PHYSICAL REVIEW VOLUME 155, NUMBER 5 25 MARCH 1967

Off-Shell Pion-Nucleon Scattering. II*

M. L. THIEBAUX, JR.
University of Massachusetts, Amherst, Massachusetts
(Received 19 October 1966)

In an earlier paper, expressions for pionic form factors were derived in an off-shell treatment of pion-nucleon scattering. The development is improved and numerical bounds are estimated for the πNN and $\pi NN^*(1238)$ pionic form factors which are found to deviate from unity by less than 1% and 0.1%, respectively, when the invariant square of the off-shell pion 4-momentum is varied between 0 and m_π^2.

I. INTRODUCTION

THIS paper is a continuation of an earlier one,[1] hereafter referred to as I, in which the nucleon-N^* (1238-MeV) reciprocal bootstrap[2] was exploited as a model to derive the pionic form factors of the πNN and πNN^* vertices. Other authors[3] have studied the same model and have shown that the positions of the singularities of the vertex functions so obtained are consistent with the Landau prescription.

A qualitative understanding of the mechanism underlying the derivation of these form factors comes about in the following way. First we may think of the existence of the N^* as due to the dynamics in the appropriate partial-wave and isotopic-spin projection of a sum of particle-exchange diagrams in pion-nucleon scattering and their higher-order corrections, and dominated by the nucleon-exchange Born term.[4] That is, this sum of diagrams is equivalent to the N^* pole diagram. The basic conjecture in I, then, is twofold: (a) Since the off-shell effects associated with an external pion of invariant mass squared $\Delta^2\neq 1$ (physical pion mass$=1$) are intrinsically buried in the structure contained in this sum of dynamical diagrams, they are equivalently buried in the pole diagram, where they may be lumped into a form factor $\Phi_3(\Delta^2)$ multiplying the πNN^* coupling constant. (b) It is sufficient to consider the dynamics of only the nucleon exchange, as suggested by its dominant role, in order to obtain a closed and tractable system. Hence the off-shell effects in the sum are approximated by lumping them into a form factor $\Phi_1(\Delta^2)$ which multiplies the πNN coupling constant g_1. The above arguments are then repeated with the roles of the nucleon and N^* interchanged, completing the conditions on, and uniquely determining, Φ_1 and Φ_3.

Here we present numerical results based on an improved version of the unitarization procedure formulated in I. In particular, the choice of phases determining the D functions is discussed explicitly, we eliminate the need for subtractions in the dispersion relations, and the threshold behavior is handled in a better way.

We write the partial-wave amplitude in the form[5]

$$f(W,\Delta^2)=\rho(W,\Delta^2)F(W,\Delta^2)\,, \tag{1}$$

where ρ absorbs kinematical zeros and singularities. We then let

$$F(W,\Delta^2)=F^B(W,\Delta^2)+C(W,\Delta^2)\,, \tag{2}$$

where F^B is the input Born amplitude freed of kinematical zeros and singularities in the W plane, and C is an amplitude which presumably contains the other particle exchanges and corrections and which, when added to F^B, gives the unitarized partial-wave amplitude. The unitarization and bootstrap requirements introduced in I evidently generate the higher-order structure lumped into C and the pionic form factors.

While C certainly has a significant dependence on Δ^2 in the physical domain, the second part of the above conjecture, now stated more precisely to suit our requirements, is that the dynamical singularities contained in C have a negligible dependence on Δ^2. For example, a part of the difference between F and F^B arises from the dependence of the vertex functions on the 4-momentum of the exchanged particle in the Born diagram. The dynamical singularities associated with this difference are carried by C, and are therefore assumed to be not strongly dependent on Δ^2.

Elastic unitarity is imposed by writing

$$F(W,\Delta^2)=N(W,\Delta^2)/[D(W)]\,, \tag{3}$$

where D has a phase representation determined by the empirically known phase shifts as discussed in I.

