Spin-Rotation Parameters and Phase-Shift Ambiguities in π-N Scattering

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It is shown that there are three phase-shift ambiguities that can occur when not all possible observables are available for the phase-shift analysis at a given energy in spin-0–spin-$\frac{1}{2}$ scattering. Two of the ambiguities, change of sign of the phases and exchange of opposite-parity phases, have been recognized for some time. The third ambiguity, which is the combination of the first two, can be resolved only by measurement of the spin-rotation parameter.

I. INTRODUCTION

ALTHOUGH there is agreement in coarse detail among most of the extensive pion-nucleon phase-shift analyses currently available,[1] there is considerable disagreement in finer detail. Further settling of these disagreements will, of course, be obtained by measuring observables (differential cross section, polarization, and spin-rotation parameters[2]) at energies and angles where they have not been measured before. The charge-exchange polarization has been measured only at one angle at one energy[3] and the spin-rotation parameters have never been measured.

It is shown in this work that in an analysis at a single energy there is an ambiguity of partial-wave amplitudes that cannot be resolved unless a spin-rotation parameter is measured. However, the energy dependence revealed by analyses at several energies or by energy-dependent analyses may serve to settle the ambiguity without measurement of a spin-rotation parameter.

II. SCATTERING EQUATIONS

Consider pions scattered by a polarized proton target with polarization $\mathbf{P}_i$. The total cross section σ_T, the differential cross section for the scattering $\sigma(\theta)$, and the polarization of the recoil nucleon $\mathbf{P}(\theta)$ in the c.m. system are given by[4]

$$\sigma_T=(4\pi/k)\,\mathrm{Im}f(0)\,,$$
$$\sigma(\theta)=|f|^2+|g|^2-2\,\mathrm{Im}f^*g(\mathbf{P}_i\cdot\hat{n})\,,$$

and

$$\sigma(\theta)\mathbf{P}(\theta)=-2\,\mathrm{Im}f^*g\hat{n}-2\,\mathrm{Re}f^*g(\hat{n}\times\mathbf{P}_i)+(|f|^2+|g|^2)(\hat{n}\cdot\mathbf{P}_i)\hat{n}-(|f|^2-|g|^2)[\hat{n}\times(\hat{n}\times\mathbf{P}_i)]\,,$$

where $\hat{n}=\mathbf{k}\times(-\mathbf{k}')/|\mathbf{k}\times(-\mathbf{k}')|$, $\mathbf{k}$ is the incident pion c.m. momentum, and $\mathbf{k}'$ is the recoil-nucleon c.m. momentum (thus, the scattered pion c.m. momentum is $-\mathbf{k}'$), and θ is the c.m. scattering angle between incident and scattered pion. The non-spin-flip amplitude is

$$f=f(\theta)=\sum_{j=1/2}^{\infty}(j+\tfrac{1}{2})[A_{j-}P_{j-1/2}+A_{j+}P_{j+1/2}]\,,$$

and the spin-flip amplitude is

$$g=g(\theta)=\sum_{j=1/2}^{\infty}[A_{j-}P_{j-1/2}{}^1-A_{j+}P_{j+1/2}{}^1]\,,$$

where $P_n{}^m=P_n{}^m(\cos\theta)$ is the Legendre functions, $A_{j\mp}$ is the partial-wave amplitude for the state specified by $j=l\pm\frac{1}{2}$, j is the total angular momentum, l is the orbital angular momentum,

$$A=(\eta e^{2i\delta}-1)/2ik$$

in terms of the scattering phase shift δ and the absorption parameter η. Further define $\hat{m}=\hat{n}\times\hat{k}$ and $\hat{s}=\hat{n}\times\hat{k}'$ (see Fig. 1). Thus

$$\hat{n}\times\hat{m}=-\hat{k}\,,$$
$$\hat{m}=-\cos\theta\hat{s}-\sin\theta\hat{k}'\,,$$

and

$$\hat{k}=+\sin\theta\hat{s}-\cos\theta\hat{k}'\,.$$

For $\mathbf{P}_i=0$ (unpolarized target):

$$\sigma(\theta)=|f|^2+|g|^2$$

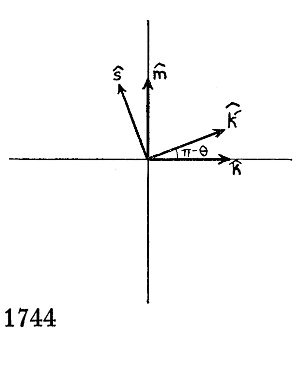

FIG. 1. Unit vectors in the c.m. system.

