Sidewise Dispersion Relations and the Axial-Vector Coupling Constant

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The axial-vector renormalization constant is related to low-energy pion-nucleon scattering amplitude by dispersing the axial-vector vertex in the nucleon mass and using the idea of threshold dominance. The low-energy pion-nucleon amplitude, in turn, can be related to the axial-vector renormalization, giving rise to a consistency condition involving $g_A/g_V$ and the pion-nucleon coupling constant. The solution to this consistency condition is in rough agreement with the observed value.

I. INTRODUCTION

RECENTLY, great progress has been made towards explaining the experimental value of the ratio $g_A/g_V$, one of the longstanding puzzles in strong-interaction physics. The well-known sum rule derived by Adler and by Weisberger on the basis of the algebra of currents relates this ratio to a dispersion integral over the pion-nucleon cross section. Although well satisfied by experiment, this sum rule still cannot be considered as a "pure" theoretical determination of $g_A/g_V$, since it requires knowledge of experimental quantities.

In this paper, we present a calculation of the ratio $g_A/g_V$ that needs no experimental information apart from the pion-nucleon coupling constant. Our approach is based on the sidewise dispersion relation for the pion-nucleon vertex first introduced by Bincer, and exploited by Drell and Pagels for calculating various anomalous magnetic moments. Recently, Suura and Simmons have derived an alternative sum rule for the axial-vector renormalization constant, using the sidewise dispersion treatment. Our approach has much in common with that of Suura and Simmons; however, we end up with a consistency equation for $g_A/g_V$, instead of a sum rule. A numerical solution of the equation for $g_A/g_V$ gives the result $g_A/g_V = 1.04$, which, in view of the approximations involved, should be considered a reasonable result.

II. THE WARD IDENTITY AND THE DISPERSION RELATION

The following basic assumptions are made in arriving at the final result.

(a) The usual partially conserved axial-vector current (PCAC) assumption,

$$\partial^\mu A^{(i)} \mu(x) = C \pi^{(i)}(x),$$

where $A^\mu$ is the axial-vector current, $\pi^{(i)}$ is the renormalized pion field, and the index $(i)$ specifies the isospin component. The constant $C$ is given by,

$$C = -2m_\mu g_A/g_v.$$

(b) The pion-nucleon vertex, defined in terms of the pion field of Eq. (1), goes to zero when the external mass of one of the nucleons goes to infinity. The assumption is crucial in what follows, and it gives partial content to Eq. (1), which could otherwise be regarded as a definition.

(c) The following current-field commutation relation is postulated:

$$\delta(x_0 - y_0) \left[ \pi^{(i)}(x), A^{(j)}(y) \right] = -i \left[ \pi^{(j)}(x), A^{(i)}(y) \right].$$

where $\psi^{(i)}$ is the renormalized nucleon field ($\beta$ specifies the isospin component), and $\gamma^\mu$'s are the usual isospin matrices.

(d) The axial-vector charges satisfy the usual $SU(2)\otimes SU(2)$ commutation relations,

$$\delta(x_0 - y_0) \left[ A^{(i)}(x), A^{(j)}(y) \right] = 2i \varepsilon^{ijk} V^{(k)}(x) \delta(x-y),$$

where $V^{(k)}$ is the fourth component of the conserved-vector current, normalized so that $g_v = 1$.

It is well known that the above assumptions are all satisfied in the "$\sigma$" model. 


