Applications of Current-Commutator Sum Rules*

NORMAN H. FUCHS
Department of Physics, Purdue University, Lafayette, Indiana
(Received 12 October 1966; revised manuscript received 12 December 1966)

The intimate relationship between current-commutator sum rules and low-energy scattering of mesons and baryons is demonstrated. The various approximations involved in the derivation are separately discussed. Finally, some comments are made on the pion-pion problem and its relation to the question of continuation in mass, which is basic to the hypothesis of partially conserved axial-vector current.

I. INTRODUCTION

THERE has been considerable interest recently in the implications of the sum rules based on the equal-time commutation relations of Gell-Mann. Several papers have appeared[1–4] which investigate the possible saturation of these sum rules in the approximation of retaining only a few intermediate states. It is found that the sum rules, in general, are not saturated in such an approximation. For example, in the case of the Adler-Weisberger sum rule,[5,6] if only the nucleon and the N^*(1236 MeV) resonance are assumed to saturate the sum rule, then the axial-vector renormalization constant g_A is calculated to be 1.4, which is much larger than the experimental value of 1.18 ± 0.02. In the paper of Cheng and Kim,[7] the saturation of various sum rules is examined by including all the observed hadrons, both particles and resonances, with masses up to about 2 GeV. They find that most of the sum rules are saturated to about 60–70% by this set of hadrons.

The original work of Adler[5] and Weisberger[6] led to a calculation of g_A from a sum rule over pion-nucleon cross sections. Extension of their work to the strangeness-changing current generators led to sum rules over K-nucleon cross sections,[8–10] from which the D/F ratio could be computed. These calculations are based on the experimental total cross sections for pion-nucleon and kaon-nucleon scattering. These cross sections are inserted into the dispersion integrals appearing in the sum rules. The sum rules evaluated in this manner are found to be valid to about 10%.

The relation between these approaches is clear. One assumes that the sum rule is essentially exact, and then one attempts to deduce properties of the hadron spectrum by approximating the sum rules by a finite set of particles.

In an apparently independent line of theoretical investigations, several authors have derived relations between the axial-vector renormalization constant g_A, the D/F ratio, and meson-baryon scattering lengths.[11–14] Furthermore, there have been some results obtained for pion-pion and pion-kaon scattering lengths.[11,12] The assumptions made in these works are basically the same as in the derivation of the various sum rules mentioned above. These are the hypothesis of partially conserved axial-vector current (PCAC), in one or another of its forms,[15,16] and the current-commutation relations of Gell-Mann for the integrated fourth components of the currents.

The relation between these results and the sum-rule relations has not been given a clear exposition as yet. It is the main purpose of this paper to demonstrate the close connection between these results. Finally, we make some comments on the pion-pion problem, which is seen to be a situation qualitatively different from all others previously considered.

II. SCATTERING LENGTHS

The starting point for our investigation is the relations derived previously[5,6,17] which involve pion-nucleon scattering amplitudes, the axial-vector renormalization constant g_A, and nucleon static electromagnetic parameters.

$$f_\pi^2\tilde{A}^{\pi N(+)}\big|_{\nu=t=0}=2Mg_A^2\,, \qquad (1)$$

$$f_\pi^2\left(\frac{\partial\tilde{A}^{\pi N(-)}}{\partial\nu}+\tilde{B}^{(-)}\right)\bigg|_{\nu=t=0}=1-g_A^2\,. \qquad (2)$$

The notation is that of Ref. 17: A and B are the usual invariant amplitudes for πN scattering, f_π is the phenomenological constant appearing in $\pi\mu\nu$ decay, and the tilde denotes that the Born term has been extracted.