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[1] For a review of the recent analyses see L. D. Roper, in *Proceedings of the Williamsburg Conference on Intermediate Energy Physics* (The College of William and Mary, Williamsburg, Virginia, 1966), Vol. II, p. 495.

[2] Y. S. Kim, Phys. Rev. **129**, 862 (1963).

[3] R. E. Hill *et al.*, Bull. Am. Phys. Soc. **9**, 410 (1964).

[4] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965).

and $P(\theta)=-2\,\mathrm{Im}f^*g/\sigma(\theta)$ (in the $\hat{n}$ direction). We shall follow convention and call these specific expressions the "differential cross section" and "polarization" observables, respectively.

For $\mathbf{P}_i=\pm P_i\hat{n}$ (target polarized perpendicular to scattering plane):

$$\sigma_\pm(\theta)=|f|^2+|g|^2\mp 2\,\mathrm{Im}f^*gP_i.$$

Thus the polarization observable can be determined directly by measuring the scattering asymmetry. Recent measurements of $P(\theta)$ have been made in this way.[5]

For $\mathbf{P}_i=\pm P_i\hat{k}$ (target polarized along direction of motion):

$$\mathbf{P}(\theta)=P(\theta)\hat{n}\pm A(\theta)P_i\hat{s}\pm R(\theta)P_i\hat{k}',$$

where

$$\sigma(\theta)A(\theta)=2\,\mathrm{Re}f^*g\cos\theta+(|f|^2-|g|^2)\sin\theta,$$
$$\sigma(\theta)R(\theta)=2\,\mathrm{Re}f^*g\sin\theta-(|f|^2-|g|^2)\cos\theta.$$

The quantities $A(\theta)$ and $R(\theta)$ are called the "spin-rotation parameters."

Now, if $\hat{s}$ where the normal to the scattering plane of a second scattering of the recoil nucleon off of a spin-zero nucleus, one could determine $A(\theta)$ by means of the difference in the cross sections of the second scattering when the polarization of the original target is switched from plus to minus. However, similar equations apply for the second scattering as do for the first scattering. These equations describe the situation in the c.m. system of the two scattering particles. So we need the polarization vector in this c.m. system.[6] Thus the polarization vector given above must be expressed in terms of unit vectors $\hat{k}_c'$ and $\hat{s}_c$. The unit vector $\hat{k}_c'$ is the incoming direction of the nucleon at the spin-zero target and $\hat{s}_c$ is the normal to the scattering plane of the second scattering, both in the second-scattering c. m. system. The angle ϕ of rotation from $\hat{k}_c'$ to $\hat{k}'$ (see Fig. 2) is determined by making a Lorentz transformation of the second-scattering momenta from the first-scattering c.m. system to the recoil-nucleon rest system. (See the Appendix.) It is given by

$$\tan\phi=\frac{k_0\sin\theta}{\gamma_r(-k_0\cos\theta+W_0k'/W')},$$

where k_0 = momentum of the spin-zero nucleus in the first-scattering c.m. system, W_0 = corresponding total energy of the spin-zero nucleus, W' = total energy of recoil proton in the first-scattering c.m. system, and $\gamma_r=(1-k'^2/W'^2)^{-1/2}$. Thus the polarization vector in the second-scattering c.m. system is

$$\mathbf{P}(\theta)=P(\theta)\hat{n}\pm P_i[A(\theta)\cos\phi+(R(\theta)/\gamma_r)\sin\phi]\hat{s}_c$$
$$\pm\gamma_cP_i[(R(\theta)/\gamma_r)\cos\phi-A(\theta)\sin\phi]\hat{k}_c'.$$

The γ_r denominators come from the Lorentz transformation of the polarization from the first-scattering c.m. system to the recoil-nucleon rest frame. The γ_c factor comes from the Lorentz transformation of the polarization from the recoil-nucleon rest frame to the second-scattering c.m. system. The details of the derivation and the definition of γ_c are given in the Appendix.

