Explicit Solution of the $SU(3)\otimes SU(3)$ Algebra of Currents*

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Explicit formulas are given for various matrix elements of the axial-vector current in an arbitrary irreducible representation of the $SU(3)\otimes SU(3)$ algebra of currents. The formulas express such quantities as the renormalization G_A of the axial-vector coupling constant, the strength G^* of the axial-vector transition between N and N^*, the D/F ratio for the axial-vector current between states of the baryon octet, etc., in terms of the pair of integers that specify an irreducible representation of $SU(3)$.

1. INTRODUCTION

THE suggestion by Gell-Mann[1] that the algebra generated by current components may be used to calculate such quantities as the renormalization G_A of the axial-vector coupling constant has been recently explored in two different ways. In one approach the current algebra is combined with the partially conserved axial-vector current hypothesis. The pioneering work along these lines was done by Weisberger[2] and Adler,[3] whose calculation yielded the value $|G_A|\cong 1.2$, in excellent agreement with experiment. In the other approach it is assumed that the sum rules derived from the current algebra are saturated by a limited set of stationary or quasistationary states. This approach was first used by Lee[4] and by Dashen and Gell-Mann,[5] using the $SU(6)$ algebra generated by the time components of the vector current and the space components of the axial-vector current, and by Gerstein,[6] using the chiral $SU(3)\otimes SU(3)$ algebra generated by the time components of the vector and axial-vector currents. All these authors obtain the unsatisfactory result $|G_A|=5/3$.

Now, in fact, the second approach is capable of yielding for G_A any value whatsoever. As was noted by Lee[4] and particularly emphasized by Ryan,[7] a consistent solution of the current-algebra equations will always be obtained provided the states used to "solve" the equations form the basis of an irreducible representation of the algebra in question. Thus an infinite set of values can be obtained for G_A, corresponding to the infinite number of different irreducible representations. Moreover, since a mixture of irreducible representations will also "solve" the equations, by adjusting the amount of mixing a value for G_A, intermediate to the values corresponding to irreducible representations, can also be obtained.

During the past year various arguments have been presented in favor of representation mixing. These may be summarized as follows[8]:

1. In any pure representation the anomalous magnetic moment of the baryon octet vanishes.[9] This argument involves two assumptions. It is first assumed that the component L_z of an internal angular momentum along the direction of the linear momentum is zero. Now if we have a pure representation it is reasonable to suppose that it involves only one value of **L**, and for the ground state of the system (to which the baryon octet belongs) it is reasonable to suppose that that one value is zero. Thus the assumption $L_z=0$ is most reasonable; nevertheless the possibility that $\mathbf{L}\neq 0$ for the baryon octet can not be ruled out. Secondly it is assumed that in the commutation relations of the relevant current densities there appear no nontrivial gradient terms. In the quark model this assumption amounts to supposing that the quarks themselves have no anomalous magnetic moment, i.e., that their electromagnetic interaction is minimal. Again, although this assumption may be reasonable the possibility that it is wrong can not be ruled out.

2. Explicit mixtures of representations have been constructed in which the numerical values of a variety of quantities come out in excellent agreement with experiment.[10,11] Although the success of the representation-mixing schemes is undeniable, it is perhaps not too surprising that better agreement with experiment is achieved when additional free parameters, the mixing angles, are introduced into the theory.

3. In our, perhaps prejudiced, opinion the most convincing argument in favor of representation mixing was presented in Ref. 12. In that work a consistency relation is derived between the renormalization of the

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1 M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964).
2 W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965).
3 S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965).
4 B. W. Lee, Phys. Rev. Letters **14**, 676 (1965).
5 R. F. Dashen and M. Gell-Mann, Phys. Letters **17**, 142 (1965); **17**, 145 (1965).
6 I. S. Gerstein, Phys. Rev. Letters **16**, 114 (1966).
7 C. Ryan, Ann. Phys. (N. Y.) 38, 1 (1966).

8 It has also been argued (Refs. 9 and 10) that we must have representation mixing because the success of the Weisberger-Adler sum rule hinges on there being contributions from states other than the $\frac{1}{2}^+$ octet and the $\frac{3}{2}^+$ decuplet. We omit this argument from our summary since it only proves that those pure representations whose $SU(3)$ content is $\mathbf{8}\oplus\mathbf{10}$ are unacceptable, i.e., the representation **56** in the case of $SU(6)$ or the representation [6,3] in the case of $SU(3)\otimes SU(3)$.
9 R. F. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy, University of Miami, 1966* (W. H. Freeman and Company, San Francisco, 1966).
10 H. Harari, Phys. Rev. Letters **16**, 964 (1966); **17**, 56 (1966).
11 I. S. Gerstein and B. W. Lee, Phys. Rev. Letters **16**, 1060 (1966).
12 A. M. Bincer, Phys. Rev. Letters **16**, 754 (1966).

axial-vector coupling constant G_A and the D/F ratio for the axial-vector current between states of the octet. The relation is derived on the assumption that the $SU(3)$ spectrum of baryon states contains just one octet but is otherwise arbitrary. Since the experimental values of G_A and D/F violate this consistency condition it follows that the $SU(3)$ spectrum of baryons must contain more than one octet. This is equivalent to the statement that we must have representation mixing because the $SU(3)$ content of any irreducible $SU(3)\otimes SU(3)$ representation has at most one octet in it with one exception. If the two representations used in $SU(3)\otimes SU(3)$ are conjugate then in the Clebsch-Gordan series that determines the $SU(3)$ content the octet will appear twice (or once). However in that case the D/F ratio is infinite or G_A vanishes and the consistency condition is violated anyway.

In the present work we refine the above results by explicitly calculating a variety of relevant quantities. In Sec. 2 we confine ourselves to isotopic spin rather than unitary spin and study the resultant chiral $SU(2)\otimes SU(2)$ algebra. In that case a complete solution of the problem is possible, i.e., we present explicit formulas for all matrix elements of the axial-vector current in an arbitrary irreducible representation of $SU(2)\otimes SU(2)$. In Sec. 3 we generalize to unitary spin and the chiral $SU(3)\otimes SU(3)$ algebra. Here we give explicit formulas for G_A, D/F, and the strength G^* of the axial-vector transition between N and N^* in an arbitrary irreducible representation of $SU(3)\otimes SU(3)$.

Our results show explicitly that for no pure representation of the chiral $SU(3)\otimes SU(3)$ algebra can the experimental value $|G_A|\cong 1.2$ be obtained. (Since the calculations refer to matrix elements at infinite momentum our conclusions for the chiral algebra hold equally well for the collinear algebra, the two being equivalent at infinite momentum.[9]) Aside from showing that we must have representation mixing, our results should be useful in the explicit construction of mixtures that are required to yield prescribed values for G_A, D/F, G^*, etc.