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Now we define the axial-vector function $\Gamma_\mu(p,p',k)$ by (see Fig. 1),
\[
\langle p,\alpha|T[A_\mu(x)p'(y)]|0\rangle \langle x,\beta|G(p,p',k)\delta(p-p'+k)
\]
\[
=\frac{i}{(2\pi)^{3/2}} \int d^4x d^4y e^{i(k-x-y)p'} \times \langle p,\alpha|T[A_\mu(x)p'(y)]|0\rangle \langle x,\beta|G(p,p',k)\delta(p-p'+k)
\]
and the pion-nucleon vertex by
\[
\frac{(m/p)_0^{(1/2)}}{(m/p')_0^{(1/2)}} a_5(p'(p',p,k)^\delta(p-p'+k)
\]
\[
X(p,p')=\frac{1}{(2\pi)^{3/2}} \int d^4x d^4y e^{i(k-x-y)p'} \times \langle p,\alpha|T[A_\mu(x)p'(y)]|0\rangle \langle x,\beta|G(p,p',k)\delta(p-p'+k)
\]
\[
G=ih\gamma_\mu A^+ + ik\gamma_\nu A^- = ih\gamma_\mu A^+ + ik\gamma_\nu A^- ,
\]
and $\Gamma_\mu$ and $G$ can be decomposed into invariant form factors as follows:
\[
G=ih\gamma_\mu A^+ + ik\gamma_\nu A^- = ih\gamma_\mu A^+ + ik\gamma_\nu A^- ,
\]
where
\[
A^+=(m^2+\gamma p')/2m, \quad A^-=(W^2+\gamma p')/2m, \quad W=\sqrt{s}, \quad s=p^2=(p+k)^2,
\]
and
\[
\Gamma_\mu=(f^+\gamma_\mu A^+ + f\gamma_\sigma A^- \gamma_\mu A^- + g\gamma_\mu A^+ + g\gamma_\sigma A^- \gamma_\mu A^- - t\gamma_\mu A^+ + t\gamma_\sigma A^- \gamma_\mu A^-)
\]
With $p^2$ kept equal to $m^2$, the form factors are functions of $k^2$ and $s=m^2$. By definition,
\[
\begin{align*}
g_\lambda &= f^+_\lambda(0,m^2), \quad g_\mu = \mu^2 f^+\lambda(\mu^2,m^2) .
\end{align*}
\]
From (a), (b), and (c), the following Ward identity can be established:
\[
kT_\mu(p,p',k) = iCG(p,p',k) - g_\lambda(m^2-\gamma p')
\]
or
\[
3m^2+s f^+_\mu - f\gamma_\mu A^+ + f\gamma_\sigma A^- \gamma_\mu A^- = Ck^+,
\]
\[
f^+_\mu - f\gamma_\mu A^+ + f\gamma_\sigma A^- \gamma_\mu A^- = -Ck^- = -2m.
\]
Hereafter, we set $k^2=0$, and consider everything as a function of only $s$. Solving (7) for $f^+_\lambda$ at $k^2=0$, we get
\[
f^+_\lambda(s) = 1 - (C/2m)[k^+(s)-k^-(s)].
\]
By assumption (b), $k^+$ and $k^-$ go to $0$ as $s \to \infty$, and, hence,
\[
f^+_\lambda(s) \to 1 as s \to \infty.
\]
$f^+_\lambda(s)$ is known to be an analytic function of in the complex $s$ plane with a cut from $s=(m+\mu)^2$ to $s=\infty$. By virtue of (9), we have the following dispersion relation:
\[
f^+_\lambda(s) = 1 + \frac{1}{\pi} \int_{m^2}^\infty ds' \frac{\text{Im}[f^+_\lambda(s')]}{s'-s}.
\]
Combining (8) and (5a), one obtains,
\[
\text{Im}[f^+_\lambda(s)] = -(C/2\sqrt{s})[k^+(s)-k^-(s)].
\]
The imaginary part of $k^\pm(s)$ can be related to $J=\frac{3}{2}$ and $s$- and $p$-wave pion-nucleon scattering amplitudes. We define the pion-nucleon scattering amplitude $T^\nu,n,\lambda(p;k,q\lambda)$ (see Fig. 2) by
\[
(m/p)_0^{(1/2)}(m/q)_0^{(1/2)} a_5(p)p(k,q\lambda)\delta(p_k-q_l)
\]
\[
= -i \frac{1}{2(2\pi)^4} \int d^4x d^4y e^{i(k-x-y)p'} \times \langle p,\alpha|T[A_\mu(x)p'(y)]|0\rangle \langle x,\beta|G(p,p',k)\delta(p-p'+k)
\]
\[
\begin{align*}
T^\nu,n,\lambda &= A^\nu,n,\lambda + B^\nu,n,\lambda \gamma_\nu \gamma_\lambda ;
\end{align*}
\]
We shall omit the isospin indices of $T$ in what follows; it will be understood that we are dealing with $I=\frac{1}{2}$ amplitude. The elastic unitarity condition results in the following equations for $\text{Im}(k^\pm)$ (see Fig. 3):
\[
\text{Im}[k^\pm(W^2)] = \rho(W^2)M^\pm(W^2)[k^\pm(W^2)]^*,
\]
where
\[
\rho(W^2) = \frac{1}{W^2} \left[ W^2 - (m-\mu)^2 \right]^{1/2},
\]
and $M^\pm$ is given by
Here, $A_{0,1}$ and $B_{0,1}$ are given as integrals over $\theta$, the scattering angle.

$$
A_0(W^2)=\int_{-1}^1 d\cos\theta \, A(W^2,\cos\theta), \quad A_1(W^2)=\int_{-1}^1 d\cos\theta \, (\cos\theta) A(W^2,\cos\theta),
$$

$$
B_0(W^2)=\int_{-1}^1 d\cos\theta \, B(W^2,\cos\theta), \quad B_1(W^2)=\int_{-1}^1 d\cos\theta \, (\cos\theta) B(W^2,\cos\theta).
$$

In the above expressions, we have taken $\mu=0$ in partial-wave projections, etc., to simplify the kinematics. Since the whole calculation is based on the smallness of the mass of the pion anyway, this simplification is justified.

