between $\hat{k}'$ and $\hat{k}_c'$ is given by

$$\tan\phi=\frac{+k_0\sin\theta}{-\gamma_r(k_0\cos\theta-\beta_r W_0)},$$

and

$$\hat{s}=\cos\phi\,\hat{s}_c-\sin\phi\,\hat{k}_c',\quad \hat{k}'=\sin\phi\,\hat{s}_c+\cos\phi\,\hat{k}_c',$$

and

$$\mathbf{P}_r=P\hat{n}\pm P_i(R\cos\phi-(A/\gamma_r)\sin\phi)\hat{s}_c \mp P_i((A/\gamma_r)\cos\phi+R\sin\phi)\hat{k}_c'.$$

Now to make the Lorentz transformation 2: [This transformation is specified by

$$\mathbf{k}_{0_c}+\mathbf{k}_c'=0=\gamma_c[\mathbf{k}_{0_r}-\beta_c(W_{0_r}+M)];$$

therefore,

$$\beta_c=\mathbf{k}_{0_r}/(W_{0_r}+M)].$$

$$\mathbf{P}_c\cdot\hat{n}=P\,,\ \mathbf{P}_c\cdot\hat{s}_c=\pm P_i(R\cos\phi-(A/\gamma_r)\sin\phi),$$

and

$$\mathbf{P}_c\cdot\hat{k}_c=\gamma_c\mathbf{P}_r\cdot\hat{k}_c=\mp\gamma_c P_i((A/\gamma_r)\cos\phi+R\sin\phi).$$

(Recall that $P_r^0=0$.) There is a time like fourth component of the polarization with which we are not concerned. Thus, the three-vector polarization in the second-scattering c.m. system is

$$\mathbf{P}_c=P\hat{n}\pm P_i(R\cos\phi-(A/\gamma_r)\sin\phi)\hat{s}_c \mp\gamma_c P_i((A/\gamma_r)\cos\phi+R\sin\phi)\hat{k}_c.$$

A similar derivation can be made for initial polarization $\mathbf{P}_i=\pm P_i\hat{k}$, yielding

$$\mathbf{P}_c=P\hat{n}\pm P_i(A\cos\phi+(R/\gamma_r)\sin\phi)\hat{s}_c \pm\gamma_c P_i((R/\gamma_r)\cos\phi-A\sin\phi)\hat{k}_c.$$

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Simplified Model for a Three-Particle Regge Trajectory*

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Final-state–interaction theory is generalized to the case of a Reggeon (that is, a particle with mass-dependent spin) which decays into three particles, any two of which can undergo a resonant S-wave interaction. It is shown that the residue functions which describe the coupling of the Reggeon to the three-particle state can be used to calculate the corresponding contribution to the imaginary part of the trajectory function. This contribution is not guaranteed to be positive. The formula also predicts the correct unstable two-particle discontinuity associated with the particle-resonance configuration of the final state.

1. INTRODUCTION

IN a previous paper[1] it was shown how, in an approximate formulation which used unstable-particle amplitudes, the three-body problem could be discussed in terms of Regge trajectory and residue functions. The specifically three-particle effects in the calculation were associated with the corresponding discontinuity of the Regge trajectories.

The analytic properties of the trajectories $\alpha(s)$, of a two-particle system, with center-of-mass energy $s^{1/2}$, are well understood. They are real analytic functions in the s plane cut along the positive real axis from threshold to infinity except for possible singularities arising from the coincidence of two trajectories. Other two-particle channels coupled to the original one do not disturb this picture. In the case of channels with three or more particles even the existence of trajectories has not yet proved. Some progress has been made recently, however, in the analysis of the problem of three-particle unitarity and complex angular momentum.[2–6]

The purpose of this paper is to see to what extent the existence of trajectories can be reconciled with three-particle unitarity or, at least an approximate version of it. It will be assumed, therefore, that trajectories do exist and that their analytic properties are the same as those of two-particle ones.

In Secs. 2 and 3, the model used as a framework for the discussion is described and simplifying assumptions made about the structure of the scattering amplitudes involved. The analytic properties of the three-particle contribution to the imaginary part of an elastic-scat-

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[1] I. T. Drummond, Phys. Rev. **140**, B482 (1965).

[2] Ya. I. Asimov, V. N. Gribov, G. S. Danilov, and I. T. Dyatlov, Yadernaya Fiz. **1**, 941 (1965) [English transl.: Soviet J. Nucl. Phys. **1**, 671 (1965)].

[3] Ya. I. Asimov, A. A. Anselm, V. N. Gribov, G. S. Danilov, and I. T. Dyatlov, Zh. Eksperim. i Teor. Fiz. **48**, 1776 (1965) [English transl.: Soviet Phys.—JETP **21**, 1189 (1965)].

[4] Ya. I. Asimov, A. A. Anselm, V. N. Gribov, G. S. Danilov, and I. T. Dyatlov, Zh. Eksperim. i Teor. Fiz. **49**, 549 (1965) [English transl.: Soviet Phys.—JETP **22**, 383 (1966)].

[5] I. T. Drummond, Phys. Rev. **140**, B1368 (1965).

[6] I. T. Drummond, Phys. Rev. (to be published).

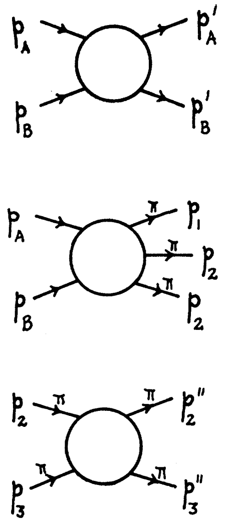

FIG. 1. Reactions considered in the paper.

tering amplitude are discussed in Secs. 4 and 5, the implications for the imaginary part of the Regge trajectory are worked out. From this analysis emerges the importance of calculating the residue functions for the coupling of a Regge pole to the three-particle state.

In Sec. 6 the two-particle discontinuity formula which provides a basis for the dynamical calculation of the residue functions is introduced and discussed. Its expression in terms of partial waves is deduced and the dynamical equation for the residue written down in Sec. 7. The analytic properties of the solutions of the equation are discussed in Sec. 8.