2. THE $SU(2)\otimes SU(2)$ ALGEBRA

We start out with the commutation relations for the time components of the vector and axial-vector currents proposed by Gell-Mann[1]:

$$[V_\mu,V_\nu]=c_{\mu\nu}{}^\lambda V_\lambda\,,\quad [V_\mu,A_\nu]=c_{\mu\nu}{}^\lambda A_\lambda\,,\quad [A_\mu,A_\nu]=c_{\mu\nu}{}^\lambda V_\lambda\,, \tag{1}$$

where the subscripts run from 1 to 8 and the $c_{\mu\nu}{}^\lambda$ are the structure constants of $SU(3)$ so that the V_μ are the eight unitary spin generators. We note that if we introduce

$$\mathbf{K}\equiv\tfrac{1}{2}(\mathbf{V}+\mathbf{A})\,,\quad \mathbf{L}\equiv\tfrac{1}{2}(\mathbf{V}-\mathbf{A})\,, \tag{2}$$

where we consider these quantities as 8-component vectors, then Eq. (1) becomes

$$[K_\mu,K_\nu]=c_{\mu\nu}{}^\lambda K_\lambda\,,\quad [L_\mu,L_\nu]=c_{\mu\nu}{}^\lambda L_\lambda\,,\quad [K_\mu,L_\nu]=0\,, \tag{3}$$

and therefore we are dealing with the group $SU(3)\otimes SU(3)$, where the first $SU(3)$ is generated by $\mathbf{K}$, the second by $\mathbf{L}$.

We shall deal with this group in Sec. 3. Here we restrict the values of the subscripts to 1, 2, 3. Then the $c_{\mu\nu}{}^\lambda$ are the structure constants of $SU(2)$, the $\mathbf{V}$ are the three isotopic spin generators, and the group that we are dealing with is $SU(2)\otimes SU(2)$.[13]

An irreducible representation of the $SU(2)$ group generated by $\mathbf{V}$ may be specified by the number v related to the dimension D of the representation by

$$D=2v+1\,. \tag{4}$$

The allowed values of v are 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, $\cdots$. A basis for this irreducible representation is provided by the set of states $|v,m\rangle$, where $m=-v,\ -v+1,\ \cdots,\ +v$ is the magnetic quantum number equal to the eigenvalue of V_3. It is these states that we identify with physical particle states since $\mathbf{V}$ is the isotopic spin and the physical particle states are supposed to form isotopic spin multiplets. Our task therefore is to calculate matrix elements of the form $\langle v,m|\mathbf{A}|v',m'\rangle$.

The dependence of the matrix elements on the magnetic quantum numbers is disposed of by means of the Wigner-Eckart theorem:

$$\langle v,m''|T_m|v',m'\rangle=\langle v\|T\|v'\rangle\langle v'm',1m|vm''\rangle\,, \tag{5}$$

where $\langle v\|T\|v'\rangle$ is the reduced matrix element independent of magnetic quantum numbers, $\langle v'm',1m|vm''\rangle$ is the Clebsch-Gordan coefficient of $SU(2)$, and T_m is the mth component in a spherical basis of a rank-one $SU(2)$ tensor, such as $\mathbf{V}$ or $\mathbf{A}$.

We calculate first the diagonal reduced matrix elements. It follows from the Wigner-Eckart theorem and the fact that the $\mathbf{V}$ are generators that

$$\langle v,m|\mathbf{V}\cdot\mathbf{T}|v,m\rangle=\langle v\|V\|v\rangle\langle v\|T\|v\rangle\,. \tag{6}$$

Consequently

$$\langle v\|A\|v\rangle=\langle v,m|\mathbf{V}\cdot\mathbf{A}|v,m\rangle[\langle v,m|\mathbf{V}^2|v,m\rangle]^{-1/2}\,. \tag{7}$$

To get the off-diagonal elements we note that by the Wigner-Eckart theorem

$$\langle v,m|\mathbf{A}^2|v,m\rangle=\sum_{v'=v-1}^{v+1}\langle v\|A\|v'\rangle^2\,. \tag{8}$$

In view of the relation

$$\langle v\|A\|v'\rangle^2(2v+1)=\langle v'\|A\|v\rangle^2(2v'+1)\,, \tag{9}$$

[13] The treatment that we give here follows closely that of W. Pauli, Ergeb. Exakt. Naturw. 37, 85 (1965).

(which follows from the reality of our matrix elements and the Hermitian nature of $\mathbf{A}$), we can convert Eq. (8) into a two-term recursion formula:

$$\langle v\|A\|v+1\rangle^2+\frac{2v-1}{2v+1}\langle v-1\|A\|v\rangle^2 =\langle v,m|\mathbf{A}^2|v,m\rangle-\frac{\langle v,m|\mathbf{V}\cdot\mathbf{A}|v,m\rangle^2}{\langle v,m|\mathbf{V}^2|v,m\rangle}. \quad (10)$$

Equations (7) and (10) are the solution of our problem given the values of $\mathbf{V}^2$, $\mathbf{A}^2$, and $\mathbf{V}\cdot\mathbf{A}$ for an arbitrary irreducible representation of $SU(2)\otimes SU(2)$. Let $k(l)$ specify the irreducible representation of the $SU(2)$ generated by $\mathbf{K}(\mathbf{L})$; the corresponding irreducible representation of $SU(2)\otimes SU(2)$ will be specified by the pair $[k,l]$ (the allowed values of either k or l being $0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$). The group $SU(2)\otimes SU(2)$ has two invariants, i.e., quantities that commute with all the operators, namely the two quadratic Casimir operators $\mathbf{K}^2$ and $\mathbf{L}^2$. In the $[k,l]$ representation their values are

$$\mathbf{K}^2=k(k+1),\quad \mathbf{L}^2=l(l+1). \quad (11)$$

It follows from Eq. (2) that

$$\mathbf{K}^2-\mathbf{L}^2=\mathbf{V}\cdot\mathbf{A}=\mathbf{A}\cdot\mathbf{V}, \quad (12)$$

$$\mathbf{K}^2+\mathbf{L}^2=\tfrac{1}{2}(\mathbf{V}^2+\mathbf{A}^2), \quad (13)$$

and therefore in the $[k,l]$ representation we have

$$\mathbf{V}\cdot\mathbf{A}=k(k+1)-l(l+1)=(k-l)(k+l+1), \quad (14)$$

$$\tfrac{1}{2}(\mathbf{V}^2+\mathbf{A}^2)=k(k+1)+l(l+1). \quad (15)$$

Finally, since $\mathbf{V}^2$ is the quadratic Casimir operator of the isotopic spin $SU(2)$ we also have

$$\langle v,m|\mathbf{V}^2|v,m\rangle=v(v+1). \quad (16)$$

Making use of Eqs. (14)–(16) in Eqs. (7) and (10) we have in the $[k,l]$ representation of $SU(2)\otimes SU(2)$:

$$\langle v\|A\|v\rangle=(k-l)(k+l+1)[v(v+1)]^{-1/2} \quad (17)$$

and

$$\langle v\|A\|v+1\rangle^2+\frac{2v-1}{2v+1}\langle v-1\|A\|v\rangle^2 =2k(k+1)+2l(l+1)-v(v+1) -(k-l)^2(k+l+1)^2/[v(v+1)], \quad (18)$$

provided that the representation v is contained in the representation $[k,l]$ (if it is not contained then the matrix elements vanish).