We now have to calculate the amplitudes $A$ and $B$ defined by Eq. (12). Following Ref. (4), we assume that the main contribution to the integral in (10) comes from the region near the threshold, in which case the threshold or the low-energy behavior of the pion-nucleon amplitude is needed. This can be computed using the low-energy theorems for soft-pion emission.11 Here, we briefly recall Weinberg’s result for the emission of two soft pions. Assumptions (a) and (d), combined with the fact that the commutator $\delta(x_0-\gamma_0)[\partial^2 A, A'(x), A'(y)]$ is small (of the order of $\mu^2$), yield the following identity:

$$
k^{\mu\nu}\int d^4xd'y(\mu^2-k^2)(\mu^2-p) \langle \beta,\gamma| T(A_{\mu}(x),A_{\nu}(y)) \rangle |q,\beta\rangle e^{il\cdot x-i\nu\cdot y} \cong C^2 \int d^4xd'y(\mu^2-k^2)(\mu^2-p) e^{il\cdot x-i\nu\cdot y}

\times \langle \beta,\gamma| T(p,\gamma) |q,\beta\rangle - (2\pi)^4 \delta^{\mu\nu} \langle \beta,\gamma| V_{\mu}(k)(0) |q,\beta\rangle (\mu^2-k^2).
$$

(15)

Now consider the limit of negligibly small pion mass and $s \to m^2$ with $-1 \leq \cos\theta \leq 1$. This clearly corresponds to both $l$ and $k$ vanishing as four-vectors. Therefore, the left-hand side of Eq. (15) goes to zero as $(s-m^2)^2$, if the direct and crossed Born terms corresponding to one-pion exchange, which become singular in this limit, are separated out. The contribution of the Born terms is then given by the pseudovector or the gradient coupling model, and we can state the final result:12 Up to terms quadratic in the threshold factor $(s-m^2)$, the pion-nucleon scattering amplitude near threshold is given by the gradient-coupled direct and crossed Born terms plus the equal-time commutator on the right-hand side of Eq. (15).

Therefore, we have

$$
\hat{U}(p) T^{\mu\nu}(pk; q\nu)(q) = \frac{g^2}{16\pi m^2} \tilde{a}(p) \left[ \gamma^\gamma \cdot k \cdot (p+k) + m \gamma^\gamma L(p\tau\tau)_{\alpha\beta} + \gamma^\gamma L(p\tau\tau)_{\alpha\beta} \right] u(q)

- \frac{ig^2}{8\pi m^2 \nabla} \tau_\alpha \tilde{a}(p) \left[ (1+F) \gamma^\gamma \cdot k \cdot \frac{F}{2m} (q+p) \right] u(q) + \text{higher order terms},
$$

(16)

where $F$ is the anomalous magnetic moment,

$$
\langle \beta,\gamma| V_{\mu}(k)(0) |q,\beta\rangle = \frac{1}{(2\pi)^2 \sqrt{m}} \left( \frac{p+q}{m} \right)^{1/2} \tau_\alpha \tilde{a}(p) f_1 g_{\mu\nu} (p-q) u(q),
$$

and

$$
F = f_3(0)/f_1(0) = f_2(0).
$$

Combining Eqs. (16) and (14), and expanding in powers of the threshold factor $(s-m^2)$, we obtain the following for $M^\pm(s)$:

$$
M^+(s) = \left( g^2/24\pi m^2 \right) (s-m^2) + O[(s-m^2)^2],
$$

$$
M^-(s) = \left( g^2/16\pi m^2 \nabla \right) (s-m^2) + O[(s-m^2)^2].
$$

(17)

On the other hand, Eqs. (10), (11), and (13) yield,

$$
g_A = 1 + \frac{\mu^2 g_A m}{\pi} \int_{(m+p)^2}^\infty \rho(s') s'^{-3/2} \sqrt{s'}

\times [M^+(s')(k^+(s')) - M^-(s')(k^-(s'))].
$$

(18)

The simplest way to evaluate the integral in Eq. (18) is to replace $M^\pm$ by their threshold values given by Eq. (17), and to replace $k^\pm$ by their threshold values, which can be obtained from (6) and (7) at $s=m^2$:

$$
k^+(m^2) = h^+(m^2) \equiv g/\mu^2,
$$

$$
k^-(m^2) = h^-(m^2) \equiv g/\mu^2 A.
$$

(19)

The integral can then be carried out using a suitable cutoff for the upper limit. The answer then turns out

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Fig. 2. Pion-nucleon scattering amplitude.