Finally in Sec. 9 it is shown that the resulting three-particle discontinuity formula predicts an unstable two-particle discontinuity which is consistent with the assumptions made.

2. THE MODEL

The model used in this paper has been discussed in two previous ones.[5,6] It involves the elastic scattering of two scalar particles of masses m_A and m_B,

$$A+B \leftrightarrow A+B, \tag{1}$$

and the production of three scalar particles of mass m_π,

$$A+B \leftrightarrow \pi+\pi+\pi. \tag{2}$$

It is assumed that the π mesons are identical. Another reaction of importance is elastic π-π scattering

$$\pi+\pi \leftrightarrow \pi+\pi. \tag{3}$$

These processes are illustrated in Fig. 1. In addition it is assumed that reaction (3) is dominated by an S-wave resonance, the ρ meson of (complex) mass m_ρ. The following reactions therefore, are also of interest:

$$A+B \leftrightarrow \pi+\rho, \tag{4}$$

$$\pi+\rho \leftrightarrow \pi+\rho. \tag{5}$$

The amplitudes describing these processes, which involve the unstable ρ meson, are defined as in Ref. 1, to be certain factors in S-matrix pole residues.

3. STRUCTURE OF THE PRODUCTION AMPLITUDE

The amplitude for reaction (2) is denoted by T. If the various momenta are as shown in Fig. 1, then

$$T=T(s;s_1,s_2,s_3,t_1,t_2,t_3), \tag{6}$$

where the arguments of T are

$$\begin{aligned} s&=(p_A+p_B)^2,\\ s_1&=(p_j+p_k)^2,\\ (i,j,k)&=\text{cyclic permutation of }(1,2,3),\\ t_i&=(p_A-p_i)^2. \end{aligned} \tag{7}$$

It is convenient to define also

$$u_i=(p_B-p_i)^2. \tag{8}$$

The kinematics of reactions (1) and (2) have already been described in detail in a previous paper.[5]

In order to create a theoretical framework for discussing T it is necessary to know something of its analytic structure. The complete structure is necessarily rather complicated since the crossing property of relativistic amplitudes requires T to describe not only reaction (2) but also such processes as

$$A+\pi \leftrightarrow \bar{B}+\pi+\pi.$$

The concern of this paper is with the constraints imposed by unitarity on the residues of Regge trajectories in the direct reaction. No attempt therefore, will be made to satisfy the requirements of crossing. Just sufficient structure will be included in order to guarantee the correct analytic properties, or what is considered a good approximation to them, for the Regge residues.

The nature of the approximation to be made is exemplified by the equations for the π-π scattering amplitude, $A_{\pi\pi}$ used as a basis for the calculation. In Ref. 1 it was supposed that $A_{\pi\pi}$ could be calculated in the Reggeized strip approximation of Chew and Jones.[7] A more drastic approximation scheme due to Cini and Fubini[8] is assumed to hold in this paper. It utilizes only the S-wave absorptive part in the three π-π reactions related by crossing and could be justified on the basis of the more elaborate calculation. Its plausibility depends of course, on the assumed dominance of the resonating S wave.

It is assumed then, that

$$A_{\pi\pi}(t,u,v)=8\pi[F(t)+F(u)+F(v)], \tag{9}$$

[7] G. F. Chew and C. E. Jones, Phys. Rev. 135, B208 (1964).
[8] M. Cini and S. Fubini, Ann. Phys. (N. Y.) 3, 352 (1960).

where

$$F(t)=\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dt'}{t'-t}f(t'),\qquad(10)$$

and

$$t=(p_2+p_3)^2=s_1,$$
$$u=(p_2-p_2'')^2,\quad v=(p_2-p_3'')^2,\qquad(11)$$

the momenta being those of Fig. 1.

The function $F(t)$ is determined by imposing S-wave unitarity on $A_{\pi\pi}$ by means of the N/D method and requiring the input and output absorptive parts to coincide. That is, a bootstrap solution is sought. The S-wave amplitude is

$$A_{\pi\pi}(t)=\frac{1}{16\pi}\int_{-1}^{1}dx\,A_{\pi\pi}(t,u,v),$$

where x is the cosine of the π-π center-of-mass scattering angle. Dynamical equations determining $A_{\pi\pi}(t)$ are

$$A_{\pi\pi}(t)=N(t)/D(t),$$
$$D(t)=1-\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dt'}{t'-t}\rho(t')N(t'),\qquad(12)$$
$$N(t)=B(t)+\int_{4m_\pi^2}^{\infty}dt'\frac{B(t')-B(t)}{t'-t}\rho(t')N(t'),$$

where

$$B(t)=A_{\pi\pi}(t)-F(t),$$

and

$$\rho(t)=[(t-4m_\pi^2)/4t]^{1/2}.\qquad(13)$$

An S-wave resonance or bound state of mass M encountered in reaction (2) contributes a term to T of the form (see Fig. 2)

$$T^{(P)}=-g_{AB}A(s_1,s_2,s_3)/(s-M^2),\qquad(14)$$

where g_{AB} is the ABM coupling constant and $A(s_1,s_2,s_3)$ is the amplitude for the processes

$$\pi+\pi\leftrightarrow\pi+M,\qquad(15)$$

and

$$M\leftrightarrow\pi+\pi+\pi,\qquad(16)$$

if M is unstable. An approximation consistent with the assumptions of the π-π calculation is to put

$$A(s_1,s_2,s_3)=8\pi[G(s_1)+G(s_2)+G(s_3)],\qquad(17)$$

where

$$G(s_1)=\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds_1'}{s_1'-s_1}g(s_1').\qquad(18)$$

For process (15) this is again the Cini-Fubini approximation[8] and for (16) it is the Khuri-Treiman approximation.[9] The function $G(s_1)$ can be determined, to within an over-all normalization, by imposing S-wave

[9] N. N. Khuri and S. B. Treiman, Phys. Rev. **119**, 1115 (1960).

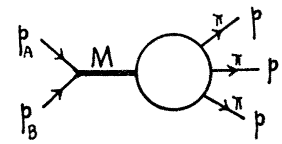

FIG. 2. Pole approximation to the production amplitude.

unitarity on reaction (14) by the N/D method and seeking a bootstrap solution.