To see whether the representation specified by v is contained in that specified by $[k,l]$ we note that according to Eq. (2)

$$\mathbf{V}=\mathbf{K}+\mathbf{L};$$

thus we have the problem of addition of two angular momenta, whose solution is the well-known rule

$$|k-l|\leq v\leq k+l. \quad (19)$$

Now, in particular, for the nucleon $v=\frac{1}{2}$ and we are only interested in $[k,l]$ representations which contain the nucleon. Hence

$$|k-l|=\tfrac{1}{2}, \quad (20)$$

and the $SU(2)\otimes SU(2)$ representations of interest may be specified by a single (integer or half-integer) number l; they are of the form $[l+\frac{1}{2}, l]$ or $[l, l+\frac{1}{2}]$. These two types are referred to as each other's mirror representations.

Now, by definition, the renormalization of the axial-vector coupling constant G_A is given by

$$G_A=\frac{\langle\frac{1}{2}\|A\|\frac{1}{2}\rangle}{\langle\frac{1}{2}\|V\|\frac{1}{2}\rangle}. \quad (21)$$

Consequently

$$G_A=\pm(1+\tfrac{4}{3}l),\quad l=0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots, \quad (22)$$

where the upper sign holds for the irreducible representation $[l+\frac{1}{2}, l]$, the lower sign for the mirror $[l, l+\frac{1}{2}]$. We note that for $l=0$ we have $|G_A|=1$, i.e., no renormalization, for $l=\frac{1}{2}$ we have $|G_A|=5/3$, i.e., the famous $SU(6)$ number. These numbers are in agreement with the known results for the representations $[\frac{1}{2},0]$, $[0,\frac{1}{2}]$ and $[1,\frac{1}{2}]$, $[\frac{1}{2},1]$.

By definition, the strength of the axial-vector transition between N and N^* is given by

$$G^*=\frac{\langle\frac{3}{2}\|A\|\frac{1}{2}\rangle}{\langle\frac{1}{2}\|V\|\frac{1}{2}\rangle}. \quad (23)$$

Using the recursion formula, Eq. (18), for $v=\frac{1}{2}$ we obtain immediately

$$G^{*2}=8l(1+\tfrac{2}{3}l)/3 \quad (24)$$

for either of the mirror representations $[l+\frac{1}{2}, l]$, $[l, l+\frac{1}{2}]$. In particular $G^*=0$ for $l=0$, as it should since $v=\frac{3}{2}$ is not contained in either $[\frac{1}{2},0]$ or $[0,\frac{1}{2}]$; and $G^{*2}=(\frac{4}{3})^2$ for $l=\frac{1}{2}$, in agreement with the known value for this representation.

3. THE $SU(3)\otimes SU(3)$ ALGEBRA

We now proceed to the case when $\mathbf{V}$ stands for the eight generators of unitary spin and the current algebra is the algebra of $SU(3)\otimes SU(3)$. Most of the equations of Sec. 2 are generalized to the present case in an obvious way but there also are some fundamental differences which make the calculation at the $SU(3)$ level considerably more complicated.

An irreducible representation of $SU(3)$ is specified by a pair of non-negative integers (p,q) related to the dimension D of the representation by

$$D=(p+1)(q+1)(p+q+2)/2, \quad (25)$$

and to the quadratic Casimir invariant $C^{(2)}$ by

$$C^{(2)}=p+q+(p^2+q^2+pq)/3. \quad (26)$$

We shall refer to a representation of the $SU(3)$ group generated by **V**, **K**, and **L**, respectively, by the symbols v, κ, and λ. Thus v stands for the pair of non-negative integers (p_v,q_v), κ stands for (p_κ,q_κ), and λ stands for (p_λ, q_λ). We shall also use for v (or κ, or λ) the actual dimension of the representation and will then distinguish conjugate representations by a star: $v\leftrightarrow(p_v,q_v)$, $v^*\leftrightarrow(q_v,p_v)$. The representations of $SU(3)\otimes SU(3)$ will be specified by $[\kappa,\lambda]$. The quadratic Casimir operators are given in the $[\kappa,\lambda]$ representation by the corresponding quadratic invariants

$$\mathbf{K}^2=C_\kappa^{(2)}, \quad \mathbf{L}^2=C_\lambda^{(2)}, \quad (27)$$

and we also have

$$\langle v\mu|\mathbf{V}^2|v\mu\rangle=C_v^{(2)}. \quad (28)$$

Here $|v\mu\rangle$ stands for the set of states that provide a basis for the irreducible representation v (μ, the "magnetic" quantum number, now involves the specification of isotopic spin, z component of isotopic spin, and hypercharge). These are the states that are now identified with physical particle states which are now supposed to form unitary spin multiplets.

As before the problem is to determine matrix elements of the form $\langle v_1\mu_1|\mathbf{A}|v_2\mu_2\rangle$. The Wigner-Eckart theorem, which disposes of the dependence on the magnetic quantum numbers, has now the more complicated form

$$\langle v_1\mu_1|T_\mu|v_2\mu_2\rangle=\begin{pmatrix} v_2 & 8 & v_{1,a} \\ \mu_2 & \mu & \mu_1\end{pmatrix}\langle v_1\|T\|v_2\rangle_a + \begin{pmatrix} v_2 & 8 & v_{1,s} \\ \mu_2 & \mu & \mu_1\end{pmatrix}\langle v_1\|T\|v_2\rangle_s, \quad (29)$$

where $\langle v_1\|T\|v_2\rangle_\gamma$ is the reduced matrix element,

$$\begin{pmatrix} v_2 & 8 & v_{1,\gamma} \\ \mu_2 & \mu & \mu_1\end{pmatrix}$$

is the $SU(3)$ Clebsch-Gordan coefficient with phase conventions as defined by de Swart,[14] and T_μ is the μth component in a spherical basis of an octet $SU(3)$ tensor, such as **A**, **K**, **L**. **V** is also an octet $SU(3)$ tensor, but being the generator of the group here considered, has only reduced matrix elements diagonal in v, and only of the type $\gamma=a$.

As a result, the analog of Eq. (6) now reads

$$\langle v\mu|\mathbf{V}\cdot\mathbf{T}|v\mu\rangle=\langle v\|V\|v\rangle\langle v\|T\|v\rangle_a, \quad (30)$$

and therefore the analog of Eq. (17) is

$$\langle v\|A\|v\rangle_a=(C_\kappa^{(2)}-C_\lambda^{(2)})/\sqrt{C_v^{(2)}}, \quad (31)$$

provided that v is contained in $[\kappa,\lambda]$, otherwise this matrix element vanishes.