II. CHOICE OF PHASES

In a theory satisfying elastic unitarity and restricted to empirical phase shifts known only for energies $W\lesssim M+5$, it is clear that the choice made for the behavior of the phase of the amplitude above the inelastic threshold will have to be governed primarily by computational expedience. For this purpose, the change in variable, $X\equiv(M+1)/W$, is useful, in that it suggests a way to extrapolate the phases to arbitrarily large energies, and also, in anticipation of numerical work, in that the entire physical region in the X plane is a finite interval.

* Supported by the National Science Foundation.
[1] M. L. Thiebaux, Jr., Phys. Rev. **144**, B1224 (1966).
[2] E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963).
[3] M. Noga and M. Dubec, Nucl. Phys. **76**, 577 (1966); **87**, 191 (1966).
[4] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).
[5] See the Appendix for an explanation of undefined quantities.

Consider the P_{11} amplitude[6] and its MacDowell reflected extension, the S_{11} amplitude, taken, respectively, as the values of a single analytic function on the right- and left-hand cuts which now constitute a single finite cut for $-1\leq X\leq 1$ in the X plane. In I, it was suggested that the phase δ be chosen such that $0\leq\delta\leq\pi$, an *ad hoc* requirement intended to ensure the convergence of integrals containing D. This requirement is unnecessarily strong, and, in fact, leads to an undesirable ambiguity.[7] The P_{11} phase shift is negative above threshold, has a zero near the inelastic threshold, and remains positive thereafter, going through a resonance[8] at $X\approx 0.715$. The S_{11} phase shift $\delta_{S_{11}}$ increases from zero at $X=-1$. Hence the left-hand phase is $\pi-\delta_{S_{11}}$, which decreases from π at $X=-1$, and may conveniently be joined to the P_{11} branch by a smooth curve. Furthermore, requiring that the joining curve pass through $\pi/2$ at $X=0$ suggests a procedure which, if on shaky physical grounds, is at least consistent for handling the complex phase shifts known along a substantial energy interval above the inelastic threshold. Thus, assuming that the partial waves ultimately become predominantly inelastic and consequently imaginary at high energy, we choose the phase of the off-shell amplitude above the inelastic threshold to equal that of the on-shell amplitude, as opposed to the real part of the phase shift as a possible alternate choice. However, this procedure results in a discontinuity of π in the phase across the threshold at $X=-1$ which emerges in the phase representation as a pole in the D function. Now, the threshold behavior of the amplitude is already presumed to be absorbed in the function ρ_1. Hence we prefer to absorb the pole into the N function, a procedure equivalent to subtracting π from the left-hand phase, leaving at $X=0$ a discontinuity of π which simply governs the asymptotic behavior of D. Fig. 1(a) shows the resulting phase, $\delta_1(X)$, for the combined S_{11} and P_{11} amplitudes.

In a similar fashion, we consider the P_{33} and D_{33} amplitudes, taken, respectively, as the values of a single analytic function on the right- and left-hand cuts. Recent phase-shifts analyses seem to point to a D_{33} phase shift which is very small and of uncertain sign.[8] Taking $\delta_{D_{33}}=0$, the phase on the left-hand cut would be π, and may be joined to the P_{33} phase by a smooth curve. In this case, it seems more natural to let the joining curve pass through π at $W=0$, since (a) the P_{33} phase is already significantly near π at the upper limit of the empirical curve in Fig. 1(b), and (b) the D_{33}, P_{33} amplitudes develop very little inelasticity over the 0–700-MeV region, in contrast to the S_{11}, P_{11} amplitudes, which are highly absorptive. Again, the discontinuity at $X=-1$ should be removed by subtracting π from the phase along the interval $-1\leq X\leq 0$, leaving, at $X=0$ a discontinuity of π as indicated in Fig. 1(b), which shows the phase $\delta_3(X)$ for the combined D_{33} and P_{33} amplitudes.

The phases shown in Fig. 1. determine the functions labeled D_0 in I. From the discontinuity of π at $X=0$; in each case, we conclude that both $D_{0,11}(W)$ and $D_{0,33}(W)\sim W$, asymptotically.