We now proceed to apply the relations (1) and (2) to the determination of low-energy pion-nucleon scattering parameters. In contrast to the work of Tomozawa,[11] Hamprecht,[12] and Weinberg,[13] the procedure we use

* Supported by the National Science Foundation.
[1] B. W. Lee, Phys. Rev. Letters **14**, 676 (1965).
[2] R. F. Dashen and M. Gell-Mann, Phys. Letters **17**, 145 (1965).
[3] H. Harari, Phys. Rev. Letters **16**, (1966).
[4] I. S. Gerstein and B. W. Lee, Phys. Rev. Letters, **16**, 1069 (1966); Phys. Rev. **152**, 1418 (1966).
[5] S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965).
[6] W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965).
[7] W. K. Cheng and C. W. Kim, Phys. Rev. (to be published).
[8] C. A. Levinson and I. J. Muzinich, Phys. Rev. Letters **15**, 715 (1965).
[9] D. Amati, C. Bouchiat, and J. Nuyts, Phys. Letters **19**, 59 (1965).
[10] L. K. Pandit and J. Schechter, Phys. Letters **19**, 56 (1965).
[11] Y. Tomozawa, Nuovo Cimento **46**, 707 (1966).
[12] B. Hamprecht, Cambridge University Report, 1966 (unpublished).
[13] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).
[14] A. P. Balachandran, M. G. Gundzik, and F. Nicodemi, Syracuse University Report (unpublished).
[15] J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento **17**, 757 (1960).
[16] S. L. Adler, Phys. Rev. **137**, B1022 (1965).
[17] N. Fuchs, Phys. Rev. **149**, 1145 (1966).

involves a clear separation of approximations. The PCAC hypothesis entails the approximation of a dispersion integral by a single pole term. This approximation was used to derive the basic equations (1) and (2) of this paper. Now we make a further approximation by assuming some reasonably smooth behavior for the scattering amplitudes. The assumption is that, for small values of ν, the amplitudes with the Born terms extracted satisfy

$$\tilde{T}^{(+)}(\nu=\mu)=\tilde{T}^{(+)}(\nu=0)\,, \tag{3}$$

$$\tilde{T}^{(-)}(\nu=\mu)=\mu(\partial\tilde{T}^{(-)}/\partial\nu)(\nu=0)\,, \tag{4}$$

where T denotes either A or B. Since $T^{(+)}$ is an even function of ν and $T^{(-)}$ is an odd function of ν, the correction terms to these approximations are expected to be of order $(\mu/M)^2$ with respect to the dominant term. Of course, the Born terms must be computed exactly, since their variation with ν will not be a slow one, owing to the proximity of the pole to threshold at $\nu=\mu$.

With these assumptions, we find the following values for pion-nucleon scattering lengths:

$$a^{(+)}=\tfrac{1}{3}(2a_3+a_1)=-0.008\text{ F}\,, \tag{5}$$

$$a^{(-)}=\tfrac{1}{3}(a_1-a_3)=+0.075\text{ F}=\frac{\mu}{4\pi f_\pi^2(1+\mu/M)}\,. \tag{6}$$

The value for $a^{(+)}$ is only an order-of-magnitude estimate, since the correction terms referred to above are of the same order. These values are in good agreement with experiment[18]:

$$a_{\text{exp}}^{(+)}=-0.012\pm0.004\text{ F}\,, \tag{7}$$

$$a_{\text{exp}}^{(-)}=+0.097\pm0.007\text{ F}\,. \tag{8}$$

It is clear that analogous results may be derived which involve pion-hyperon scattering by taking matrix elements between Λ, Σ, and Ξ states which correspond to the matrix elements taken between proton states that led to Eqs. (1) and (2). For example, the result for $\pi\Lambda$ scattering corresponding to Eq. (5) would be the order-of-magnitude estimate

$$a_{\pi\Lambda}=+\frac{m_p+m_n}{m_\Lambda+m_\Sigma}\left(\frac{g_A{}^{\Lambda\Sigma}}{g_A}\right)^2 a_{\pi N}{}^{(+)}\cong-0.002\text{ F}\,. \tag{9}$$

Since pion-hyperon scattering lengths are not known experimentally, we do not pursue this further.

