Analyticity in Momentum Transfer and Short-Range Interactions

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Omnes's work on the relation between short-range forces and analyticity in the momentum transfer is reviewed and some corrections pointed out. A simpler and more physical proof of his results is given.

I. INTRODUCTION

ANALYTIC properties of scattering amplitudes in momentum transfer t have been studied for about ten years.[1] Within the last year, Martin,[2] using axiomatic field theory results and unitarity, has extended the domain of analyticity of the absorptive part for $\pi\pi$ scattering to include a circle of radius $4\mu^2$ centered at $t=0$. Unlike the Lehmann ellipse,[3] this domain does not shrink to $t=0$ as the energy becomes very large. This has enabled Martin[4] to establish the Froissart bound on total cross sections without assuming the Mandelstam representation.

We will follow a different path originally started by Omnes.[5] Until this recent paper by Omnes, the proofs of analyticity were not explicitly based on a simple physical picture. Omnes proposed that the underlying physics of this analyticity was the short-range nature of the forces in strong interactions. Essentially his idea is the following: Suppose one has short-range forces, and one considers the scattering of an incident wave packet with larger and larger impact parameter. Then, with some technical assumptions about the width of the packet, one expects that the amplitude of the scattered wave should decrease exponentially with increasing impact parameter.

The physical idea underlying the proof of analyticity properties is simple. Large impact parameters correspond semiclassically to high partial waves. A wave packet with average momentum k and impact parameter a contains mainly angular momenta near $L=ka$. Therefore, if the scattered wave decreases exponentially with a, one obtains an exponential decrease of the partial-wave scattering amplitude with large l, which leads to the desired domain of analyticity. In the proof supplied by Omnes, this simple picture is obscured. We shall use this picture to give a more direct proof of Omnes's results.

Omnes's paper, although basically correct, is somewhat imprecise. Under his assumptions, one might prove analyticity in a domain which is too large at low energies. Moreover, it is also necessary to make a somewhat stronger assumption than he made in order to establish his connection between short range forces and analyticity.

In Sec. II, we briefly review Omnes's work and point out some corrections to it. In Sec. III, we present what we feel is the simpler and physically more transparent derivation of his results. Section IV presents some conclusions.

II. OMNES'S WORK

Omnes begins by considering the scattering of an incident Gaussian wave packet with width b in configuration space, average momentum $\mathbf{k}$, and impact parameter $\mathbf{a}$ with $\mathbf{k}\cdot\mathbf{a}=0$. In momentum space the wave packet is

$$\phi(\mathbf{p},t=0)=\left(\frac{b}{\pi}\right)^{3/2} e^{-(\mathbf{p}-\mathbf{k})^2b^2/2}e^{-i\mathbf{p}\cdot\mathbf{a}}. \tag{1}$$

Omnes shows that in potential theory, the probability of scattering by a potential with an exponential tail $e^{-\mu r}$ decreases exponentially with the impact parameter $a=|\mathbf{a}|$ provided that the width increases with the impact parameter according to

$$b^2=\lambda a \tag{2}$$

for any positive λ. If b were independent of a, the spreading of the wave packet would make it overlap the scattering center. Since the spreading is small if the width of the packet is large, relation (2) suppresses the spreading of the packet. Since the width varies only as the square root of the impact parameter, it is still true that the bulk of the incident packet will lie outside the range of the forces.

Having established the exponential decrease of the scattered wave in potential theory, Omnes then imposes this condition on the physical situation and shows that this implies certain analytic properties of the absorptive part of the T matrix as a function of momentum trans-

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[1] See for example A. Martin, The XIIIth International Conference on High-Energy Physics, Berkeley, 1966 (unpublished).
[2] A. Martin, Nuovo Cimento **42**, 930 (1966).
[3] H. Lehmann, Nuovo Cimento **10**, 579 (1958).
[4] A. Martin, Phys. Rev. **129**, 1432 (1963).
[5] R. Omnes, Phys. Rev. **146**, 1123 (1966).

fer. His method of proof is somewhat involved, and we refer the reader to his paper.[5]