The analog of Eq. (18) now reads

$$\langle v\|A\|v\rangle_s{}^2+\sum_{v'\neq v}\langle v\|A\|v'\rangle^2=2C_\kappa^{(2)}+2C_\lambda^{(2)}-C_v^{(2)} -(C_\kappa^{(2)}-C_\lambda^{(2)})^2/C_v^{(2)}, \quad (32)$$

where again a term such as $C_\kappa^{(2)}$ appearing on the right-hand side actually stands for $\langle v\mu|\mathbf{K}^2|v\mu\rangle$, which is equal to $C_\kappa^{(2)}$ if v is contained in $[\kappa,\lambda]$, and is zero otherwise. The summation over v' runs over all values, other than v, contained in the Clebsch-Gordan series for $8\otimes v$. Thus Eq. (32) is equivalent to a multiterm recursion formula, in contrast to the $SU(2)$ case, and is of little value unless all but one of the reduced matrix elements appearing in it have been determined in some other way.

We must therefore search for additional relations involving the same reduced matrix elements. One such set of relations is obtained directly from the fundamental Eq. (1). In the spherical basis the last of Eq. (1) is written as

$$[A_\alpha,A_\beta]=-\sqrt{3}\sum_\rho\begin{pmatrix} 8 & 8 & 8_a \\ \alpha & \beta & \rho\end{pmatrix}V_\rho. \quad (33)$$

By taking the v, v'' matrix element of this equation, using the Wigner-Eckart theorem and properties of the Clebsch-Gordan coefficients, we arrive at the relations

$$\sum_{v'\gamma'\gamma''}\langle(v''8)v'_{\gamma''},8v_{\gamma'}|v''(88)N_\gamma v_{\gamma'''}\rangle \times\langle v\|A\|v'\rangle_{\gamma'}\langle v'\|A\|v''\rangle_{\gamma''}$$
$$=(\sqrt{3}/2)(C_v^{(2)})^{1/2}\delta_{v'',v}\delta_{N_\gamma,8_a}\delta_{\gamma''',a};$$
$$N_\gamma=\mathbf{8}_a,\ \mathbf{10},\ \mathbf{10}^*. \quad (34)$$

For $v''=v$ these relations involve the same reduced matrix elements that appear in Eqs. (31) and (32).

The $SU(3)$ Racah coefficient appearing in Eq. (34) is defined by[15]

$$\langle(v_1v_2)v_{12,\alpha}v_3v_\gamma|v_1(v_2v_3)v_{23,\beta}v_{\gamma'}\rangle =\sum_{\substack{\mu_1\mu_2\mu_3\\ \mu_{12}\mu_{23}}}\begin{pmatrix} v_1 & v_2 & v_{12,\alpha} \\ \mu_1 & \mu_2 & \mu_{12}\end{pmatrix}\begin{pmatrix} v_{12} & v_3 & v_\gamma \\ \mu_{12} & \mu_3 & \mu\end{pmatrix} \times\begin{pmatrix} v_2 & v_3 & v_{23,\beta} \\ \mu_2 & \mu_3 & \mu_{23}\end{pmatrix}\begin{pmatrix} v_1 & v_{23} & v_{\gamma'} \\ \mu_1 & \mu_{23} & \mu\end{pmatrix} \quad (35)$$

being the standard recoupling coefficient that arises when the product of the three representations $v_1\otimes v_2\otimes v_3$ is reduced in the two different ways: $(v_1\otimes v_2)\otimes v_3$ and $v_1\otimes(v_2\otimes v_3)$.

[14] J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963).

[15] The particular Racah coefficients that will be needed in this work were tabulated by M. Krammer, Acta Phys. Austriaca, Suppl. 1, 183 (1964).

We note that whereas for $N_\gamma = \mathbf{8}_a$, $\gamma''' = a$ Eq. (34) has an $SU(2)$ analog, for $N_\gamma = \mathbf{8}_a$, $\gamma''' = s$ and for $N_\gamma = \mathbf{10}, \mathbf{10}^*$ it does not. Another set of relations, also without an $SU(2)$ analog, is obtained by considering cubic Casimir operators. In $SU(3)$ one can construct a cubic Casimir operator, which [in contrast to the $SU(2)$ case] is functionally independent of the quadratic operator. We denote the cubic operators by $\mathbf{K}^3$, $\mathbf{L}^3$, and $\mathbf{V}^3$ respectively for the three $SU(3)$ groups. In the $[\kappa,\lambda]$ representation we have

$$\mathbf{K}^3 = C_\kappa^{(3)}, \quad \mathbf{L}^3 = C_\lambda^{(3)}. \tag{36}$$

Also

$$\langle v\mu | \mathbf{V}^3 | v\mu \rangle = C_v^{(3)}. \tag{37}$$

Here $C^{(3)}$ is the cubic Casimir invariant whose value in the (p,q) representation of $SU(3)$ is

$$C^{(3)} = 6(p+q+1)(p-q) + 2(p^2q - pq^2) + \tfrac{4}{3}(p^3 - q^3). \tag{38}$$

In terms of components in the spherical basis our cubic operator is given by[16]

$$\mathbf{V}^3 = (4\sqrt{15}) \sum_{\alpha\beta\rho} \begin{pmatrix} 8 & 8 & 8_s \\ \alpha & \beta & \rho \end{pmatrix} V_\alpha V_\beta V_\rho^\dagger, \tag{39}$$

and similarly for $\mathbf{K}^3$ and $\mathbf{L}^3$.

We note parenthetically that the quadratic operator can be cast into a very similar form:

$$\mathbf{V}^2 = \sum_\rho V_\rho V_\rho^\dagger = -(2\sqrt{3}) \sum_{\alpha\beta\rho} \begin{pmatrix} 8 & 8 & 8_a \\ \alpha & \beta & \rho \end{pmatrix} V_\alpha V_\beta V_\rho^\dagger. \tag{40}$$

By taking matrix elements of $\mathbf{K}^3$ and $\mathbf{L}^3$ in the representation v we obtain two additional relations for the reduced matrix elements of $\mathbf{A}$. Because of the cubic nature of these invariants these relations contain terms up to trilinear in $\mathbf{A}$, and consequently reduced matrix elements beyond those occurring in Eqs. (31), (32), and (34). We note however, that the combination $\mathbf{K}^3 + \mathbf{L}^3$ is an even function of $\mathbf{A}$, hence does not contain trilinear (or linear) terms and therefore involves only the desired reduced matrix elements. We have

$$\mathbf{K}^3 + \mathbf{L}^3 = \tfrac{1}{4}\mathbf{V}^3 + (3\sqrt{15}) \sum_{\alpha\beta\rho} \begin{pmatrix} 8 & 8 & 8_s \\ \alpha & \beta & \rho \end{pmatrix} A_\alpha A_\beta V_\rho^\dagger, \tag{41}$$

which upon taking of matrix elements can be manipulated into

$$(3\sqrt{15}) \sum_{v'\gamma\gamma'} \langle (v8)v'_{\gamma'} 8 v_\gamma | v(88)8_s v_a \rangle \times \langle v \| A \| v' \rangle_\gamma \langle v' \| A \| v \rangle_{\gamma'} = (C_\kappa^{(3)} + C_\lambda^{(3)} - \tfrac{1}{4} C_v^{(3)}) / \sqrt{C_v^{(2)}}, \tag{42}$$

where again $C_\kappa^{(3)} + C_\lambda^{(3)}$ must be replaced by 0 if the representation v is not contained in $[\kappa,\lambda]$.