A second scattering of the recoil nucleons off of some spin-zero analyzing target in the plane specified by $\hat{n}$ and $\hat{k}_c'$ yields $A(\theta)\cos\phi+[R(\theta)/\gamma_r]\sin\phi$ by means of the difference in the cross sections of the second scattering when the polarization of the original target is switched from plus to minus.

For $\mathbf{P}_i=\pm P_i\hat{m}$ (target polarized in scattering plane perpendicular to direction of motion):

$$\mathbf{P}(\theta)=P(\theta)\hat{n}\pm P_i[R(\theta)\cos\phi-(A(\theta)/\gamma_r)\sin\phi]\hat{s}_c$$
$$\mp\gamma_cP_i[(A(\theta)/\gamma_r)\cos\phi+R(\theta)\sin\phi]\hat{k}_c'.$$

In the same way as above, a second scattering yields the quantity

$$R(\theta)\cos\phi-(A(\theta)/\gamma_r)\sin\phi.$$

III. PARTIAL-WAVE AMPLITUDE AMBIGUITIES IN THE OBSERVABLES

The object of measuring the values of the various combinations of f and g given above at different energies and angles is to determine the partial-wave amplitudes that comprise f and g. We now investigate the ambiguities that exist in the relationships between the observable combinations of f and g and the partial-wave amplitudes. The relevant combinations written in terms of the partial-wave amplitudes (after much

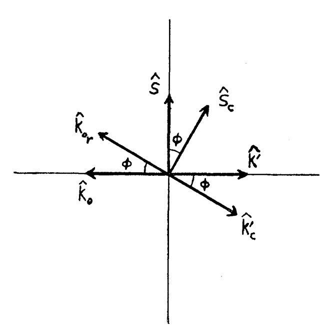

FIG. 2. Unit vectors in all three frames used in the Appendix.

[5] For example, see O. Chamberlain *et al.*, Phys. Letters **7**, 293 (1963).

[6] W. S. C. Williams, *An Introduction to Elementary Particles* (Academic Press Inc., New York, 1961), pp. 364–366; H. P. Stapp, Phys. Rev. **103**, 425 (1956).

algebraic manipulation) are[7]

$$\sigma(\theta)=\sum_{j=1/2}^{\infty}\sum_{j'=1/2}^{\infty}\frac{(j+\frac{1}{2})(j'+\frac{1}{2})}{\sin^2\theta}\{[A_{j-}^*A_{j'-}+A_{j+}^*A_{j'+}-\cos\theta(A_{j-}^*A_{j'+}+A_{j+}^*A_{j'-})]\times(P_{j-1/2}P_{j'-1/2}+P_{j+1/2}P_{j'+1/2})+[A_{j-}^*A_{j'+}+A_{j+}^*A_{j'-}-\cos\theta(A_{j-}^*A_{j'-}+A_{j+}^*A_{j'+})]\times(P_{j-1/2}P_{j'+1/2}+P_{j+1/2}P_{j'-1/2})\},$$

$$i\sigma(\theta)P(\theta)=\sum_{j=1/2}^{\infty}\sum_{j'=1/2}^{\infty}\frac{(j+\frac{1}{2})(j'+\frac{1}{2})}{\sin\theta}[(A_{j-}^*A_{j'+}-A_{j+}^*A_{j'-})(P_{j-1/2}P_{j'-1/2}-P_{j+1/2}P_{j'+1/2})+(A_{j-}^*A_{j'-}-A_{j+}^*A_{j'+})(P_{j-1/2}P_{j'+1/2}-P_{j+1/2}P_{j'-1/2})],$$

$$\sigma(\theta)R(\theta)=\sum_{j=1/2}^{\infty}\sum_{j'=1/2}^{\infty}\frac{(j+\frac{1}{2})(j'+\frac{1}{2})}{\sin^2\theta}\{[A_{j-}^*A_{j'+}+A_{j+}^*A_{j'-}-\cos\theta(A_{j-}^*A_{j'-}+A_{j+}^*A_{j'+})]\times(P_{j-1/2}P_{j'-1/2}+P_{j+1/2}P_{j'+1/2})+A_{j-}^*A_{j'-}+A_{j+}^*A_{j'+}-\cos\theta(A_{j-}^*A_{j'+}+A_{j+}^*A_{j'-})]\times(P_{j-1/2}P_{j'+1/2}+P_{j+1/2}P_{j'-1/2})\},$$