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Fig. 3. Elastic unitarity.

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to be strongly cutoff-dependent since the integral is quadratically divergent. We overcome this problem by imposing elastic unitarity on the threshold expansions given by (17) and (19). The simplest way to achieve this is to use an effective-range parameterization.\textsuperscript{12} The parametrization that turns out to be most convenient for our purposes is the following:

\[ M^\pm(s) = (s - m^2)[a^\pm - i(s - m^2)\rho(s)]^{-1} \]

which satisfies the unitarity equation,

\[ \text{Im} M^\pm = \rho M^\pm (M^\pm)^* \].

The threshold condition given by (17) fixes \( a^\pm \) as

\[ a^\pm = 24\pi m^2/g^2, \quad a^\pm = 16\pi m^2 g_A^2/g^2 \] \hspace{1cm} (20b)

The unitarity condition (13), together with (20) for \( M^\pm \) and (19) for \( k^\pm(m^2) \), fixes the effective-range formulas for \( k^\pm \) as

\[ k^\pm(s) = k^\pm(m^2)a^\pm/[a^\pm - i(s - m^2)\rho(s)]. \] \hspace{1cm} (21)

Instead of (18), one now has

\[ g_A = 1 + \frac{8m^2 g_A}{9g} \int_{m+\mu}^{\infty} \frac{d\rho(s')}{\sqrt{s'}} \times \left[ \frac{3}{(24\pi m^2/g^2)^2 + (s' - m^2)^2\rho^2(s')} - \frac{2g_A}{16\pi m^2 g_A^2/g^2 + (s' - m^2)^2\rho^2(s')} \right]. \] \hspace{1cm} (22)

While this section, we will discuss the numerical evaluation of Eq. (22).

### III. Numerical Evaluation of the Axial-Vector Coupling Constant

For simplicity, we set \( \mu = 0 \) in (22), which is consistent with our previous approximations, and we also introduce a cutoff for the upper limit to investigate the contribution from the high-energy region. Taking \( g^2/4\pi \approx 15 \), we get, after a change of variable,

\[ g_A = 1 + 0.13g_A \int_0^\Lambda \frac{dx}{(1+x)^{1/2}} \times \left[ \frac{1}{0.16(1+x)+x^2} - \frac{2g_A}{0.071(1+x)g_A^4+x^4} \right], \] \hspace{1cm} (23)

where \( \Lambda \) is a suitable cutoff. The integral in (23) can be done in closed form, yielding a transcendental equation for \( g_A \). Since the equation thus obtained is fairly manageable, it is easier to solve for \( g_A \) by trial and error, using numerical methods. If one uses the ununitarized integrand in Eq. (23), which amounts to setting the term \( x^2 \) in the denominators equal to zero, a cubic equation for \( g_A \) results, which has only one real root for reasonable values of the cutoff parameter \( 0 < \Lambda < 2 \). We hope that this situation also persists in the unitarized equation. When both sides of Eq. (23) are evaluated for \( \Lambda = \infty \) using trial values of \( g_A \), it is found that a consistent solution is obtained for \( g_A = 1.04 \). This value decreases as \( g_A \) approaches about \( g_A = 1.03 \) for \( \Lambda = 2 \). The following remarks are of interest:

(a) The resulting value of \( g_A \) is insensitive to contributions from the high-energy region in Eq. (22), as expected.

(b) The sign of \( (g_A - 1) \) comes out correctly, but the magnitude is too small compared with the experimental value \( g_A - 1 \approx 0.2 \). This shows that the threshold contribution does not completely dominate the dispersion integral.

Using experimental pion-nucleon phase shifts, Suura and Simmons\textsuperscript{6} also get too small a value for \( g_A - 1 \). Since they also neglect inelastic contributions, it would seem that inelastic contributions to the dispersion integral in Eq. (10) need to be included in order to improve the agreement between theory and experiment.

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