If interest were restricted simply to fixed s-plane poles there would be no justification for going beyond Eqs. (14), (17), and (18) in constructing an approximation to T. Since, however, the aim is to accommodate Regge poles in the approximation it is necessary to include a dependence of T on the momentum-transfer variables. This can be achieved by writing

$$T=T_1+T_2+T_3,\qquad(19)$$

where

$$T_i=A(s,s_i,t_i,u_i),\quad i=1,2,3.\qquad(20)$$

Because T_1, for example, depends only on (s_1,t_1,u_1) it describes the production of (π_2,π_3) in a relative S wave. Similar remarks apply to T_2 and T_3.

The existence of Regge poles is guaranteed by requiring the asymptotic behavior for large t_i of each T_i to be governed by terms of the form

$$T_i\cong\Gamma(s,s_i)(t_i^{\alpha(s)}\pm u_i^{\alpha(s)}),\qquad(21)$$

where $\pm$ takes account of the signature of the trajectory. Of necessity the same trajectory controls the asymptotic behavior of all three contributions to T and because of the identity of the π mesons the residue function $\Gamma(s,s_i)$, is the same in each case also.

A more precise statement of the above requirements on T is as follows. It is being assumed that the structure of T_i in the momentum transfer plane may be described in terms of t_i-type and u_i-type singularities. Examples of these would be poles and normal thresholds in t_i and u_i, respectively. Singularities corresponding to other Landau curves will exist but the assumption is that they can still be classified in the manner stated. A simple realization of this situation is the model illustrated in Fig. 3 in which T_1 is given by a sum over what are essentially ladder diagrams. The contributions in the first line produce t_1-type singularities while those in the second produce singularities of u_1 type. This model is basically the same as one discussed by Azimar *et al.*[2] Because of the relations

$$t_i=m_A^2+m_\pi^2-2p_{A0}p_{i0}+2p_Ap_iz_i,$$
$$u_i=m_B^2+m_\pi^2-2p_{A0}p_{i0}-2p_Ap_iz_i,\qquad(22)$$

where

$$z_i=p_A\cdot p_i\qquad(23)$$

the t_i- and u_i-type singularities are reflected in the z_i plane where they may be referred to as singularities of right type and left type. In the limits $(s\pm i\epsilon,\ s_i\pm i\epsilon)$ the example of the box diagram of Fig. 4 which was dis-

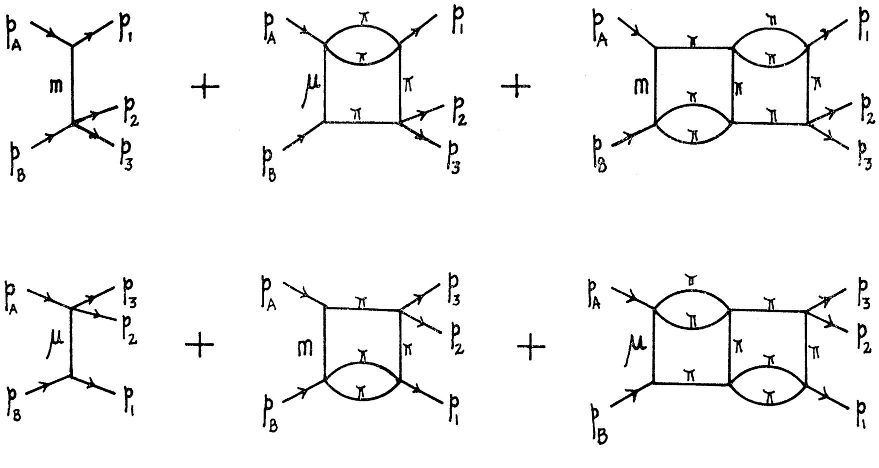

FIG. 3. Model for the production amplitude.

cussed in a previous paper,[6] shows that the right-type singularities may indeed lie on the positive real z_i axis. At least they are expected to remain away from the negative real z_i axis. The same example shows however, in the limits $(s\pm i\epsilon, s_i\mp i\epsilon)$ that the right-type singularities *can* lie on the negative z_i axis. Similar remarks apply *mutatis mutandis* to the left-type singularities.

This classification of singularities permits T_i to be written as

$$T_i=A(s,s_i,z_i)=A^R(s,s_i,z_i)+A^L(s,s_i,z_i), \quad (24)$$

where A^R is a dispersion integral over right-type cuts and A^L the same over left-type cuts. Amplitudes of definite signature may be defined by

$$T_i^{(\pm)}=A^{(\pm)}(s,s_i,z_i)=A^R(s,s_i,z_i) \pm A^L(s,\, s_i,\, -z_i). \quad (25)$$

From the above discussion it follows that these amplitudes do not have singularities on the negative real z_i axis in the limits $(s\pm i\epsilon, s_i\pm i\epsilon)$. The asymptotic behavior of the amplitudes of definite signature is of course, controlled by Regge trajectories of corresponding signature.

Finally it is assumed that the residue function satisfies a subenergy dispersion relation,

$$\Gamma(s,s_i)=\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds_i'}{s_i'-s_i}g(s,s_i'). \quad (26)$$

It turns out that this is consistent with the unitarity equations used.

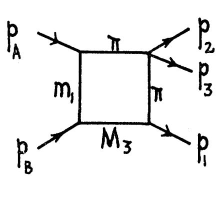

FIG. 4. Box diagram.

4. THE THREE-PARTICLE DISCONTINUITY OF THE ELASTIC-SCATTERING AMPLITUDE

The amplitude for reaction (1) will be denoted by $T_{AB}(s,z)$, where

$$z=\hat{p}_A\cdot\hat{p}_A'. \quad (27)$$

Unitarity and Hermitian analyticity imply that the three-particle discontinuity of T_{AB} is

$$\Delta_{3\pi}T_{AB}(s,z)=i(2\pi)^4\int d\rho(3)T(s_+,s_{1+},s_{2+},s_{3+},z_1,z_2,z_3) \times T(s_-,s_{1-},s_{2-},s_{3-},z_1',z_2',z_3'), \quad (28)$$

where

$$d\rho(3)=\frac{1}{(2\pi)^4}\prod_{i=1}^{3}d^4p_i\,\delta(p_i^2-m_\pi^2) \times\delta^{(4)}(p_1+p_2+p_3-p_A-p_B). \quad (29)$$

The $\pm$ subscripts attached to the energy variables indicate the senses of the limits onto the real axes.