and

$$\sigma(\theta)A(\theta)=\sum_{j=1/2}^{\infty}\sum_{j'=1/2}^{\infty}\frac{(j+\frac{1}{2})(j'+\frac{1}{2})}{\sin\theta}[-(A_{j-}^*A_{j'-}-A_{j+}^*A_{j'+})(P_{j-1/2}P_{j'-1/2}-P_{j+1/2}P_{j'+1/2})-(A_{j-}^*A_{j'+}-A_{j+}^*A_{j'-})(P_{j-1/2}P_{j'+1/2}-P_{j+1/2}P_{j'-1/2})];$$

where we have used

$$P_{j-1/2}{}^1=\frac{j+\frac{1}{2}}{\sin\theta}(\cos\theta P_{j-1/2}-P_{j+1/2})$$

and

$$P_{j+1/2}{}^1=\frac{j+\frac{1}{2}}{\sin\theta}(P_{j-1/2}-\cos\theta P_{j+1/2}).$$

We shall call $\sigma(\theta)$, $P(\theta)$, $R(\theta)$, and $A(\theta)$ "observables."

We consider three physically allowable transformations of the partial-wave amplitudes, and determine which of the four observables are and are not invariant under these transformations. The transformations are:

(1) $A^*\leftrightarrow -A$. (The subscripts are suppressed.) That is $\mathrm{Re}A\to-\mathrm{Re}A$ and $\mathrm{Im}A\to\mathrm{Im}A$ for each partial wave. Since

$$\mathrm{Re}A=\frac{1}{2k}\eta\sin2\delta$$

and

$$\mathrm{Im}A=\frac{1}{2k}(1-\eta\cos2\delta),$$

this transformation is equivalent to $\delta\to-\delta$ and $\eta\to\eta$. This leads to a well-known ambiguity in the differential cross section as shown below.[8]

(2) $A_{j-}\leftrightarrow A_{j+}$. That is, $\delta_{j-}\leftrightarrow\delta_{j+}$ and $\eta_{j-}\leftrightarrow\eta_{j+}$. This leads to the well-known Minami ambiguity as shown below.[7]

(3) $A_{j-}^*\leftrightarrow -A_{j+}$. That is, $\delta_{j-}\leftrightarrow-\delta_{j+}$ and $\eta_{j-}\leftrightarrow\eta_{j+}$. This transformation is just the simultaneous application of the first two. This leads to a third ambiguity as shown below.

The various terms that occur in the expressions for the observables and their behavior under these three transformations are given in the top part of Table I. Upon combining the appropriate terms, one obtains the behavior of the observables under the transformations as given in the bottom part of Table I.

From Table I we see that differential cross sections at a given energy cannot distinguish either of the three transformations. Transformation (1) (ambiguity of phase-shift sign) was distinguished at low energies by utilizing Coulomb interference.[9] At high energies the Coulomb interference is negligible and cannot be used to resolve this ambiguity. It was shown by Minami,[7] as in Table I, that in order to distinguish transformation (2) (ambiguity of opposite-parity states) measurements of the polarization $P(\theta)$ must be made at the energy of concern. Likewise, we see from Table I that in order to distinguish transformation (3) (ambiguity of opposite-parity states and change of phase-shift sign) measurements of the spin-rotation parameter $A(\theta)$ must be made at the energy of concern. The other spin-rotation parameter $R(\theta)$ is, like $\sigma(\theta)$, completely ambiguous with respect to all three transformations. However, in the previous section it was shown that $A(\theta)$ and $R(\theta)$ cannot be measured separately. These ambiguities may not be important when partial-wave amplitudes are determined at several energies; the energy dependence of the amplitudes may be enough to distinguish them, particularly for transformations

[7] S. Minami, Progr. Theoret. Phys. (Kyoto) **11**, 213 (1954).
[8] E. Fermi, Phys. Rev. **91**, 947 (1953).
[9] L. Van Hove, Phys. Rev. **88**, 1358 (1952); J. Orear, *ibid.* **96**, 1417 (1954).