III. EXTENSION TO K-NUCLEON SCATTERING

The preceding results were derived under the assumption that the weak hadronic currents obeyed the equal-time commutation relations of $SU(2)\times SU(2)$. If we extend this hypothesis to include the strangeness-changing currents, so that the set of 16 currents obey the equal-time commutation relations of $SU(3)\times SU(3)$, and if we extend the PCAC hypothesis to include the strangeness-changing current as well, then we will obtain further results. The new relations will typically be of the form

$$f_K{}^2\tilde{A}^{Kn(+)}=(m_\Sigma+m_n)(g_A{}^{n\Sigma^-})^2\,, \tag{10}$$

$$f_K{}^2\tilde{A}^{Kp(+)}=(m_\Sigma+m_p)(g_A{}^{p\Sigma^0})^2+(m_\Lambda+m_p)(g_A{}^{p\Lambda})^2\,, \tag{11}$$

$$f_K{}^2\frac{\partial\tilde{A}^{Kn(-)}}{\partial\nu}+\tilde{B}^{Kn(-)}=1-(g_A{}^{n\Sigma^-})^2\,, \tag{12}$$

$$f_K{}^2\frac{\partial\tilde{A}^{Kp(-)}}{\partial\nu}+\tilde{B}^{Kp(-)}=2-(g_A{}^{p\Sigma^0})^2-(g_A{}^{p\Lambda})^2\,, \tag{13}$$

where f_K is the phenomenological constant appearing in $K\mu\nu$ decay, and $f_K=f_\pi$, using the Cabibbo form[19] of the weak hadronic current. The above relations all involve KN and $\bar{K}N$ amplitudes evaluated at $\nu=t=0$. This point is between the cuts in ν, and therefore there is no contradiction with the reality of the right-hand sides of the equations. Now, if we wish to obtain some threshold values for the amplitudes, as we did above for the pion-nucleon amplitudes, we must be more careful. We certainly cannot assume that the amplitudes vary slowly if we continue them towards the $\bar{K}N$ threshold, since there are poles $(Y_0{}^*,Y_1{}^*)$ and cuts $(\Lambda\pi,\Lambda\pi\pi)$ which are encountered in that direction. However, we might try to obtain some KN scattering lengths by continuing in the opposite direction with the approximations analogous to those used in the pion-nucleon case, i.e., Eqs. (3) and (4). We should not expect this to be as good an approximation here as it was previously, since we are neglecting terms of order $(\mu_K/M)^2$ rather than $(\mu/M)^2$; moreover, the $Y_0{}^*$ and $Y_1{}^*$ poles which lie below the $\bar{K}N$ threshold may give the amplitudes a faster variation than we can account for with our crude approximation method. However, if we do proceed in this manner, we find the following values for the KN scattering lengths.

$$4\pi f_K{}^2(1+\mu_K/M)a_1=-2m_K-m_K{}^2\left[\frac{(g_A{}^{p\Sigma^0})^2}{m_p+m_\Sigma-m_K}+\frac{(g_A{}^{p\Lambda})^2}{m_p+m_\Lambda-m_K}\right];$$

$$a_1\simeq-0.46\text{ F}\,; \tag{14}$$

$$4\pi f_K{}^2(1+\mu_K/M)a_0=\left[\frac{(g_A{}^{p\Sigma^0})^2}{m_p+m_\Sigma-m_K}+\frac{(g_A{}^{p\Lambda})^2}{m_p+m_\Lambda-m_K}-\frac{2(g_A{}^{n\Sigma^-})^2}{m_n+m_\Sigma-m_K}\right];$$

$$a_0\simeq+0.04\text{ F}\,. \tag{15}$$

The value for a_0 is an order-of-magnitude estimate, while that for a_1 is expected to be good to about 25%

[18] V. K. Samaranayake and W. S. Woolcock, Phys. Rev. Letters 15, 936 (1965).

[19] N. Cabibbo, lecture notes at Brandeis Summer School, 1965 (unpublished).

(if the PCAC hypothesis were exact). The experimental values[20]

$$a_0=+0.03\pm0.03\,, \tag{16}$$

$$a_1=-0.22\pm0.01\,, \tag{17}$$

show that our estimate of a_0 is good, while our value for a_1 is poor, although the sign is predicted correctly.