If the elastic scattering amplitude of definite signature $T_{AB}^{(+)}$, is defined in the usual way, then the arguments of Ref. 5 show, when the identity of the π mesons is taken into account, that

$$\Delta_{3\pi}T_{AB}^{(+)}(s,z)=3i(2\pi)^4\int d\rho(3)A^{(+)}(s,s_{1+},z_1) \times T_1^{(+)}(s_-,s_{1-},s_{2-},s_{3-},z_1,z_2,z_3), \quad (30)$$

where

$$T_1^{(+)}(s,s_1,s_2,s_3,z_1,z_2,z_3) =A^{(+)}(s,s_1,z_1)+A^{(+)}(s,\, s_2,\, -z_2) +A^{(+)}(s,\, s_3,\, -z_3). \quad (31)$$

The difference between the situation in this paper and that in Ref. 5 is that in this case it is not immediately clear that the various contributions to the unitarity integral in Eq. (3) have singularities only in the right-half z plane. That they do in the case of the model of Fig. 3 follows from the fact that they are all contribution to the three-particle discontinuity of the scattering amplitude illustrated in Fig. 5 which, at least for restricted ranges of the masses, satisfies the Mandelstam representation.

A simple example of this sort of model has been discussed in Ref. 6 where it is shown that the simplicity of the final result may be due to cancellations occurring within the unitarity integral. Indeed this may be considered the usual state of affairs. The reason can be understood by writing the discontinuity equation in the form

$$T_{AB}^{(+)}(s,z)=\frac{3i}{(2\pi)^5}\int_{4m_\pi^2}^{(s^{1/2}-m_\pi)^2} ds_1\left(\frac{p_1}{4s^{1/2}}\right)\left(\frac{\pi q_2}{s_1^{1/2}}\right) \times\int d\Omega_1\, A_1^{(+)}(s,s_{1+},z_1) \times M_{1,0}^{(+)}(s_-,s_{1-},z_1'), \quad (32)$$

where

$$M_{1,0}^{(+)}(s_-,s_{1-},z_1) =\frac{1}{4\pi}\int d\Omega_{23}\, T_1^{(+)}(s_-,s_{1-},s_{2-},s_{3-},z_1,z_2,z_3), \quad (33)$$

and q_2 is the momentum of π_2 in the (2,3) center-of-mass frame, $d\Omega_{23}$ is the differential solid angle of $\hat{q}_2$ and $d\Omega_1$ is the same for $\hat{p}_1$.

From the discussion of Ref. 6 it can be seen that although the singularities of $M_{1,0}^{(+)}(s,s_1,z_1)$ are by definition all of right type they may actually lie on the negative real z_1 axis. Therefore, the result of performing the $d\Omega_1$ integration in Eq. (32) can produce an integrand which, as a function of z, has singularities on the negative as well as the positive real z axis. That the superposition which results from performing the s_1 integration has singularities only on the latter part must be due to a cancellation. Reference 6 discusses an example of this phenomenon in detail.

5. PARTIAL WAVES

The standard definition of the partial waves of a production amplitude yields

$$T_{1,\Lambda}^{(+)}(J)=\frac{1}{8\pi^2}\int T_1^{(+)}D_{0\Lambda}^J(R)dR, \quad (34)$$

where R is the rotation which carries the initial state axes with z axis along $\hat{p}_A$ into the final-state ones with z axis along $\hat{p}_1$. For $\Lambda=0$, as shown in Ref. 5.

$$T_{1,0}^{(+)}(s;s_1,s_2,s_3) =A^{(+)}(J,s,s_1)+A^{(+)}(J,s,s_2)P_J(-z_{12}) +A(J,s,s_3)P_J(-z_{13}), \quad (35)$$

where

$$A^{(+)}(J,s,s_i)=\frac{1}{2}\int_{-1}^{1} dz_i\, P_J(z_i)A^{(+)}(s,s_i,z_i). \quad (36)$$

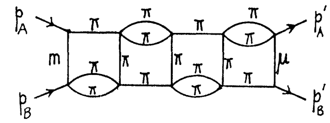

FIG. 5. Model for the elastic amplitude.

The partial-wave version of the discontinuity equation is

$$\Delta_{3\pi}T_{AB}^{(+)}(J,s)=3i(2\pi)^4\int d\bar{p}(3)A^{(+)}(J,s_+,s_{1+}) \times T_{1,0}^{(+)}(J,s_-,s_{1-},s_{2-},s_{3-}), \quad (37)$$

where

$$\int d\bar{p}(3)=\frac{1}{(2\pi)^4}\frac{\pi^2}{4s}\int_{4m_\pi^2}^{(s^{1/2}-m_\pi)^2} ds_1\int_{s_2^{(-)}}^{s_2^{(+)}} ds_2, \quad (38)$$

and

$$s_2^{(\pm)}=2m_\pi^2+(s-s_1-m_\pi^2) \pm\tfrac{1}{2}[(s_1-4m_\pi^2)\lambda(s,s_1,m_\pi^2)/s_1]^{1/2}. \quad (39)$$

The phase-space integral, therefore, is over the Dalitz region shown in Fig. 6. The value of s_1 for which $s_2^{(+)}$ attains a maximum of $(s^{1/2}-m_\pi)^2$ is denoted by s_c and the lower end point $s_2^{(-)}$ attains a minimum of $4m_\pi^2$ when $s_1=(s-m_\pi^2)/2$.

For continuation to arbitrary J, $A^{(+)}(J,s,s_i)$ is calculated from the Froissart-Gribov prescription,

$$A^{(+)}(J,s,s_i)=\frac{1}{\pi i}\int_V dz_1\, Q_J(z_i)A^{(+)}(s,\, s_i\cdot z_i), \quad (40)$$

where V is the contour shown in Fig. 7. Because of the method of construction $A^{(+)}(s,s_i,z_i)$ does not have any singularities on the negative real z_i axis. Equation (40) therefore, yields a continuation which is bounded by $e^{\lambda|J|}$ for large J with $\lambda<\pi$, and the continuation is unique in its asymptotic behavior.