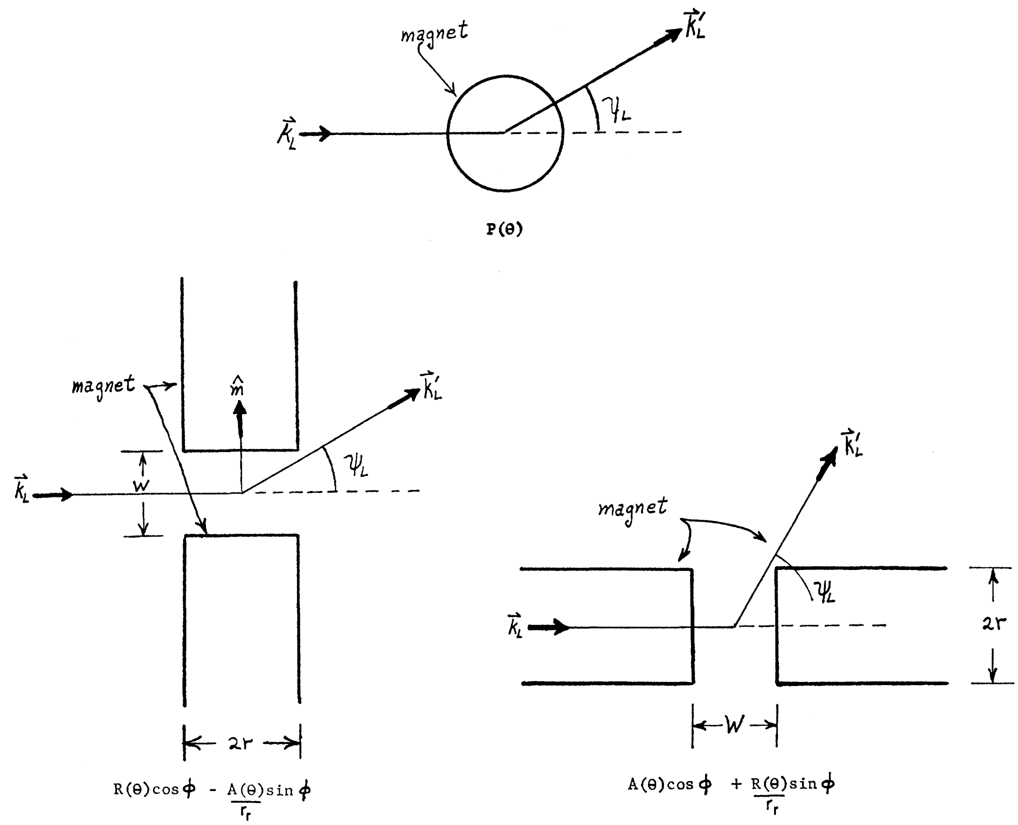

FIG. 3. Laboratory arrangement for measuring observables.

(2) and (3) because of threshold behavior ($\delta \propto k^{2l+1}$), which is always important for some partial waves at any energy.

IV. EXPERIMENTAL CONSIDERATIONS

Figure 3 shows the arrangement of the polarizing magnet poles relative to the original scattering plane for the measurement of $P(\theta)$, $R(\theta)$, and $A(\theta)$. No great obstacle exists in the measurement of $P(\theta)$, and indeed, several measurements of it using a polarized target have been made.[5] In measuring $R(\theta)\cos\phi - [A(\theta)/\gamma_r]\sin\phi$, the magnet-pole faces restrict the lab nucleon-recoil angle ψ_2 to two ranges in the 0–180° total range that is needed: $0° \leq \psi_L \leq \tan^{-1}(\frac{1}{2}w/r)$ and $\tan^{-1}(-\frac{1}{2}w/r) \leq \psi_L \leq 180°$.

TABLE I. Phase-shift ambiguities in pion-nucleon observables.