Since we do not expect the PCAC hypothesis for the strangeness-changing current to be as good as that for the strangeness-conserving current (the K-meson pole is relatively nearer to the $K\pi\pi$ cut than the π-meson pole is to the 3π cut), we do not interpret this disagreement as being very significant, as it may be due to a combination of the two main approximations which we have made to obtain the scattering lengths. We would expect the relations (10)–(13) to be more accurate, since they do not depend upon the approximation of small variation in ν. These could be tested by computing the amplitudes at the unphysical point $\nu=t=0$ by means of forward dispersion relations. Indeed, this method could be used for the $\bar{K}N$ amplitudes as well. But this, of course, is exactly what is called a sum rule.

IV. REMARKS ON PION-PION SCATTERING PREDICTIONS

The problem of pion-pion scattering is of a different character in the context of this work, since there is no small parameter on which to base an approximation scheme. In other words, the target mass is not much larger than the projectile's mass, as was the case for πN scattering. Nevertheless, several papers have appeared which treat this problem under various assumptions closely analogous to those used for πN scattering. The reason for the discrepancies between the results obtained by the several authors is simply that different initial assumptions are made. In the case of πN scattering, the different forms of the PCAC hypothesis (Adler[16] and Bernstein *et al.*[15]) lead to essentially the same conclusions, because of the slow variation of the matrix elements as a function of the external pion mass. Furthermore, the ambiguity discussed in Ref. 17 does not matter much, since the subtraction constant for the even isotopic-spin πN amplitude is small in any case. Thus, the Adler-Weisberger relation and Adler's consistency condition are not mutually inconsistent whether we apply PCAC to the matrix element of one or two factors of $\partial_\mu A^\mu$ or not. Therefore, no problems (in practice) are encountered in πN-scattering relations. However, for $\pi\pi$ scattering, these difficulties do appear. One does not know the magnitude of the subtraction constant for the even isotopic-spin combination of scattering amplitudes. There is no obvious way to estimate the order of magnitude of errors in the various approximations which are made: this includes both the PCAC hypothesis and the extrapolation from an unphysical point to threshold.

Weinberg assumes an Adler type of PCAC, extrapolates to threshold by using only the terms linear in s, t, and u, and assumes that the equal-time commutator of A_0 and $\partial_\mu A^\mu$ is given by a quark model. We feel that all Weinberg's assumptions seem to be necessary for his results. If we use the pole-dominance form of PCAC, a linear extrapolation, and a zero subtraction constant, we find no consistent solution. If we use the pole-dominance form of PCAC, a linear extrapolation, and Weinberg's assumption on the subtraction constant (i.e., the assumption on the commutator mentioned above), we find two relations:

$$2a_0-5a_2=0.69\text{ F}\,, \tag{18}$$

which agrees with Weinberg[13] and with Meiere and Sugawara,[21] and comes from the Adler-Weisberger–type relation for pion-pion scattering; however,

$$a_2+2a_0=0\,, \tag{19}$$

which gives

$$a_0=+0.06\text{ F}\,, \tag{20}$$

$$a_2=-0.12\text{ F}\,, \tag{21}$$

which is in violent disagreement with these authors' results. On the other hand, if we assume that the constant term in the expansion of $M_{ab,cd}$ about the point $\nu=t=0$ is proportional to $(\delta_{ad}\delta_{bc}+\delta_{bd}\delta_{ac})$ instead of $\delta_{ab}\delta_{cd}$, as Weinberg assumes, then we find

$$a_0=0.35\text{ F}\,, \tag{22}$$

$$a_2=0\,. \tag{23}$$

In short, since the different approximations lead to different results here, unlike the situation in all other cases (πN, $\pi\Lambda$, KN, etc.), we must conclude that we cannot trust any calculation in this framework unless an estimate of the theoretical error entailed by the extrapolation procedures is provided. Such an estimate is found, for example, for the πN case in Weinberg's paper, and for the $\pi\pi$ case in the paper of Meiere and Sugawara.[22]

[20] S. Goldhaber *et al.*, Phys. Rev. Letters **9**, 135 (1962); V. J. Stenger *et al.*, Phys. Rev. **134**, B1111 (1965).

[21] F. T. Meiere and M. Sugawara, Phys. Rev. **153**, 1702 (1967).

[22] F. T. Meiere and M. Sugawara, Phys. Rev. **153**, 1709 (1967).