III. BOOTSTRAP REQUIREMENT

Specializing (2) and (3) to the D_{33}, P_{33} projection of the nucleon-exchange amplitude, we write a Cauchy integral for the quantity CD, and with a few manipulations find

$$\begin{aligned}F_3(W,\Delta^2)=F_3{}^B(W,\Delta^2)&+[2\pi iD_3(W)]^{-1}\int_V dW'D_3(W')\,\mathrm{Disc}C_3(W',1)/(W'-W)\\&-[\pi D_3(W)]^{-1}\int_U dW'F_3{}^B(W',1)\,\mathrm{Im}D_3(W')/(W'-W-i\epsilon)\\&+[2\pi iD_3(W)]^{-1}\int_V dW'D_3(W')\Delta\,\mathrm{Disc}C_3(W',\Delta^2)/(W'-W)\\&-[\pi D_3(W)]^{-1}\int_U dW'\mathrm{Im}D_3(W')\Delta F_3{}^B(W',\Delta^2)/(W'-W-i\epsilon)\,,\end{aligned}\tag{4}$$

where $D_{0,33}(W)$ is shortened to $D_3(W)$, $\mathrm{Disc}C_3$ denotes the discontinuity of C_3 across the dynamical cuts V, the unitary cuts are denoted by U, we use the notation $\Delta x(\Delta^2)\equiv x(\Delta^2)-x(1)$, and

$$F_3{}^B(W,\Delta^2)=g_1{}^2\Phi_1(\Delta^2)H_3{}^B(W,\Delta^2)/\rho_3(W,\Delta^2)\,.\tag{5}$$

[6] The notation for pion-nucleon states is $l_{2T,2J}$.

[7] Adding π to the negative portion of the P_{11} phase shift introduces two discontinuities which are equivalent to multiplying D by a ratio of first-degree polynomials. One of these is associated with the threshold and must be absorbed into N, while the other is associated with the zero in the P_{11} amplitude. If, instead of absorbing the zero into N, it is explicitly retained, equivalent also to a subtraction in the dispersion integral for the amplitude, there is no *a priori* knowledge of how its position depends on Δ^2.

[8] Phase-shift data come from L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. 138, B190 (1965).

Here, we differ from I, where the kinematical zeros and singularities for $W<0$ were ignored, and make the correct choice

$$\rho_3 = 1/[R_+^2(W,\Delta^2)R_-(W,\Delta^2)]\,. \tag{6}$$

In the on-shell limit of (4), the last two integrals are identically zero, and there must presumably occur implicit cancellations between the remaining terms to give the correct amplitude. In view of the difficulties encountered in (a) continuing Eq. (4) onto the unphysical energy sheet, in which case we might directly equate the residues of the pole which must exist near the N^* resonance in the unitarized Born amplitude and that in the N^* pole diagram, and (b) the lack of guarantee that the residues, now complex, have the same phase to ensure real form factors, we prefer to equate the amplitudes at the physical value $W=N$, the real N^* mass. However, this is equivalent to assuming both amplitudes are well approximated by $R/(W-N+i\Gamma/2)$ where R is a constant in the W plane, and the pole position $N-i\Gamma/2$ is independent of Δ^2, in analogy to the fixed nucleon pole, and then equating residues. At this point, the phases of the pole diagram and the unitarized Born amplitude are both $\pi/2$, and, in any event, the N^* pole must lie close to $W=N$ (approximately 60 MeV from the real axis.)

The above prescription yields

$$2iN[(N-M)^2-1]^{-3/2}[(N+M)^2-1]^{-5/2}[\Phi_3(\Delta^2)-1] = \Delta F_3{}^B(N,\Delta^2) - [\pi D_3(N)]^{-1}\int_U dW'\,\mathrm{Im}D_3(W')\Delta F_3{}^B(W',\Delta^2)/(W'-N-i\epsilon)\,, \tag{7}$$

an inhomogeneous linear equation in Φ_1 and Φ_3. In accordance with our conjecture, we have dropped the integral in (4) containing $\Delta\mathrm{Disc}C_3(W',\Delta^2)$, since it is over the dynamical cut only.

