtracting, and rearranging,

$$-\lambda_0\frac{1}{(\mathcal{E}_{0\xi}+1)\mathfrak{D}}\mathcal{E}_{10}\mathcal{E}_{0\xi}=\frac{d}{d\xi}\frac{1}{(\mathcal{E}_{0\xi}+1)}\times\left\{\mathcal{E}_{10}\frac{d}{d\xi}\mathcal{E}_{0\xi}-\mathcal{E}_{0\xi}\frac{d}{d\xi}\mathcal{E}_{10}\right\}\,. \tag{C3}$$

Here $\mathcal{E}_{0\xi}$ has only one zero and we set the origin of the coordinates there. Integrating both sides over the range $(-\infty,0)$,

$$\lambda_0=-\mathcal{E}_{10}(0)\mathcal{E}_{0\xi\xi}(0)\bigg/\int_{-\infty}^{0}\frac{\mathcal{E}_{10}\mathcal{E}_{0\xi}}{(\mathcal{E}_{0\xi}+1)\mathfrak{D}}d\xi\,, \tag{C4}$$

since $\mathcal{E}_{10}$, $\mathcal{E}_{0\xi}\to 0$ as $\xi\to-\infty$. Now $\lambda_0$ so evaluated is manifestly positive. For a high-field domain the curvature is negative at the top; and furthermore $\mathcal{E}_{0\xi}$ does not change signature in the range of integration and is positive. The converse holds for low-field domains, but $\mathcal{E}_{0\xi}<0$. Of course, $\mathcal{E}_{10}$ has no zero. $\mathcal{E}_{0\xi\xi}(0)$ can be evaluated directly from (17):

$$\mathcal{E}_{0\xi\xi}(0)=(v(\mathcal{E}_{\delta'})-c)/\mathfrak{D}(\mathcal{E}_{\delta'})\,. \tag{C5}$$

As $f\to f_c$, then $\mathcal{E}_{\delta'}\to\mathcal{E}_\gamma$ and the configuration of the domain approaches that of an accumulation and depletion layer back to back. In this limit, the length increases without bound. Also, $\mathcal{E}_{10}$ becomes localized in the region about each domain edge; i.e., in the negative resistance regions. (See Fig. 4.) The amplitude of $\mathcal{E}_{10}$ drops off exponentially in the interior of the domain. Since the zero in the first-excited state occurs well beyond the "turning point," $\mathcal{E}_{10}$ and $\mathcal{E}_{0\xi}$ can be regarded as "bonding" and "antibonding" combinations of the wave packets localized around each edge. Thus,

$$\mathcal{E}_{10}\xrightarrow[\xi\to-\infty]{}\mathcal{E}_{0\xi}$$
$$\xrightarrow[\xi\to+\infty]{}\alpha|\mathcal{E}_{0\xi}|\,. \tag{C6}$$

The proportionality constant $\alpha$ is required to account for the asymmetry around the origin. $\alpha$ will be determined by the constraint of orthogonality between $\mathcal{E}_{10}$ and $\mathcal{E}_{0\xi}$.

Approximate solutions are easily found in the interior by a WKB technique:

$$\mathcal{E}_{0\xi}\cong\frac{(v(\mathcal{E}_{\delta'})-c)/\mathfrak{D}(\mathcal{E}_{\delta'})}{(q_1(\xi)q_1(0)/(\mathcal{E}_{0\xi}+1))^{1/2}}\times\sinh\int_0^\xi(\mathcal{E}_{0\xi}+1)q_1d\xi', \quad \text{(C7)}$$

$$\mathcal{E}_{10}\cong\frac{A}{(q_0(\xi)/(\mathcal{E}_{0\xi}+1))^{1/2}}\times\cosh\left(\int_0^\xi(\mathcal{E}_{0\xi}+1)q_0d\xi'+\vartheta\right), \quad \text{(C8)}$$

where

$$q_m^2=\frac{d}{d\mathcal{E}}\left(\frac{v-c}{\mathfrak{D}}\right)\bigg|_{\mathcal{E}_0}(\mathcal{E}_{0\xi}+1)+\frac{\lambda_m}{\mathfrak{D}}. \quad \text{(C9)}$$

The matching conditions (C6) determine the amplitude $A$ and phase $\vartheta$. The result is

$$\mathcal{E}_{10}\cong\surd\alpha'\frac{(v(\mathcal{E}_{\delta'})-c)/\mathfrak{D}(\mathcal{E}_{\delta'})}{(q_0(\xi)q_0(0)/(\mathcal{E}_{0\xi}+1))^{1/2}}\times\cosh\left[\int_0^\xi(\mathcal{E}_{0\xi}+1)q_0d\xi'+\tfrac{1}{2}\ln\alpha\right]. \quad \text{(C10)}$$