It follows from this equation that

$$A^{(+)}(J,s,s_i)=(p_Ap_i)^JB^{(+)}(J,s,s_i), \quad (41)$$

where the reduced amplitude $B^{(+)}$, is analytic in s_i where

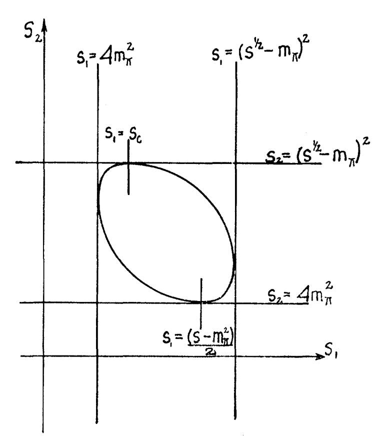

FIG. 6. Dalitz region.

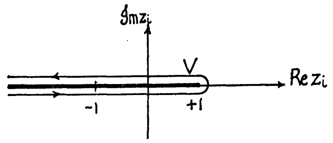

FIG. 7. Contour V for continuing partial-wave amplitudes.

$p_i=0$. Because of the asymptotic behavior assumed in Eq. (21), $B^{(+)}(J,s,s_i)$ has a pole at $J=\alpha(s)$ with residue $8\pi\Gamma(s,s_i)$.

The continuation of $T_{1,0}^{(+)}(J,s,s_1,s_2,s_3)$ is achieved in the manner described in Ref. 5 with the result that Eq. (35) may be used also for arbitrary J. The same is true of the discontinuity Eq. (37) provided, as is also explained in Ref. 5, the flat physical s_2 contour is replaced by a suitably deformed one for $s_1>s_c$. For the term

$$\int d\bar{p}(3)A^{(+)}(J,s_+,s_{1+})P_J(-z_{12})A^{(+)}(J,s_-,s_{2-}) \quad (42)$$

the deformed contour is illustrated in Fig. 8.

A proof of this result may be constructed along lines laid down in Refs. 5 and 6 by writing the discontinuity formula in the form

$$\Delta_{3\pi}T_{AB}^{(+)}(J,s)=\frac{3i}{(2\pi)^5}\int_{4m_\pi^2}^{(s^{1/2}-m_\pi)^2}\frac{\pi^2p_1q_2}{(ss_1)^{1/2}} \times A^{(+)}(J,s_+,s_{1+})M_{1,0}^{(+)}(J,s_-,s_{1-}), \quad (43)$$

where

$$M_{1,0}^{(+)}(J,s,s_1)=\frac{1}{\pi i}\int_V dz_1\,Q_J(z_1)M_{1,0}^{(+)}(s,s_1,z_1) \quad (44)$$

and using the relationship

$$M_{1,0}^{(+)}(J,s,s_1)=\frac{1}{4q_1q_2}\int_{s_2^{(-)}}^{s_2^{(+)}}ds_2\,T_{1,0}^{(+)}(J,s,s_1,s_2,s_3), \quad (45)$$

q_1 being the momentum of π_1 in the (2,3) center-of-mass frame. The s_2 contour in this last equation is straightforward for $s_1<s_c$, and is the deformed one for $s_1>s_c$. As pointed out in Ref. 6 the complicated structure of $M_{1,0}^{(+)}(s,s_1,z_1)$ means that V must be distributed with care and that $M_{1,0}^{(+)}(J,s,s_1)$ can diverge as fast as $e^{\pi|J|}$ for large J. That the integral in Eq. (43) does not is the result of cancellations which occur when the s_1 integration is performed.

An important point to note concerns the behavior of $M_1(J,s,s_1)$ as $s_1\to(s^{1/2}-m_\pi)^2$. Because of its definition by means of Eq. (44)

$$M_{1,0}^{(+)}(J,s,s_1)=(p_Ap_1)^JN_{1,0}^{(+)}(J,s,s_1), \quad (46)$$

where $N_{1,0}^{(+)}(J,s,s_1)$ is analytic in s_1 when $p_1\to 0$. This result could be overthrown by pathological behavior of the z_1-plane singularities of $M_{1,0}^{(+)}(s,s_1,z_1)$. However, the example considered in Ref. 6 suggests that their behavior is sufficiently innocuous to allow Eq. (46) to be true. This has the rather surprising consequence that sufficient cancellations occur when the s_2 integration is performed to permit the result,

$$\int_{s_2^{(-)}}^{s_2^{(+)}}ds_2\,(p_Ap_2)^JB^{(+)}(J,s_-,s_{2-}) \times P_J(-z_{12})\propto(P_Ap_1)^J \quad \text{as} \quad p_1\to 0. \quad (47)$$

6. THREE-PARTICLE DISCONTINUITY OF THE REGGE TRAJECTORY

An approximation to $\Delta_{3\pi}\alpha(s)$ can be obtained by a generalization of a method due to Cheng and Sharp.[10] Both sides of Eq. (37) have poles at $J=\alpha(s_\pm)$. It is convenient to consider the residue at $J=\alpha(s_+)$. The result is, the signature superfix being dropped for simplicity,

$$\gamma_{AB}(s_+)(p_A^2)^{\alpha(s+)} =3i(2\pi)^4\int d\bar{p}(3)(p_Ap_1)^{\alpha(s+)}\Gamma(s_+,s_{1+}) \times T_{1,0}(\alpha(s_+),s_-,s_{1-},s_{2-},s_{3-}), \quad (48)$$

where $8\pi\gamma_{AB}(s)$ is the residue of $T_{AB}(J,s)$ at the pole and of course,

$$T_{1,0}(\alpha(s_+),s_-,s_{1-},s_{2-},s_{3-}) =p_A^{\alpha(s+)}[p_1^{\alpha(s+)}B(\alpha(s_+),s_-,s_{1-}) +p_2^{\alpha(s+)}B(\alpha(s_+),s_-,s_{2-})P_{\alpha(s_+)}(-z_{12}) +p_3^{\alpha(s+)}B(\alpha(s_+),s_-,s_{3-})P_{\alpha(s_+)}(-z_{13})]. \quad (49)$$