We proceed in an analogous way with the S_{11}, P_{11} projection of the N^* exchange amplitude. The equation corresponding to Eq. (4) is

$$\begin{aligned}F_1(W,\Delta^2) = {} & F_1{}^B(W,\Delta^2) + [2\pi i(W-M)D_1(W)]^{-1}\int_V dW'(W'-M)D_1(W')\,\mathrm{Disc}C_1(W',1)/(W'-W)\\ & -[\pi(W-M)D_1(W)]^{-1}\int_U dW' F_1{}^B(W',1)(W'-M)\,\mathrm{Im}D_1(W')/(W'-W-i\epsilon)\\ & +[2\pi i(W-M)D_1(W)]^{-1}\int_V dW'(W'-M)D_1(W')\Delta\mathrm{Disc}C_1(W',\Delta^2)/(W'-W)\\ & -[\pi(W-M)D_1(W)]^{-1}\int_U dW'(W'-M)\,\mathrm{Im}D_1(W')\Delta F_1{}^B(W',\Delta^2)/(W'-W-i\epsilon)\\ & -[G(0,\Delta^2)+WdG(0,\Delta^2)/dW]/W^2(W-M)D_1(W)\,, \end{aligned} \tag{8}$$

where the complete D function is now the product of the nucleon pole term $W-M$ and $D_{0,11}(W)$, shortened here to $D_1(W)$,

$$F_1{}^B(W,\Delta^2) = g_3{}^2\Phi_3(\Delta^2)H_1{}^B(W,\Delta^2)/\rho_1(W,\Delta^2)\,, \tag{9}$$

$$G(W,\Delta^2) = W^2(W-M)D_1(W)F_1{}^B(W,\Delta^2)\,, \tag{10}$$

and the last term is due to the second-order pole in $H_1{}^B$. The correct choice for ρ_1 is

$$\rho_1(W,\Delta^2) = 1/R_-(W,\Delta^2)\,. \tag{11}$$

Because of the Feynman-diagram treatment of the high-spin exchange and the asymptotic form of D_1, the integrals do not all converge. In the belief that the important Δ^2-dependent effects are associated with the low-energy region, we choose to suppress the doubtful high-energy behavior of the N^* exchange amplitude by a cutoff factor $[1+(W/a)^2]^{-1}$, sufficient to give convergence, thus introducing a parameter a into the procedure. The normalization of unity at $W=0$, instead of at threshold, reduces the apparent sensitivity of the dependence of the right-hand side of (8) on a, since the cutoff will occur only in the integrals after the nucleon-pole residue is extracted.

Equating the residues of the nucleon-pole amplitude and the unitarized, cutoff Born amplitude, we obtain

$$(3g_1{}^2/16\pi M^2)[\Phi_1(\Delta^2)-1] = [\pi D_1(M)]^{-1}\int_U dW'\,\mathrm{Im}D_1(W')\Delta F_1{}^B(W',\Delta^2)[1+(W'/a)^2]^{-1} + \Delta[G(0,\Delta^2)+MdG(0,\Delta^2)/dW]/M^2D_1(M)\,, \tag{12}$$

a second inhomogeneous linear equation in Φ_1 and Φ_3. Again, we have dropped the dynamical-cut integral in (8) containing $\Delta\mathrm{Disc}C_1(W',\Delta^2)$. Equations (7) and (12) are readily solved for the form factors:

$$\Phi_1(\Delta^2)=1-[j_1(\Delta^2)\Delta j_3(\Delta^2)+\Delta j_1(\Delta^2)]/[1+j_1(\Delta^2)j_3(\Delta^2)], \tag{13}$$

and

$$\Phi_3(\Delta^2)=1-[j_3(\Delta^2)\Delta j_1(\Delta^2)-\Delta j_3(\Delta^2)]/[1+j_1(\Delta^2)j_3(\Delta^2)], \tag{14}$$