	Expression	Transformation: (1) $A^* \leftrightarrow -A$ $(j \leftrightarrow j')^a$	(2) $A_{j-} \leftrightarrow A_{j+}$	(3) $A_{j-}^* \leftrightarrow -A_{j+}$ $(j \leftrightarrow j')^a$
(1)	$(A_{j-}^*A_{j'-}+A_{j+}^*A_{j'+})(P_{j-1/2}P_{j'-1/2}+P_{j+1/2}P_{j'+1/2})$	Even	Even	Even
(2)	$(A_{j-}^*A_{j'-}+A_{j+}^*A_{j'+})(P_{j-1/2}P_{j'+1/2}+P_{j+1/2}P_{j'-1/2})$	Even	Even	Even
(3)	$(A_{j-}^*A_{j'+}+A_{j+}^*A_{j'-})(P_{j-1/2}P_{j'-1/2}+P_{j+1/2}P_{j'+1/2})$	Even	Even	Even
(4)	$(A_{j-}^*A_{j'+}+A_{j+}^*A_{j'-})(P_{j-1/2}P_{j'+1/2}+P_{j+1/2}P_{j'-1/2})$	Even	Even	Even
(5)	$(A_{j-}^*A_{j'-}-A_{j+}^*A_{j'+})(P_{j-1/2}P_{j'-1/2}-P_{j+1/2}P_{j'+1/2})$	Even	Odd	Odd
(6)	$(A_{j-}^*A_{j'-}-A_{j+}^*A_{j'+})(P_{j-1/2}P_{j'+1/2}-P_{j+1/2}P_{j'-1/2})$	Odd	Odd	Odd
(7)	$(A_{j-}^*A_{j'+}-A_{j+}^*A_{j'-})(P_{j-1/2}P_{j'-1/2}-P_{j+1/2}P_{j'+1/2})$	Odd	Odd	Even
(8)	$(A_{j-}^*A_{j'+}-A_{j+}^*A_{j'-})(P_{j-1/2}P_{j'+1/2}-P_{j+1/2}P_{j'-1/2})$	Even	Odd	Odd
	Observables			
	$\sigma(\theta)[\propto (1), (2), (3), (4)]$	Even	Even	Even
	$P(\theta)[\propto (6), (7)]$	Odd	Odd	Even
	$R(\theta)[\propto (1), (2), (3), (4)]$	Even	Even	Even
	$A(\theta)[\propto (5), (8)]$	Even	Odd	Odd

[a] Change of dummy sum indices.

The corresponding restrictions that this puts on the c.m. scattering angle θ is a function of the energy of the incoming pion. In order to measure $A(\theta)\cos\phi+[R(\theta)/\gamma_r]\sin\phi$, one magnet-pole piece must have a hole through the middle of it to admit the incoming pions. The advent of superconducting magnets has made this experiment feasible.[10] Again, the magnet pole faces restrict the laboratory nucleon-recoil angle to

$$\tan^{-1}(r/\tfrac{1}{2}w)\leq\psi_L\leq\tan^{-1}(-r/\tfrac{1}{2}w).$$

V. CONCLUSION

The resolution of disagreements among current phase-shift sets in pion-nucleon scattering or any spin-0–spin-$\frac{1}{2}$ scattering will undoubtedly require measurement of the spin-rotation parameters. Indeed, we see above that in an analysis at a single energy there is an ambiguity in determining the partial-wave amplitudes that can be resolved only by measuring a spin-rotation parameter. However, required energy dependences may resolve the ambiguity.

ACKNOWLEDGMENTS

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APPENDIX

We derive here the polarization in the second-scattering c.m. system in terms of the polarization in the first-scattering c.m. system. We do it by first Lorentz transforming the polarization from the first-scattering c.m. system to the recoil-nucleon rest system, followed by a Lorentz transformation from the recoil-nucleon rest system to the second-scattering c.m. system.[6]