We now have a sufficient number of relations and may proceed to an explicit solution of the algebra.

4. EXPLICIT SOLUTION OF THE $SU(3)\otimes SU(3)$ ALGEBRA

Since $\mathbf{V} = \mathbf{K} + \mathbf{L}$, the representations v that are contained in $[\kappa,\lambda]$ are those that occur in the reduction of the direct product $\kappa\otimes\lambda$. An arbitrary representation $[\kappa,\lambda]$ may be specified by the four integers p_κ, q_κ, p_λ, q_λ; if however $[\kappa,\lambda]$ is to contain $v = \mathbf{8}$ then certain relations must be satisfied by these four integers. The desired representations can be grouped into seven classes,[17] specified within each class by just two integers:

$$\text{Class I:} \quad \kappa = (m+2, n-1), \quad \lambda = (n,m), \quad n \neq 0; \tag{43}$$

$$\text{Class II:} \quad \kappa = (m-1, n+2), \quad \lambda = (n,m), \quad m \neq 0; \tag{44}$$

$$\text{Class III:} \quad \kappa = (m+1, n+1), \quad \lambda = (n,m); \tag{45}$$

$$\text{Class IV:} \quad \kappa = (m,n), \quad \lambda = (n,m); \tag{46}$$

here m, n are arbitrary non-negative integers. Three more classes are obtained from classes I, II, and III by mirroring (the mirror of class IV is again class IV).

The quantities of interest are D/F, G_A, and G^* defined by

$$D/F = \frac{\begin{pmatrix} 8 & 8 & \big| & 8_s \\ \tfrac{1}{2}\,1 & 1\,0 & \big| & \tfrac{1}{2}\,1 \end{pmatrix} \langle 8 \| A \| 8 \rangle_s}{\begin{pmatrix} 8 & 8 & \big| & 8_a \\ \tfrac{1}{2}\,1 & 1\,0 & \big| & \tfrac{1}{2}\,1 \end{pmatrix} \langle 8 \| A \| 8 \rangle_a} = \frac{3}{\sqrt{5}} \frac{\langle 8 \| A \| 8 \rangle_s}{\langle 8 \| A \| 8 \rangle_a}, \tag{47}$$

$$G_A = \left\{ \langle 8 \| A \| 8 \rangle_a + \frac{3}{\sqrt{5}} \langle 8 \| A \| 8 \rangle_s \right\} \Big/ \sqrt{3}, \tag{48}$$

$$G^* = (\tfrac{4}{3})^{1/2} \begin{pmatrix} 8 & 8 & \big| & 10 \\ \tfrac{1}{2}\,1 & 1\,0 & \big| & \tfrac{3}{2}\,1 \end{pmatrix} \langle 10 \| A \| 8 \rangle = -(\tfrac{2}{3})^{1/2} \langle 10 \| A \| 8 \rangle. \tag{49}$$

These definitions specify the quantities in question unambiguously provided that the representations $\mathbf{8}$ and $\mathbf{10}$ occur in a given $[\kappa,\lambda]$ no more than once. The representation $\mathbf{10}$, being triangular, can never occur more than once; the representation $\mathbf{8}$ can never occur more than twice.[18] Moreover, in the classes I, II, and III the octet in fact occurs just once.[17] By solving Eqs. (31), (32), (34), and (42) for $v = \mathbf{8}$, we then find for

[16] There seems to be no unanimity in the literature on the definition of these Casimir operators. For reference we give the connection between our $\mathbf{V}^2$ and $\mathbf{V}^3$ and the M_2 and M_3 of Okubo [S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962)]:

$$\mathbf{V}^2 = \tfrac{1}{2} M_2; \quad \mathbf{V}^3 = 6M_3 - 9M_2.$$

[17] D. Lurie and A. J. Macfarlane, J. Math. Phys. **5**, 565 (1964).

[18] These are special cases of the statement that the maximum multiplicity M of an $SU(3)$ representation (p,q) is given by $M = 1 + \min(p,q)$. See, for example, S. Bergia and K. Zalewski, Nuovo Cimento **44**, 542 (1966), and references cited therein.

these classes (for details see Appendix)

$$D/F=\tfrac{1}{10}(C_\kappa^{(3)}+C_\lambda^{(3)})(C_\kappa^{(2)}-C_\lambda^{(2)})^{-2}, \tag{50}$$

$$G_A=\tfrac{1}{3}[C_\kappa^{(2)}-C_\lambda^{(2)}+\tfrac{1}{10}(C_\kappa^{(3)}+C_\lambda^{(3)})/(C_\kappa^{(2)}-C_\lambda^{(2)})], \tag{51}$$

$$G^{*2}=(4/15)[C_\kappa^{(2)}+C_\lambda^{(2)}-\tfrac{1}{3}(C_\kappa^{(2)}-C_\lambda^{(2)})^2+(C_\kappa^{(3)}+C_\lambda^{(3)})/18]$$
$$=(2/45)\{[C_\kappa^{(2)}-C_\lambda^{(2)}+\tfrac{1}{6}(C_\kappa^{(3)}+C_\lambda^{(3)})/(C_\kappa^{(2)}-C_\lambda^{(2)})]^2-9\}. \tag{52}$$

Quite explicitly then, in terms of the integers m and n that determine the $[\kappa,\lambda]$ representation, we have

Class I:

$$D/F=\tfrac{3}{5}(m+2n+3)/(m+2), \tag{53}$$

$$G_A=\tfrac{1}{3}(m+2)+\tfrac{1}{5}(m+2n+3), \tag{54}$$

$$G^{*2}=(8/45)(m+n+1)(m+n+4). \tag{55}$$

The $[\mathbf{6,3}]$ representation belongs to this class and corresponds to $m=0$, $n=1$, which gives the famous $SU(6)$ numbers $D/F=\frac{3}{2}$, $G_A=5/3$, $G^{*2}=(4/3)^2$.

Class II:

$$D/F=-\tfrac{3}{5}(n+2m+3)/(n+2), \tag{56}$$

$$G_A=\tfrac{1}{3}(n+2)-\tfrac{1}{5}(n+2m+3), \tag{57}$$

$$G^{*2}=(8/45)(m+2)(m-1). \tag{58}$$

We note that (aside from relabeling of m and n) class-II representations are conjugate to class I and thus have D/F ratios equal in magnitude and opposite in sign.

Class III:

$$D/F=\tfrac{3}{5}(m-n)/(m+n+3), \tag{59}$$

$$G_A=\tfrac{1}{3}(m+n+3)+\tfrac{1}{5}(m-n), \tag{60}$$

$$G^{*2}=(8/45)m(m+3). \tag{61}$$

For the representations that are mirrors of the above the same results hold with G_A replaced by $-G_A$.