Now when $J\cong\alpha(s_-)$

$$B(J,s_-,s_{i-})\cong 8\pi\Gamma(s_-,s_{i-})/(J-\alpha(s))J-\alpha(s_-). \quad (50)$$

But if $\Delta_{3\pi}\alpha(s)$ is small as is expected when s is near threshold then $\alpha(s_+)\cong\alpha(s_-)$ so that

$$B(\alpha(s_+),s_-,s_{i-})\cong 8\pi\Gamma(s_-,s_{i-})/\Delta_{3\pi}\alpha(s). \quad (51)$$

It follows that

$$\gamma_{AB}(s)\Delta_{3\pi}\alpha(s)=12i(2\pi)^5\int d\bar{p}(3)p_1^{\alpha(s)}\Gamma(s_+,s_{1+}) \times G_{1,0}(s_-,s_{1-},s_{2-},s_{3-}), \quad (52)$$

where

$$G_{1,0}(s,s_1,s_2,s_3) =p_1^{\alpha(s)}\Gamma(s,s_1)+p_2^{\alpha(s)}\Gamma(s,s_2)P_{\alpha(s)}(-z_{12}) +p_3^{\alpha(s)}\Gamma(s,s_3)P_{\alpha(s)}(-z_{13}). \quad (53)$$

The + suffix on s has been dropped when s appears as an

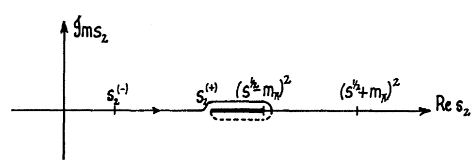

FIG. 8. Distorted s_2 contour, upper end.

[10] H. Cheng and D. Sharp, Ann. Phys. (N. Y.) **22**, 481 (1963).

argument of α or γ_{AB} on the grounds that these functions are essentially real in the approximation being made. Of course, the phase-space contour is the deformed one described in the previous section. The discontinuity can also be expressed in terms of the residue of $N_{1,0}(J,s,s_1)$ which is $8\pi\hat{G}_{1,0}(s,s_1)$, where

$$p_1^{\alpha(s)}G_{1,0}(s,s_1)=\frac{1}{4q_1q_2}\int_{s_2^{(-)}}^{s_2^{(+)}} ds_2 G_{1,0}(s,s_1,s_2,s_3)\,, \quad (54)$$

which explains the notation. That is

$$\gamma_{AB}(s)\Delta_{3\pi}\alpha(s)=\frac{12i}{(2\pi)^4}\int_{4m_\pi^2}^{(s^{1/2}-m_\pi)^2} ds_1\frac{\pi^2 p_1 q_2}{(s,s_1)^{1/2}}p_1^{2\alpha(s)} \times\Gamma(s,s_1)\hat{G}_{1,0}(s,s_1)\,. \quad (55)$$

Clearly $\hat{G}_{1,0}$ is essentially the S-wave projection of $G_{1,0}$ in the s_1 channel.

When $\alpha(s)=0$ and the Regge pole corresponds to an S-wave resonance or bound state, $G_{1,0}$ reduces to the right side of Eq. (17). In order to be able to calculate $\Delta_{3\pi}\alpha(s)$ it is necessary to obtain the amplitudes $\Gamma(s,s_i)$ describing the coupling of the Regge pole to the three-particle state. These amplitudes may be thought of as describing the decay of a Reggeon (a particle of mass-dependent spin) into three particles. The theory for calculating them, which is set out in the following sections is then a generalization of the CFKT approximation [Eqs. (17), (18)] to the decay of an unstable particle of fixed spin.

7. TWO-PARTICLE DISCONTINUITY

The discontinuity of interest is that across the cut beginning at $s_1=4m_\pi^2$. It is given by

$$\Delta_{2\pi}T(s,s_1,s_2,s_3,t_1,t_2,t_3) =i(2\pi)^4\int d\rho(2,3)T(s,s_{1+},s_2'',s_3'',t_1,t_2'',t_3'') \times A_{\pi\pi}(s_{1-},u,v)\,, \quad (56)$$

where $d\rho(2,3)$ is the (π_2,π_3) phase space and

$$\begin{aligned} s_2''&=(p_1+p_3'')^2, & s_3''&=(p_1+p_2'')^2, \\ t_2''&=(p_A-p_2'')^2, & t_3''&=(p_A-p_3'')^2. \end{aligned} \quad (57)$$

Since unitarity has been imposed only on the S wave of π-π scattering attention can be restricted to the implication of this equation for the partially projected amplitudes:

$$M_{1,0}(s,s_1,z_1)=\frac{1}{4\pi}\int d\Omega_{23}T\,. \quad (58)$$

Equation (56) yields

$$\Delta_{2\pi}M_{1,0}(s,s_1,z_1)=2i\rho(s_1)M_{1,0}(s,s_{1+},z_1)A_{\pi\pi}(s_{1-})\,. \quad (59)$$

When both sides of this equation are separated into terms of left type and right type it can be seen that

$$\Delta_{2\pi}M_{1,0}(s,s_1,z_1)=2i\rho(s_1)M_{1,0}^{(+)}(s,s_{1+},z_1)A_{\pi\pi}(s_{1-})\,. \quad (60)$$

Bronzan and Kacser[11] and Bonnevay[12] have pointed out ambiguities associated with the partial-wave projection in Eq. (58). The results of their discussion will be considered in the next section. For the moment the ambiguities may be resolved by requiring the projection to be straightforward when $s_1\cong 4m_\pi^2$. The meaning of the functions for other values of s_1 is determined by continuation from threshold.