Define: $\mathbf{P}$ is the three-vector polarization in the first-scattering c.m. system; P^0 is the fourth component of the polarization in the first-scattering c.m. system; W' and $\mathbf{k}'$ are the recoil-nucleon total energy and momentum in the first-scattering c.m. system; $\boldsymbol{\beta}_r=\mathbf{k}'/W'$ and is the frame velocity in the first-scattering c.m. system of the recoil-nucleon rest frame; $\gamma_r=(1-\beta_r^2)^{-1/2}$; $\mathbf{P}_r$ is the three-vector polarization in the recoil-nucleon rest system; $P_r^0=0$ and is the fourth component of the polarization in the recoil-nucleon rest system; W_0 and $\mathbf{k}_0$ are the second-scattering spin-zero nucleus total energy and momentum in the first-scattering c.m. system; W_{0_r} and $\mathbf{k}_{0_r}$ are the second-scattering spin-zero nucleus total energy and momentum in the recoil-nucleon rest system; $\boldsymbol{\beta}_c=\mathbf{k}_{0_r}/(W_{0_r}+M)$ and is frame velocity in the recoil-nucleon rest system of the second-scattering c.m. frame; $\gamma_c=(1-\beta_c^2)^{-1/2}$, M and is nucleon rest mass; $\mathbf{k}_{0_c}$ is the second-scattering spin-zero nucleus momentum in the second-scattering c.m. system; $\mathbf{k}_c'$ is the recoil-nucleon momentum in the second-scattering c.m. system, $\hat{s}_c=\hat{n}\times\hat{k}_c'$; $\mathbf{P}_c$ is the three-vector polarization in the second-scattering c.m. system.

1. Lorentz Transformation of Polarization from the First-Scattering c.m. System to the Recoil-Nucleon Rest System

For initial polarization $\mathbf{P}_i=\pm P_i\hat{m}$ the three-vector polarization in the first-scattering c.m. system is

$$\mathbf{P}=P\hat{n}\pm RP_i\hat{s}\mp AP_i\hat{k}'.$$

(See main body of this report.) We assume that the polarization also has a time like fourth component P^0 to be determined below.

The Lorentz transformation is along $\hat{k}'$, the recoil-nucleon momentum in the first-scattering c.m. system. So,

$$\mathbf{P}_r\cdot\hat{n}=P,\quad \mathbf{P}_r\cdot\hat{s}=\pm RP_i,$$
$$P_r\cdot\hat{k}'=\gamma_r[\mp AP_i-\beta_rP^0],$$

and

$$P_r^0=\gamma_r[P^0-\beta_r(\mp AP_i)]=0 \text{ by definition}.$$

Polarization is defined to be a three-vector $(P_r^0=0)$ in the rest frame of the particle. Therefore,

$$P^0=\mp\beta_rAP_i,$$

and

$$\mathbf{P}_r\cdot\hat{k}'=\mp(A/\gamma_r)P_i.$$

Thus, the polarization in the recoil-nucleon rest frame is $\mathbf{P}_r=P\hat{n}\pm RP_i\hat{s}\mp(A/\gamma_r)P_i\hat{k}'$.

2. Lorentz Transformation of Polarization from the Recoil-Nucleon Rest System to the Second-Scattering c.m. System

The Lorentz transformation is along $\mathbf{k}_{0_r}$, the spin-zero nucleus (analyzer) momentum in the recoil-nucleon rest frame. So we need to know the direction of $\hat{k}_{0_r}$ relative to $\hat{n}$, $\hat{s}$, and $\hat{k}'$ in order to perform the transformation of the polarization since the polarization given above is expressed in terms of these three unit vectors.

By Lorentz transformation 1 above (see Fig. 1 and note that $\mathbf{k}_0$ is opposite $\hat{k}$):

$$\mathbf{k}_{0_r}\cdot\hat{n}=0,\quad \mathbf{k}_{0_r}\cdot\hat{s}=k_0\sin(\pi-\theta)=k_0\sin\theta,$$
$$\mathbf{k}_{0_r}\cdot\hat{k}'=\gamma_r[(-k_0)\cos(\pi-\theta)-\beta_rW_0]$$
$$=\gamma_r(k_0\cos\theta-\beta_rW_0),$$

and

$$W_{0_r}=\gamma_r[W_0-\beta_r(-k_0)\cos(\pi-\theta)]$$
$$=\gamma_r(W_0-\beta_rk_0\cos\theta).$$

Therefore,

$$\mathbf{k}_{0_r}=k_0\sin\theta\hat{s}+\gamma_r(k_0\cos\theta-\beta_rW_0)\hat{k}'.$$

Using this equation and Fig. 2, we see that the angle ϕ

[10] H. Desportes and B. Tsai, Saclay Report, 1966 (unpublished).