We divide class-IV representations into two subclasses: class IV_1 containing just one octet, which happens when $\kappa=\lambda^*$ is triangular, and class IV_2 containing two octets.

Class IV_1: ($n=0$ or $m=0$)

$$D/F=\infty, \tag{62}$$

$$G_A=\tfrac{1}{10}C_\kappa^{(3)}/C_\kappa^{(2)}=\tfrac{1}{5}(2m+3) \quad \text{if} \quad n=0$$
$$=-\tfrac{1}{5}(2n+3) \quad \text{if} \quad m=0, \tag{63}$$

$$G^*=0. \tag{64}$$

Class IV_2: ($n\neq0$, $m\neq0$)

$$(D/F)_1=(D/F)_2=\infty, \tag{65}$$

$$(G_A)_1=-(G_A)_2=\tfrac{1}{5}[(2m+2n+3)^2-4mn]^{1/2}, \tag{66}$$

$$(G^{*2})_{1,2}=(2/15)\{2m+2n+\tfrac{2}{3}(m^2+n^2+mn)\mp[6(m+n+1)(m-n)+2(m^2n-mn^2)+\tfrac{4}{3}(m^3-n^3)]\times[(2m+2n+3)^2-4mn]^{-1/2}\}. \tag{67}$$

Here the subscript 1 and 2 refers to the two octets $\mathbf{8}_1$ and $\mathbf{8}_2$ which are defined by the requirement that they not be mixed by the axial coupling: $\langle 8_1\|A\|8_2\rangle_s=0$. Since $\langle 8_1\|A\|8_1\rangle=-\langle 8_2\|A\|8_2\rangle$ for any choice of 8_1 and 8_2, the identification is made complete by choosing $\langle 8_1\|A\|8_1\rangle>0$.

5. CONCLUSIONS

From the explicit formulas for G_A given in the preceding section it is clear that the experimental value can never be obtained if the nucleon is assigned to an octet from any one irreducible representation of $SU(3)\otimes SU(3)$ except possibly for class-IV_2 representations. In that case if we identify the nucleon with either $\mathbf{8}_1$ or $\mathbf{8}_2$ the above conclusion still holds but we have the option of taking some mixture of $\mathbf{8}_1$ and $\mathbf{8}_2$. However for all class-IV representations the F-type coupling vanishes and so these representations do violence to the experimental value of the D/F ratio.

We may remark that at the $SU(2)$ level the same conclusion for G_A follows even more transparently. The comparison of the results for G_A and G^* at the $SU(2)$ and $SU(3)$ level leads us to a remark which may well be obvious to experts in the field. Since $SU(2)$ is a subgroup of $SU(3)$, one might be tempted to "derive" $SU(3)$ results by dealing with the simpler $SU(2)$. After all G_A, G^* refer to nonstrange particles and $SU(2)$ should be sufficient. However when we compare the results for G_A, G^* at the $SU(2)$ and $SU(3)$ levels we do not find, in general, the same expressions. In other words an irreducible representation of $SU(3)\otimes SU(3)$ when reduced with respect to its $SU(2)\otimes SU(2)$ subgroup gives rise, in general, to a reducible representation. An exception is the $[\mathbf{6,3}]$ representation of $SU(3)\otimes SU(3)$, but this is just due to its low dimensionality.

Conversely, since it is clear that we need a reducible representation of $SU(3)\otimes SU(3)$ we should perhaps look to a higher group having $SU(3)\otimes SU(3)$ as a subgroup. Then an irreducible representation of that group would, in general, be reducible at the $SU(3)$ level. Such an approach would avoid the introduction of arbitrary mixing parameters.

APPENDIX

When the fact that a given representation υ may occur in the product $\kappa\otimes\lambda$ more than once is taken into account Eqs. (31), (32), (34), and (42) become (for brevity we use in the Appendix the notation $\langle v_\alpha \| v_\beta' \rangle_\gamma$ for the reduced matrix element $\langle v_\alpha \| A \| v_\beta' \rangle_\gamma$)

$$\langle v_\alpha \| v_\beta \rangle_a = (C_\kappa^{(2)} - C_\lambda^{(2)})(C_v^{(2)})^{-1/2} \delta_{\alpha,\beta}; \qquad \text{(A1)}$$

$$\sum_{\alpha'} \{ \langle v_\alpha \| v_{\alpha'} \rangle_s + \langle v_\beta \| v_{\alpha'} \rangle_s + \sum_{v' \neq v} \langle v_\alpha \| v'_{\alpha'} \rangle \langle v_\beta \| v'_{\alpha'} \rangle \}$$
$$= \{ 2C_\kappa^{(2)} + 2C_\lambda^{(2)} - C_v^{(2)} - (C_\kappa^{(2)} - C_\lambda^{(2)})^2 / C_v^{(2)} \} \delta_{\alpha,\beta}; \qquad \text{(A2)}$$

$$\sum_{v'\alpha'\gamma'\gamma''} \langle (v''8) v'_{\gamma''} 8 v_{\gamma'} | v''(88) N_\gamma v_{\gamma'''} \rangle \times \langle v_\alpha \| v'_{\alpha'} \rangle_{\gamma'} \langle v'_{\alpha'} \| v''_\beta \rangle_{\gamma''}$$
$$= (\tfrac{1}{2}\sqrt{3})(C_v^{(2)})^{1/2} \delta_{v,v''} \delta_{\alpha,\beta} \delta_{N\gamma, 8_a} \delta_{\gamma''', a},$$
$$N_\gamma = 8_a,\ \mathbf{10},\ \mathbf{10^*}; \qquad \text{(A3)}$$

$$3(\sqrt{15}) \sum_{v'\alpha'\gamma\gamma'} \langle (v8) v'_{\gamma'} 8 v_\gamma | v(88) 8_s v_a \rangle \times \langle v_\alpha \| v'_{\alpha'} \rangle_\gamma \langle v'_{\alpha'} \| v_\beta \rangle_{\gamma'}$$
$$= (C_\kappa^{(3)} + C_\lambda^{(3)} - \tfrac{1}{4} C_v^{(3)})(C_v^{(2)})^{-1/2} \delta_{\alpha,\beta}. \qquad \text{(A4)}$$