Equation (60) implies that

$$\Delta_{2\pi}M_{1,0}^{(+)}(J,s,s_1) =2i\rho(s_1)M_{1,0}^{(+)}(J,s,s_{1+})A_{\pi\pi}(s_{1-})\,. \quad (61)$$

Equating the residues at the pole $J=\alpha(s)$ it follows that

$$\Delta_{2\pi}\hat{G}_{1,0}(s,s_1)=2i\rho(s_1)\hat{G}_{1,0}(s,s_{1+})A_{\pi\pi}(s_{1-})\,. \quad (62)$$

Because of this discontinuity equation

$$\hat{G}_{1,0}(s,s_1)=n_{1,0}(s,s_1)/D(s_1)\,, \quad (63)$$

where $n_{1,0}(s,s_1)$ does not have the subenergy normal threshold cut. The numerator function is given by

$$n_{1,0}(s,s_1)=H_{1,0}(s,s_1)+\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds_1'\frac{H_{1,0}(s,s_1')-H_{1,0}(s,s_1)}{s_1'-s_1} \times\rho(s_i)N(s_1')\,, \quad (64)$$

where

$$H_{1,0}(s,s_1)=\hat{G}_{1,0}(s,s_1)-\Gamma(s,s_1)\,. \quad (65)$$

It follows that $H_{1,0}$ is itself determined by $\Gamma(s,s_i)$. A bootstrap solution is obtained by requiring the input Γ to coincide with that obtained from the output.

It should be noted that the dynamical equations, (62)–(65) are homogeneous and do not determine the normalization of $\Gamma(s,s_i)$. A convenient way of fixing this is to require $\Gamma(s,s_i)$ to have the correct residue $g\gamma(s)/\sqrt{3}$, at the subenergy pole $s_i=m_\rho^2$, where $\gamma(s)$ is the Regge residue obtained from the amplitude for reaction (4) and g is the $\rho\pi\pi$ coupling constant. The factorization property of Regge residues requires that

$$[\gamma(s)]^2=\gamma_{AB}(s)\gamma_{\pi\rho}(s)\,, \quad (66)$$

where $\gamma_{\pi\rho}$ is the residue obtained from the amplitude for reaction (5). Thus

$$\Gamma(s,s_i)=g\gamma(s)\gamma(s,s_i)/\sqrt{3}\,, \quad (67)$$

where $\gamma(s,s_i)$ is a solution of the dynamical equations with unit residue at $s_i=m_\rho^2$. Substituting Eq. (67) into Eq. (52) the result is

$$\Delta_{3\pi}\alpha(s)=4i(2\pi)^5 g^2\gamma_{\pi\rho}(s)\int d\bar{\rho}(3)p_1^{\alpha(s)}\gamma(s_+,s_{1+}) \times g_{1,0}(s_-,s_{1-},s_{2-},s_{3-})\,, \quad (68)$$

[11] J. B. Bronzan and C. Kaczer, Phys. Rev. **132**, 2703 (1963); C. Kaczer, *ibid.* **132**, 2712 (1963).

[12] G. Bonneveay, Nuovo Cimento **30**, 1325 (1963).

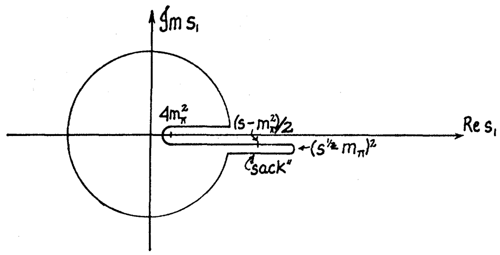

FIG. 9. Cuts in the s_1 plane of $H_{1,0}(s,s_1)$.

where $g_{1,0}$ bears the same relation to $\gamma(s,s_1)$ as $G_{1,0}$ does to Γ. Because of the factorizability property of residues, Eq. (68) makes no reference to γ_{AB}. By making the identification

$$R(\alpha(s),s)=2(2\pi)^5g^2\int d\bar{p}(3)p_1{}^{\alpha(s)}\gamma(s_+,s_{1+})\times g_{1,0}(s_-,s_{1-},s_{2-},s_{3-}),$$

the claim made in a previous paper[1] about $\Delta_{3\pi}\alpha(s)$ can be substantiated.

In the next section the analytic structure of $\hat{G}_{1,0}$ is considered in detail particular regard being paid to the relationship between the function calculated in this section and that which appears on the right side of Eq. (55).

8. THE ANALYTIC STRUCTURE OF $\hat{G}_{1,0}(s,s_1)$

The singularities of $\hat{G}_{1,0}(s,s_1)$ may be deduced from the fact that

$$\hat{G}_{1,0}(s,s_1)=\Gamma(s,s_1)+H_{1,0}(s,s_1). \quad (69)$$

The first term gives rise to a normal threshold cut which runs $4m_\pi{}^2<s_1<\infty$, and the other singularities arise from the second term. Because of the identity of the π mesons this may be expressed as

$$p_1{}^{\alpha(s)}H_{1,0}(s,s_1)=\frac{1}{2q_1q_2}\int_{s_2^{(-)}}^{s_2^{(+)}}ds_2\,p_2{}^{\alpha(s)}\Gamma(s,s_2)P_{\alpha(s)}(-z_{12}). \quad (70)$$

It follows that $H(s,s_1)$ is analytic in the s_1 plane cut along the image of the s_2 normal threshold cut which is generated by the equations

$$s_2{}^{(\pm)}=s_2,\quad 4m_\pi{}^2\leqslant s_2<\infty. \quad (71)$$

This mapping has been studied by Bronzan and Kacser and by Bonnevay.[12] In this case, Eq. (70) is made precise when $s_1\cong 4m_\pi{}^2$ by adding $-i\epsilon$ to both s and s_2 on the right side. Therefore, it is convenient to distribute the cuts as shown in Fig. 9, where they have been drawn, so that analytic continuation from threshold can be effected by moving along the real s_1 axis.