where

$$\begin{aligned} j_1(\Delta^2)=&[256\pi\Gamma M^2N^5/3g_1^2\Lambda_1(M)][(N-M)^2-1]^{-3/2}[(N+M)^2-1]^{-5/2}\\ &\times\{[-(4N^2-\Delta^2+1)+Nd(2N^2+4NM-2M^2+1+\Delta^2)]/12N^2\\ &+\int_U dW'\sin\delta_1(W')\Lambda_1(W')H_1{}^B(W',\Delta^2)R_-(W',\Delta^2)[1+(W'/a)^2]^{-1}\}, \end{aligned} \tag{15}$$

$$\begin{aligned} j_3(\Delta^2)=&[g_1^2/2\pi N\Lambda_3(N)][(N-M)^2-1]^{3/2}[(N+M)^2-1]^{5/2}\\ &\times P\int_U dW'\sin\delta_3(W')\Lambda_3(W')H_3{}^B(W',\Delta^2)R_+{}^2(W',\Delta^2)R_-(W',\Delta^2)/(W'-N), \end{aligned} \tag{16}$$

$$\Lambda_{1,3}(W)=\exp[-(W/\pi)P\int dW'\delta_{1,3}(W')/(W'-W)W'], \tag{17}$$

and

$$d=\int_U dW'\delta_1(W')/\pi W'^2. \tag{18}$$

IV. NUMERICAL RESULTS AND DISCUSSION

The uncertainty inherent in the essentially arbitrary extrapolations of the phase shifts suggests that a precise and cumbersome numerical evaluation of the integrals in (15) and (16) is unwarranted. A sensible approach within this limited framework seems to be to invoke the following approximations designed to estimate bounds on the departure of the form factors from unity:

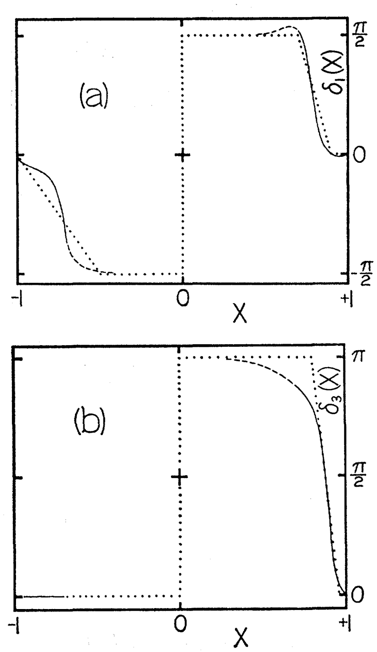

FIG. 1. Plots of the amplitude phases. (a) The combined S_{11} and P_{11} phases. (b) The combined D_{33} and P_{33} phases. The solid lines are derived from the known phase shifts, the dashed lines are the high-energy extrapolations, and the dotted lines are straight-line approximations, Eqs. (18) and (19), used in carrying out the numerical estimates in the text.

1. Simple analytic expressions for the function Λ_1 and Λ_3 are readily obtained when the phases are approximated by

$$\begin{aligned} \delta_1(X)&=-\pi(X+1), & -1\ &\le X<-0.5\\ &=-\pi/2, & -0.5\ &\le X<0\\ &=\pi/2, & 0\ &\le X<0.7\\ &=2.5\pi(0.9-X), & 0.7\ &\le X<0.9\\ &=0, & 0.9\ &\le X\le 1.0 \end{aligned} \tag{19}$$

and

$$\begin{aligned} \delta_3(X)&=0, & -1\ &\le X<0\\ &=\pi, & 0\ &\le X<0.8\\ &=6.25\pi(0.96-X), & 0.8\ &\le X<0.96\\ &=0, & 0.96&\le X\le 1.0. \end{aligned} \tag{20}$$

2. The integral of Eq. (15) is estimated with $a\approx 4(M+1)$, by evaluating a few points on the interval $-1\le X\le 1$.

3. The rapid variation of δ_3 near the 3,3 resonance justifies the approximation:

$$\begin{aligned} P\int dW'f(W')\sin\delta_3(W')/(W'-N)=&(4/\pi)\Gamma df(N)/dW\\ &+O(\Gamma^3)d^3f(N)/dW^3 \end{aligned} \tag{21}$$

in the integral of (16).