Here the subscripts α, β serve to distinguish the representations that occur more than once. As always, the Casimir invariants with the subscript κ or λ must be replaced by zero if the representation v or v'' does not occur in $\kappa \otimes \lambda$. We know that, by construction, the octet always is present. Hence if we set $v = v'' = 8$ we get from Eqs. (A2)–(A4):

$$\sum_{\alpha'} (\langle 8_\alpha \| 27_{\alpha'} \rangle \langle 8_\beta \| 27_{\alpha'} \rangle + \langle 8_\alpha \| 8_{\alpha'} \rangle_s \langle 8_\beta \| 8_{\alpha'} \rangle_s) + \langle 8_\alpha \| 10 \rangle \langle 8_\beta \| 10 \rangle + \langle 8_\alpha \| 10^* \rangle \times \langle 8_\beta \| 10^* \rangle + \langle 8_\alpha \| 1 \rangle \langle 8_\beta \| 1 \rangle$$
$$= \{ 2C_\kappa^{(2)} + 2C_\lambda^{(2)} - 3 - (C_\kappa^{(2)} - C_\lambda^{(2)})^2/3 \} \delta_{\alpha,\beta}, \qquad \text{(A5)}$$

$$\sum_{\alpha'} \{ (\tfrac{1}{2}\sqrt{6}) \langle 8_\alpha \| 27_{\alpha'} \rangle \langle 27_{\alpha'} \| 8_\beta \rangle + \langle 8_\alpha \| 8_{\alpha'} \rangle_s \langle 8_{\alpha'} \| 8_\beta \rangle_s \}$$
$$- \tfrac{1}{2}\sqrt{2} \langle 8_\alpha \| 1 \rangle \langle 1 \| 8_\beta \rangle = \{ 3 - \tfrac{1}{3}(C_\kappa^{(2)} - C_\lambda^{(2)})^2 \} \delta_{\alpha,\beta}, \qquad \text{(A6)}$$

$$\sum_{\alpha'} \{ (\tfrac{1}{8}\sqrt{6}) \langle 8_\alpha \| 27_{\alpha'} \rangle \langle 27_{\alpha'} \| 8_\beta \rangle - \langle 8_\alpha \| 8_{\alpha'} \rangle_s \langle 8_{\alpha'} \| 8_\beta \rangle_s \}$$
$$+ (\tfrac{1}{4}\sqrt{5}) \{ \langle 8_\alpha \| 10^* \rangle \langle 10^* \| 8_\beta \rangle - \langle 8_\alpha \| 10 \rangle \langle 10 \| 8_\beta \rangle \}$$
$$- (5\sqrt{2}/8) \langle 8_\alpha \| 1 \rangle \langle 1 \| 8_\beta \rangle = 0, \qquad \text{(A7)}$$

$$(2/\sqrt{3})(C_\kappa^{(2)} - C_\lambda^{(2)}) \langle 8_\alpha \| 8_\beta \rangle_s + \langle 8_\alpha \| 10 \rangle \langle 10 \| 8_\beta \rangle + \langle 8_\alpha \| 10^* \rangle \langle 10^* \| 8_\beta \rangle = 0, \qquad \text{(A8)}$$

$$(2/\sqrt{3})(C_\kappa^{(2)} - C_\lambda^{(2)}) \langle 8_\alpha \| 8_\beta \rangle_s - \langle 8_\alpha \| 10 \rangle \langle 10 \| 8_\beta \rangle - \langle 8_\alpha \| 10^* \rangle \langle 10^* \| 8_\beta \rangle$$
$$= (2/9\sqrt{5})(C_\kappa^{(3)} + C_\lambda^{(3)}) \delta_{\alpha,\beta}. \qquad \text{(A9)}$$

From the reality of the matrix elements, the Hermitian nature of $\mathbf{A}$, and the de Swart phase conventions it follows that

$$\langle N_\alpha \| M_\beta \rangle_\gamma = \langle M_\beta \| N_\alpha \rangle_\gamma (M/N)^{1/2} \times \xi_1(8NM_\gamma) \xi_1(8MN_\gamma) \xi_2(8NM_\gamma) \xi_3(8MN_\gamma), \qquad \text{(A10)}$$

so that

$$\langle 1 \| 8_\beta \rangle = -2\sqrt{2} \langle 8_\beta \| 1 \rangle,$$
$$\langle 8_\alpha \| 8_\beta \rangle_\gamma = \langle 8_\beta \| 8_\alpha \rangle_\gamma,$$
$$\langle 10 \| 8_\beta \rangle = -(2/\sqrt{5}) \langle 8_\beta \| 10 \rangle,$$
$$\langle 10^* \| 8_\beta \rangle = (2/\sqrt{5}) \langle 8_\beta \| 10^* \rangle,$$
$$\langle 27_\alpha \| 8_\beta \rangle = -\frac{2\sqrt{2}}{3\sqrt{3}} \langle 8_\beta \| 27_\alpha \rangle.$$

Using these relations in Eqs. (A5)–(A9), and including Eq. (A1) evaluated for $v = 8$, we get after some simple manipulations

$$\langle 8_\alpha \| 8_\beta \rangle_a = (C_\kappa^{(2)} - C_\lambda^{(2)}) 3^{-1/2} \delta_{\alpha,\beta}, \qquad \text{(A11)}$$

$$\sum_{\alpha'} \langle 8_\alpha \| 8_{\alpha'} \rangle_s \langle 8_\beta \| 8_{\alpha'} \rangle_s = \tfrac{1}{5} [2(C_\kappa^{(2)} + C_\lambda^{(2)}) + 3 - (C_\kappa^{(2)} - C_\lambda^{(2)})^2] \delta_{\alpha,\beta}, \qquad \text{(A12)}$$

$$(6\sqrt{15})(C_\kappa^{(2)} - C_\lambda^{(2)}) \langle 8_\alpha \| 8_\beta \rangle_s = (C_\kappa^{(3)} + C_\lambda^{(3)}) \delta_{\alpha,\beta}, \qquad \text{(A13)}$$

$$\langle 8_\alpha \| 10 \rangle \langle 8_\beta \| 10 \rangle - \langle 8_\alpha \| 10^* \rangle \langle 8_\beta \| 10^* \rangle = (1/18)(C_\kappa^{(3)} + C_\lambda^{(3)}) \delta_{\alpha,\beta}, \qquad \text{(A14)}$$

$$\langle 8_\alpha \| 10 \rangle \langle 8_\beta \| 10 \rangle + \langle 8_\alpha \| 10^* \rangle \langle 8_\beta \| 10^* \rangle + 4 \langle 8_\alpha \| 1 \rangle \langle 8_\beta \| 1 \rangle = [C_\kappa^{(2)} + C_\lambda^{(2)} - \tfrac{1}{3}(C_\kappa^{(2)} - C_\lambda^{(2)})^2] \delta_{\alpha,\beta}, \qquad \text{(A15)}$$

$$\sum_{\alpha'} \langle 8_\alpha \| 27_{\alpha'} \rangle \langle 8_\beta \| 27_{\alpha'} \rangle - 3 \langle 8_\alpha \| 1 \rangle \langle 8_\beta \| 1 \rangle = \tfrac{1}{5} [3(C_\kappa^{(2)} + C_\lambda^{(2)}) - 18 + (C_\kappa^{(2)} - C_\lambda^{(2)})^2] \delta_{\alpha,\beta}. \qquad \text{(A16)}$$