As pointed out in Refs. 11 and 12, when continuation is made to values of $s_1>(s-m_\pi{}^2)/2$ the lower end of the s_2 contour in Eq. (70) is wrapped around the branch point $s_2=4m_\pi{}^2$. This situation is illustrated in Fig. 10(a) (the deformation round $p_2=0$ which affects the upper end is omitted for simplicity). For $s_1>(s^{1/2}-m_\pi)^2$ the contour can run between complex values. The function $\hat{G}_{1,0}$ defined by these carefully arranged s_2 contours is the one calculated in the previous section. The function used in Eq. (55) however, derives originally from $M_{1,0}{}^{(+)}(s,s_1,z_1)$ calculated in Eq. (33) by means of a physical subenergy S-wave projection which utilized flat integration contours even when $s_1>(s-m_\pi{}^2)/2$. The result is that in this case $\hat{G}_{1,0}$ is calculated always with the flat s_2 contour illustrated in Fig. 10(b). The two functions, therefore, while they coincide for $s_1<(s-m_\pi{}^2)/2$ differ when s_1 is greater than this value. It is easy to verify that this difference is just the discontinuity across the cut running below the real axis in Fig. 9. A convenient way of encompassing the difference between the two functions is to permit $\hat{G}_{1,0}$ to be the same in both cases but to require the s_1 contour in Eq. (55) to run along the real axis from threshold to $(s-m_\pi{}^2)/2$ and then to cross the cut as shown in Fig. 9 and lie inside the "sack" up to $(s^{1/2}-m_\pi)^2$.

In the next section essential use is made of this intimate connection between the two definitions of $\hat{G}_{1,0}(s,s_1)$.

9. $\Delta_{3\pi}\alpha(s)$ AND THE π-ρ THRESHOLD

Because of the existence of a π-ρ intermediate state it is to be expected that $\alpha(s)$ has a complex normal threshold at $s=(m_\rho+m_\pi)^2$. This should appear not on the physical sheet but on the unphysical one reached by continuing from physical values into the lower half s plane. It follows that $\Delta_{3\pi}\alpha(s)$ must exhibit this same singularity.

Equation (55) predicts that such a complex normal threshold follows because it is generated when the end point $(s^{1/2}-m_\pi)^2$ coincides with the pole of $\Gamma(s,s_1)$ at $s_1=m_\rho{}^2$. The discontinuity is easily evaluated since the difference between continuing one way and another round this singularity is expressed by a loop integral round the pole. It follows that

$$\gamma_{AB}(s)\Delta_{\pi\rho}\alpha(s)=(-\sqrt{3}/2\pi)\rho(m_\rho{}^2)\times(p_1{}^{2\alpha(s)+1}/s^{1/2})g\gamma(s)\hat{G}_{1,0}(s,m_\rho{}^2). \quad (72)$$

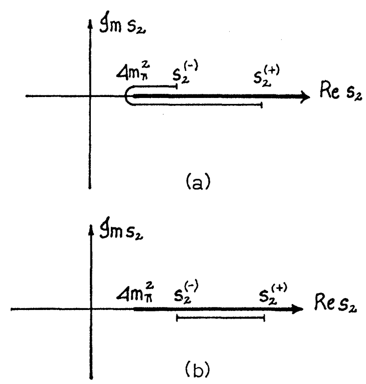

FIG. 10. (a) Distorted s_2 contour, lower end. (b) Undistorted contour.

The problem, therefore, reduces to calculating $G_{1,0}(s,m_\rho^2)$, remembering that the value sought is obtained by continuing from inside the "sack."

It is simplest to evaluate $\hat{G}_{1,0}$, not on the sheet directly of interest but on the one outside the sack which is reached by continuing directly from the underside of the normal threshold cut. If $\hat{G}_{1,0}{}^{\mathrm{II}}$ and $A_{\pi\pi}{}^{\mathrm{II}}$ denote the values of these functions on the upper side of this cut then on the lower side,

$$\hat{G}_{1,0}(s,s_1)=\hat{G}_{1,0}{}^{\mathrm{II}}(s,s_1)/[1-2i\rho(s_1)A_{\pi\pi}{}^{\mathrm{II}}(s_1)]. \quad (73)$$

As $s_1\to m_\rho^2$ both the numerator and denominator behave like poles of known residue and the result is

$$\hat{G}_{1,0}(s,m_\rho^2)=-i(4\pi/\sqrt{3})\gamma(s)/g\rho(m_\rho^2). \quad (74)$$

If this value is substituted into Eq. (72) the result is

$$\Delta_{\pi\rho}\alpha(s)=2i\gamma_{\pi\rho}(s)p_1{}^{2\alpha(s)+1}/s^{1/2}, \quad (75)$$

which is exactly what is expected for the approximate value of a two-particle discontinuity of a Regge trajectory.[10]

It is still to be established that this value is correct even though evaluated on the wrong sheet. This follows from a property of the N/D equations that the N function when evaluated at a pole position has the same value on all sheets. Equation (64) can be written

$$n_{1,0}(s,s_1)=H_{1,0}(s,s_1)D(s_1)+\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds_1'}{s_1'-s_1}H_{1,0}(s,s_1')\rho(s_1')N(s_1'). \quad (76)$$

At a pole position $D(s_1)$ vanishes and the cuts of $H_{1,0}$ cease to influence the value of $n_{1,0}(s,s_1)$. It follows that $n_{1,0}(s,m_\rho^2)$, and hence $\hat{G}_{1,0}(s,m_\rho^2)$ has the same value inside and outside the "sack." This proves the result.

10. CONCLUSION

In this paper it has been shown how to discuss a generalization of the theory of a final-state S-wave interaction to the case where the production amplitude exhibits a Regge-type dependence on momentum transfers. In particular it was possible to obtain a generalization of the CFKT approximation to calculate the residue functions which describe the coupling of the Regge pole to the three-particle state.

It was shown that the three-particle discontinuity of the trajectory function $\alpha(s)$ could be calculated by means of a unitarity-type integral which involved these residue functions. The unitarity-type integral was one of the reorganized kind discussed in a previous paper and involved deformed contours which included points outside the physical region. For these reasons it cannot be concluded that the three-particle contribution to $\mathrm{Im}\alpha(s)$ is positive. Nevertheless, the formula for this contribution is such that it predicts correctly (within the approximation) the unstable two-particle discontinuity of $\alpha(s)$ associated with a particle-resonance configuration of the final state. This last result depends on imposing subenergy unitarity conditions correctly. It suggests that, to the extent that particle-resonance configurations dominate three-particle states, the contribution to $\mathrm{Im}\alpha(s)$ coming from these states will in fact be positive where it is of any size.