Advantage has been taken of

$$\lambda_0\ll\frac{d}{d\mathcal{E}}\left(\frac{v-c}{\mathfrak{D}}\right)\bigg|_{\mathcal{E}_0}(\mathcal{E}_{0\xi}+1)\mathfrak{D}, \quad \text{(C11)}$$

so that

$$q_0\cong q_1.$$

Thus,

$$\mathcal{E}_{10}(0)=-\frac{1}{2}\frac{(v(\mathcal{E}_{\delta'})-c)/\mathfrak{D}(\mathcal{E}_{\delta'})}{\left[\frac{d}{d\mathcal{E}}\left(\frac{v-c}{\mathfrak{D}}\right)\Big|_{\mathcal{E}_{\delta'}}\right]^{1/2}}(\alpha+1). \quad \text{(C12)}$$

The integral appearing in the denominator of (C4) can be evaluated as $\mathfrak{D}\to 0$. We do this by introducing a scale factor for the diffusion constant $\mathfrak{D}\to\eta\mathfrak{D}$ and letting $\eta\to 0$. The major contribution to this integral occurs where $\mathcal{E}_{10}\cong\mathcal{E}_{0\xi}$. Therefore we investigate

$$\int_{-\infty}^0\frac{\mathcal{E}_{0\xi}^2}{(\mathcal{E}_{0\xi}+1)\mathfrak{D}}d\xi.$$

Now the slope of the trailing edge of the domain steepens as $\eta\to 0$, and indeed from (23) we see that

$$\mathcal{E}_{0\xi}\cong\frac{1}{\eta}\int_{\mathcal{E}_\alpha}^{\mathcal{E}_0}\left(\frac{v-c}{\mathfrak{D}}\right)d\mathcal{E}\equiv\frac{1}{\eta}u(\mathcal{E}_0), \quad \text{(C13)}$$

since $\mathcal{E}_{0\xi}$ dominates $\ln(\mathcal{E}_{0\xi}+1)$. Consequently,

$$\int_{-\infty}^0\frac{\mathcal{E}_{0\xi}^2}{(\mathcal{E}_{0\xi}+1)\mathfrak{D}}d\xi\cong\int_{-\infty}^0\frac{\mathcal{E}_{0\xi}d\xi}{\mathfrak{D}}=\int_{\mathcal{E}_\alpha}^{\mathcal{E}_{\delta'}}\frac{d\mathcal{E}}{\mathfrak{D}(\mathcal{E})}, \quad \text{(C14)}$$

which calculates the finite jump across the shock discontinuity.

We prove now that $\lim_{\eta\to 0}\alpha=0$. By orthogonality,

$$\alpha\cong\int_{-\infty}^0\frac{\mathcal{E}_{0\xi}^2}{(\mathcal{E}_{0\xi}+1)\mathfrak{D}}d\xi'\bigg/\int_0^\infty\frac{\mathcal{E}_{0\xi}^2}{(\mathcal{E}_{0\xi}+1)\mathfrak{D}}d\xi'. \quad \text{(C15)}$$

The numerator diverges as $\eta^{-1}$; therefore the contention is sustained if the denominator has a dominant divergence. On the leading edge as $\eta\to 0$, the slope approaches $-1$. In fact, from (23)

$$-\ln(\mathcal{E}_{0\xi}+1)\cong(1/\eta)u(\mathcal{E}_0);$$

therefore,

$$\mathcal{E}_{0\xi}\cong-1+e^{-(1/\eta)u(\mathcal{E}_0)}. \quad \text{(C16)}$$

Thus the integral in the denominator of (C15)

$$\int_0^\infty\frac{\mathcal{E}_{0\xi}^2}{(\mathcal{E}_{0\xi}+1)\mathfrak{D}}d\xi'\cong\int_0^\infty e^{(1/\eta)u(\mathcal{E}_0)}d\xi'. \quad \text{(C17)}$$

Using a saddle-point expansion

$$\int_0^\infty e^{(1/\eta)u(\mathcal{E}_0)}d\xi'\cong\left[\frac{2\pi\eta}{\left|\left(\frac{v-c}{\mathfrak{D}}\right)\Big|_{\mathcal{E}_\beta}\right|}\right]^{1/2}e^{(1/\eta)u(\mathcal{E}_\beta)}. \quad \text{(C18)}$$

Therefore $\alpha$ vanishes exponentially with the diffusion constant.