between $\hat{k}'$ and $\hat{k}_c'$ is given by

$$\tan\phi = \frac{+k_0 \sin\theta}{-\gamma_r(k_0 \cos\theta - \beta_r W_0)},$$

and

$$\hat{s} = \cos\phi \hat{s}_c - \sin\phi \hat{k}_c', \quad \hat{k}' = \sin\phi \hat{s}_c + \cos\phi \hat{k}_c',$$

and

$$\mathbf{P}_r = P\hat{n} \pm P_i(R\cos\phi - (A/\gamma_r)\sin\phi)\hat{s}_c \mp P_i((A/\gamma_r)\cos\phi + R\sin\phi)\hat{k}_c'.$$

Now to make the Lorentz transformation 2: [This transformation is specified by

$$\mathbf{k}_{0_c} + \mathbf{k}_c' = 0 = \gamma_c[\mathbf{k}_{0_r} - \beta_c(W_{0_r} + M)];$$

therefore,

$$\beta_c = \mathbf{k}_{0_r}/(W_{0_r} + M)].$$

$$\mathbf{P}_c \cdot \hat{n} = P, \; \mathbf{P}_c \cdot \hat{s}_c = \pm P_i(R\cos\phi - (A/\gamma_r)\sin\phi),$$

and

$$\mathbf{P}_c \cdot \hat{k}_c = \gamma_c \mathbf{P}_r \cdot \hat{k}_c = \mp\gamma_c P_i((A/\gamma_r)\cos\phi + R\sin\phi).$$

(Recall that $P_r^0 = 0$.) There is a time like fourth component of the polarization with which we are not concerned. Thus, the three-vector polarization in the second-scattering c.m. system is

$$\mathbf{P}_c = P\hat{n} \pm P_i(R\cos\phi - (A/\gamma_r)\sin\phi)\hat{s}_c \mp \gamma_c P_i((A/\gamma_r)\cos\phi + R\sin\phi)\hat{k}_c.$$

A similar derivation can be made for initial polarization $\mathbf{P}_i = \pm P_i \hat{k}$, yielding

$$\mathbf{P}_c = P\hat{n} \pm P_i(A\cos\phi + (R/\gamma_r)\sin\phi)\hat{s}_c \pm \gamma_c P_i((R/\gamma_r)\cos\phi - A\sin\phi)\hat{k}_c.$$

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Simplified Model for a Three-Particle Regge Trajectory*

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Final-state–interaction theory is generalized to the case of a Reggeon (that is, a particle with mass-dependent spin) which decays into three particles, any two of which can undergo a resonant S-wave interaction. It is shown that the residue functions which describe the coupling of the Reggeon to the three-particle state can be used to calculate the corresponding contribution to the imaginary part of the trajectory function. This contribution is not guaranteed to be positive. The formula also predicts the correct unstable two-particle discontinuity associated with the particle-resonance configuration of the final state.

1. INTRODUCTION

In a previous paper[1] it was shown how, in an approximate formulation which used unstable-particle amplitudes, the three-body problem could be discussed in terms of Regge trajectory and residue functions. The specifically three-particle effects in the calculation were associated with the corresponding discontinuity of the Regge trajectories.

The analytic properties of the trajectories $\alpha(s)$, of a two-particle system, with center-of-mass energy $s^{1/2}$, are well understood. They are real analytic functions in the s plane cut along the positive real axis from threshold to infinity except for possible singularities arising from the coincidence of two trajectories. Other two-particle channels coupled to the original one do not disturb this picture. In the case of channels with three or more particles even the existence of trajectories has not yet proved. Some progress has been made recently, however, in the analysis of the problem of three-particle unitarity and complex angular momentum.[2–6]

The purpose of this paper is to see to what extent the existence of trajectories can be reconciled with three-particle unitarity or, at least an approximate version of it. It will be assumed, therefore, that trajectories do exist and that their analytic properties are the same as those of two-particle ones.

In Secs. 2 and 3, the model used as a framework for the discussion is described and simplifying assumptions made about the structure of the scattering amplitudes involved. The analytic properties of the three-particle contribution to the imaginary part of an elastic-scat-

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