The representations belonging to classes I, II, and III, and their mirrors, are distinguished by the fact that they contain just one octet and no singlet. The solution for these classes follows immediately from the above equations:

$$\langle 8 \| 1 \rangle = 0, \qquad \text{(A17)}$$

$$\langle 8 \| 8 \rangle_a = (C_\kappa^{(2)} - C_\lambda^{(2)})/\sqrt{3}, \qquad \text{(A18)}$$

$$\langle 8 \| 8 \rangle_s = (C_\kappa^{(3)} + C_\lambda^{(3)}) \times [(6\sqrt{15})(C_\kappa^{(2)} - C_\lambda^{(2)})]^{-1}, \qquad \text{(A19)}$$

$$\langle 8 \| 10 \rangle^2 = \tfrac{1}{2}(C_\kappa^{(2)} + C_\lambda^{(2)}) - \tfrac{1}{6}(C_\kappa^{(2)} - C_\lambda^{(2)})^2 + (1/36)(C_\kappa^{(3)} + C_\lambda^{(3)}), \qquad \text{(A20)}$$

$$\langle 8 \| 10^* \rangle^2 = \tfrac{1}{2}(C_\kappa^{(2)} + C_\lambda^{(2)}) - \tfrac{1}{6}(C_\kappa^{(2)} - C_\lambda^{(2)})^2 - (1/36)(C_\kappa^{(3)} + C_\lambda^{(3)}), \qquad \text{(A21)}$$

$$\sum_\alpha \langle 8 \| 27_\alpha \rangle^2 = \tfrac{1}{5} [3(C_\kappa^{(2)} + C_\lambda^{(2)}) - 18 + (C_\kappa^{(2)} - C_\lambda^{(2)})^2]. \qquad \text{(A22)}$$

We note that for these classes, $C_\kappa^{(2)}-C_\lambda^{(2)}$ can not vanish so that Eq. (A19) is meaningful; we also remark that Eq. (A12) is superfluous—combined with Eq. (A13) it results in a relation among the Casimir invariants which is an identity within the classes under consideration.

Combining appropriately Eqs. (A18)–(20) we obtain Eqs. (50)–(52) in the text.

Class-IV representations contain the singlet and contain, in general, two octets. Moreover for this class we have (since $\kappa=\lambda^*$):

$$C_\kappa^{(2)}-C_\lambda^{(2)}=0\,,\qquad C_\kappa^{(3)}+C_\lambda^{(3)}=0\,,\tag{A23}$$

and Eq. (A13) becomes a useless identity. However now we may obtain additional equations by taking $\upsilon=1$ in Eqs. (A2) and (A3) [Eqs. (A1) and (A4) are identities for $\upsilon=1$]. Moreover we may replace the cubic invariant equation (A4) by an equation for $\mathbf{K}^3-\mathbf{L}^3$ evaluated in the singlet state. This combination of cubic operators contains a term trilinear in $\mathbf{A}$ and was therefore not considered before since when evaluated in the octet state it would contain as intermediate states representations as complicated as those occurring in $8\otimes8\otimes8$. Here, however, we evaluate it in the singlet state so that the intermediate states must be octets only.

The additional equations that are obtained in this way are

$$\sum_\alpha \langle 1\|8_\alpha\rangle^2=4C_\kappa^{(2)}\,,\tag{A24}$$

$$\sum_\alpha \langle 10\|8_\alpha\rangle\langle 1\|8_\alpha\rangle=0\,,\tag{A25}$$

$$\sum_\alpha \langle 10^*\|8_\alpha\rangle\langle 1\|8_\alpha\rangle=0\,,\tag{A26}$$

$$\sum_{\alpha\beta} \langle 1\|8_\alpha\rangle\langle 8_\alpha\|8_\beta\rangle_s\langle 1\|8_\beta\rangle=(2/\sqrt{15})C_\kappa^{(3)}.\tag{A27}$$

We divide class-IV representations into two subclasses, IV_1 and IV_2.

Class IV_1: one octet only. Although class-IV representations contain in general two octets if $\kappa=\lambda^*$ is triangular [i.e., $\kappa=(p,q)$ has either $p=0$ or $q=0$] then we get only one octet.[17] We easily find the solution:

$$\langle 8\|1\rangle^2=\tfrac{1}{2}C_\kappa^{(2)}\,,\tag{A28}$$

$$\langle 8\|8\rangle_a=0\,,\tag{A29}$$

$$\langle 8\|8\rangle_s=C_\kappa^{(3)}[(2\sqrt{15})C_\kappa^{(2)}]^{-1}\,,\tag{A30}$$

$$\langle 8\|10\rangle=\langle 8\|10^*\rangle=0\,,\tag{A31}$$

$$\sum_\alpha \langle 8\|27_\alpha\rangle^2=(9/10)(3C_\kappa^{(2)}-4)\,.\tag{A32}$$

Here again one of the three equations, Eqs. (A12), (A24), (A27), is superfluous; they may be combined to yield a relation among the Casimir invariants which is an identity within this class.

From Eqs. (A29)–(A31) we obtain Eqs. (62)–(64) in the text.

Class IV_2: two octets. If $\kappa=\lambda^*$ is not triangular we have two octets. These two octets may be identified in a variety of ways. In the present context the most reasonable choice seems to be to require that $\mathbf{A}$ be diagonal within the subspace of the two octets, i.e., that they not be mixed by the axial coupling. So we define $\mathbf{8}_1$ and $\mathbf{8}_2$ by

$$\langle 8_1\|8_2\rangle_s=0\,.\tag{A33}$$

With this definition we get the solution

$$\langle 8_\alpha\|8_\beta\rangle_a=0\quad \alpha,\beta=1,2\,,\tag{A34}$$

$$\langle 8_1\|8_1\rangle_s=(\tfrac{4}{5}C_\kappa^{(2)}+\tfrac{3}{5})^{1/2}\,,\tag{A35}$$

$$\langle 8_2\|8_2\rangle_s=-(\tfrac{4}{5}C_\kappa^{(2)}+\tfrac{3}{5})^{1/2}\,,\tag{A36}$$

$$\langle 8_1\|10\rangle^2=\langle 8_1\|10^*\rangle^2=2\langle 8_2\|1\rangle^2 =\tfrac{1}{4}\{2C_\kappa^{(2)}-C_\kappa^{(3)}(12C_\kappa^{(2)}+9)^{-1/2}\}\,,\tag{A37}$$

$$\langle 8_2\|10\rangle^2=\langle 8_2\|10^*\rangle^2=2\langle 8_1\|1\rangle^2 =\tfrac{1}{4}\{2C_\kappa^{(2)}+C_\kappa^{(3)}(12C_\kappa^{(2)}+9)^{-1/2}\}\,,\tag{A38}$$

$$\sum_\alpha \langle 8_1\|27_\alpha\rangle^2=(6/5)(C_\kappa^{(2)}-3)+\tfrac{3}{4}C_\kappa^{(2)} +\tfrac{3}{8}C_\kappa^{(3)}(12C_\kappa^{(2)}+9)^{-1/2}\,,\tag{A39}$$

$$\sum_\alpha \langle 8_2\|27_\alpha\rangle^2=(6/5)(C_\kappa^{(2)}-3)+\tfrac{3}{4}C_\kappa^{(2)} -\tfrac{3}{8}C_\kappa^{(3)}(12C_\kappa^{(2)}+9)^{-1/2}.\tag{A40}$$

From Eqs. (A34)–(A37) we get Eqs. (65)–(67) in the text.