The result's turn out to be quite flat form factors

$$|d\Phi_1/d\Delta^2|\quad<0.01, \tag{22}$$

$$|d\Phi_3/d\Delta^2| \quad <0.001\,, \tag{23}$$

for the range $0\leq\Delta^2\leq 1$.

The uncertainty in the sign is traced to appreciable phase-dependent cancellations which occur in the integrals. In view of the observed sensitivity of these bounds (and, in fact, of the sign of the derivative) on the choice of phases, it has become clear that further improvements of the model, at least this particular one, based on the nucleon-N^* bootstrap, will require a special emphasis on determining how the phases should be handled above the inelastic threshold.

The small values of $d\Phi_{1,3}/d\Delta^2$ are, of course, to be expected in any sensible theory, since there are no observed low-mass resonances communicating with the pion. Reinforcing this statement, there is the result[9] that in the π-exchange–dominated process $\pi^-+p\to\rho^-+p$ with final-state absorptive effects included, the vertex function and propagator t dependence (lumped into a form factor) turn out, empirically, to be unimportant, and in fact, the lower bound on the effective mass in a simple pole approximation for the form factor is not inconsistent with (22).

Since the bootstrap model apparently in no way resembles a conventional calculation in which the form factor is represented as a dispersion integral over real states connecting the pion and $N\bar{N}$ or $N^*\bar{N}$ systems, the above consistency may be taken as either an accident related to the phase-dependent cancellations, aided by the smallness of the pion mass, or else an indication of the essential correctness of the liberal interpretation of bootstrap dynamics stated in the introduction.

ACKNOWLEDGMENTS

The author wishes to thank Professor B. A. Jacobsohn and Professor E. M. Henley for their hospitality at the 1966 National Science Foundation Summer Institute on Theoretical Physics held at the University of Washington, where much of this work was done.

[9] M. Bander and G. L. Shaw, Phys. Rev. **139**, B956 (1965).

APPENDIX

Here we list quantities defined or derived in I:

W is the total energy in pion-nucleon c.m. system;
M is the nucleon mass;
N is the N^* (1238) mass;
g_1 is the πNN coupling constant $(g_1^2/4\pi\approx 14)$;
Γ is the N^* width;
g_3 is the πNN^* coupling constant and
$=16\pi\Gamma N^5[(N-M)^2-1]^{-3/2}[(N+M)^2-1]^{-5/2}$;
$R_\pm(W,\Delta^2)=[(W\pm M)^2-1]^{-1/2}[(W\pm M)^2-\Delta^2]^{-1/2}$;
$H_3{}^B(W,\Delta^2)=(1/4\pi)[(W-M)R_-Q_1(z_1)+(W+M)R_+Q_2(z_1)]$;
$H_1{}^B(W,\Delta^2)=(1/12\pi N^2)[V_2/R_-W^2+2Y_1R_-Q_1(z_3)+2Y_2R_+Q_0(z_3)]$;
$z_1=R_+R_-[(M^2-1)^2-W^2(W^2-2)+(\Delta^2-1)(W^2-M^2+1)]$;
$z_3=z_1+2R_+R_-W^2(M^2-N^2)$;
$Q_l(z)=\frac{1}{2}\int_{-1}^{1}dx\,P_l(x)/(z-x)$;
$V_2=-A''+(W+M)B''$;
$Y_1=A'+(W-M)B'$;
$Y_2=-A'+(W+M)B'$;
$A'=6W^2N^2(N+M)+2[N^5-2N^3(2M^2+1)-2MN^2(1+M^2)+2N(M^2-1)-M(M^2-1)^2]+2(\Delta^2-1)[-N^3-MN^2+N(M^2-2)+M(M^2-1)]$;
$A''=2N(N^2-M^2+1)+(\Delta^2-1)(N+M)$;
$B'=-6W^2N^2-2[2N^4+2MN^3-4N^2+2M(M^2-1)N-(M^2-1)^2]+(\Delta^2-1)[4N^2+2MN-2(M^2-1)]$;
$B''=-4N^2+\Delta^2-1$.