Putting the partial results (C5), (C12), and (C14) together,

$$\lambda_0{}_{f\to f_c,\,\mathfrak{D}\to 0}\cong\frac{1}{2}\left(\frac{v(\mathcal{E}_{\delta'})-c}{\mathfrak{D}(\mathcal{E}_{\delta'})}\right)^2\left[\frac{d}{d\mathcal{E}}\left(\frac{v-c}{\mathfrak{D}}\right)\bigg|_{\mathcal{E}_{\delta'}}\right]^{-1/2}\times\left(\int_{\mathcal{E}_\alpha}^{\mathcal{E}_{\delta'}}\frac{d\mathcal{E}}{\mathfrak{D}(\mathcal{E})}\right)^{-1}. \quad \text{(C19)}$$

## APPENDIX D: LAYERS; EVALUATION OF $c$ IN THE LIMIT $D\to 0$

Equations (53) and (B2) state

$$\frac{dc}{df}=\frac{c_1}{f_1}=-\int_{-\infty}^\infty\frac{\mathcal{E}_{0\xi}}{(\mathcal{E}_{0\xi}+1)}\frac{d\xi}{\mathfrak{D}}\bigg/\int_{-\infty}^\infty\frac{\mathcal{E}_{0\xi}^2}{(\mathcal{E}_{0\xi}+1)}\frac{d\xi}{\mathfrak{D}}. \quad \text{(D1)}$$

For a depletion layer, $c_1\to f_1$ as $D\to 0$. This result follows for $f$ in the neighborhood of $f_c$, for then $\mathcal{E}_{0\xi}$ is given by Eq. (C16) and

$$\mathcal{E}_{0\xi}\underset{\eta\to 0}{\to}-1+e^{-(1/\eta)u(\mathcal{E}_0)}. \quad \text{(D2)}$$

This implies that in the region which contributes most to the integrals

$$\mathcal{E}_{0\xi} \cong -(\mathcal{E}_{0\xi})^2; \quad \text{(D3)}$$

consequently, the numerator and denominator of (D1) are equal in the limit, giving

$$dc/df = 1. \quad \text{(D4)}$$

On the other hand, for the isolated accumulation layer $c_1 \rightarrow 0$. This follows directly from (C13) inasmuch as in the boundary

$$\mathcal{E}_{0\xi} \underset{\eta \to 0}{\longrightarrow} (1/\eta) u(\mathcal{E}_0), \quad \text{(D5)}$$

which becomes indefinitely large as the thickness decreases. The numerator varies as $\eta^{-1}$ and the denominator as $\eta^{-2}$. Therefore $c_1 \sim O(\eta)$ which vanishes in this limit of $\eta \rightarrow 0$, giving

$$dc/df = 0. \quad \text{(D6)}$$

The conclusion is that in the limit of slight diffusion, the depletion layer has a velocity

$$c = f = v(\mathcal{E}_a), \quad \text{(D7)}$$

while the velocity of the accumulation layer is the constant

$$c = f_c, \quad \text{(D8)}$$

independent of $f$.

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# Nature of the ac Transition in the Superconducting Surface Sheath in Pb-2% In

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The response of the superconducting sheath state of Pb–2wt% In to low-frequency (20–500 cps), small-amplitude ($10^{-2}$ Oe $< h_0 <$ 20 Oe) ac fields has been studied. Direct observation of the wave form shows that screening currents are developed in the sheath up to a critical value $J_c$, when the field begins to penetrate. The screening current continues to increase until a saturation value $J_s$ is reached, at which point the current in the sheath stays constant for the remainder of the half-cycle. Fourier analysis of a model wave form for which $J_m = J_c = J_s$ indicates the harmonic content, and nonlinear nature, of the response. Measurements of the real and imaginary parts of the ac permeability, $\mu'$ and $\mu''$, at the fundamental frequency as a function of the ac field amplitude at constant dc field shows that the transition is characterized by only one parameter, dependent on the dc field, and that the model gives an excellent description of the nonlinearity. Well-defined quantitative critical-current data can be obtained for comparison with the theories of Abrikosov, Park, and Fink and Barnes. The relationship between these observations and other measurements of $\mu'$ and $\mu''$ as a function of dc field at constant ac field amplitude is discussed. It is shown that the transition is sensitive to misalignment of the specimen relative to the dc field. The change in the transition as the frequency is changed is explored, and a preliminary conclusion is that $J_s$ is much more sensitive to frequency than $J_c$.

## 1. INTRODUCTION

SUPERCONDUCTORS have been studied by ac magnetic field techniques for a considerable time.[1] We are concerned in this paper with experiments of the type[2] in which a small axial ac magnetic field $h(t) = h_0 \cos\omega t$ with a frequency in the range 20 cps $< \omega <$ 500 cps is superimposed upon a coaxial dc field $H_0$. If $H_0$ is in the range $H_{c2} < H_0 < H_{c3}$ for a type-II superconductor, or in the range $H_c < H_0 < H_{c3}$ for an appropriate type-I superconductor, the specimen is in the superconducting sheath state discussed by St. James and de Gennes[3] and experimentally confirmed by various investigators.[4,5] The response of a specimen in this regime to such an ac field probe has in the past been measured in the form of the in- and out-of-phase permeability components $\mu'$ and $\mu''$ as a function of the dc field $H_0$ at constant ac field amplitude $h_0$ and at the fundamental frequency. Characteristic features[5–10] of the response

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