We have two objections to his derivation. First in the potential-theory case, consider the scattered wave ψ, which is related to the incident wave ϕ by

$$\psi = T\phi\,, \tag{3}$$

where T is the T matrix. Omnes showed that

$$\|\psi\|^2 = \|T\phi\|^2 < Ce^{-2\sigma a} \tag{4}$$

provided that

$$k\lambda \geqslant 1\,, \tag{5}$$

where

$$\sigma = \mu\{1+\mu\lambda - [(1+\mu\lambda)^2-1]^{1/2}\}\,. \tag{6}$$

For small λ, this reduces to

$$\sigma \approx \mu\,. \tag{7}$$

In formulating his condition for the physical case, he imposes a stronger rate of decrease $e^{-\mu a}$, and treats the case where

$$\lambda < \mu^{-1}\,. \tag{8}$$

This restriction is unnecessary. One can assume the bound given in (4) with σ given by (6) and λ restricted only by the constraint (5).

It is important to note that taking $\sigma=\mu$ independent of λ will lead to too large an analyticity domain for low energy. We shall comment on this in the next section.

The second objection is more serious. In fact, one cannot derive the required analyticity from condition (4), because any normalizable wave packet contains a superposition of various energies, so that (4) involves properties of the T matrix integrated over some energy range, whereas we wish to derive properties of the T matrix at a fixed energy.[6] It is sufficient to make the following stronger assumption: If $\Psi_\alpha(\mathbf{p}_1',\mathbf{p}_2',\cdots,\mathbf{p}_n')$ represents the scattered wave in the channel α, then

$$\int |\Psi_\alpha(\mathbf{p}_1',\cdots \mathbf{p}_n')|^2 \frac{d^3p_1'}{p_1'^0}\cdots\frac{d^3p_n'}{p_n'^0} \times\delta(E-p_1'^0\cdots - p_n'^0) < C_1 e^{-2\sigma a}. \tag{9}$$

Expressed in words, for *each fixed total energy*, the integral of the square of the scattered wave over all other variables decreases exponentially with increasing impact parameter. This then allows one to derive a condition on the absorptive part of the T matrix for a given energy.

One can justify (9) as follows. Let P_E^α denote the projection operator on states of total energy less than E in the channel α. Then the left side of (9) can be written as

$$(d/dE)\|P_E^\alpha T\phi\|^2. \tag{10}$$

(This exists as a measure in E, since $\|P_E^\alpha T\phi\|^2$ is monotonic in E.)

If $f(E)$ is an infinitely differentiable function of compact support defined for positive energies, then

$$\left|\int f(E)\frac{d}{dE}\|P_E^\alpha T\phi\|^2 dE\right| = \left|-\int \frac{df(E)}{dE}\|P_E^\alpha T\phi\|^2 dE\right|$$
$$\leq \max_E \|P_E^\alpha T\phi\|^2 \int \left|\frac{df(E)}{dE}\right| dE$$
$$\leq \|T\phi\|^2 \int \left|\frac{df(E)}{dE}\right| dE$$
$$< Ce^{-2\sigma a}\int \left|\frac{df(E)}{dE}\right| dE\,. \tag{11}$$

Consequently, as a distribution in E, (10) decreases exponentially with a. All subsequent equations involving the behavior of the scattering amplitude at a fixed energy are to be interpreted as a statement about distributions in the energy. We shall establish that $A(E,\cos\theta)$ is analytic in $\cos\theta$ provided that one integrates with some test function in E, where $A(E,\cos\theta)$ is the absorptive part of the T matrix, E the total energy, and θ the scattering angle. We will then assume that $A(E,\cos\theta)$ is a *continuous function* of E for fixed θ. Then it follows that for each energy, $A(E,\cos\theta)$ is analytic in $\cos\theta$, and therefore in t.

III. DERIVATION OF ANALYTICITY

In this section we combine the bound on the scattered wave with the physical ideas outlined in the introduction to produce a proof of the desired analytic properties.

At first we follow Omnes's discussion. Consider the reaction

$$a_1 + a_2 \rightarrow a_1' + \cdots + a_n'\,,$$

where the set of final particles is called the channel α. The scattered wave is

$$\psi_\alpha(\mathbf{p}_1'\cdots\mathbf{p}_n') = \int T_\alpha(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1'\cdots\mathbf{p}_n') \times\delta^4(p_1+p_2-p_1'\cdots - p_n')\psi_i(\mathbf{p}_1,\mathbf{p}_2)\frac{d^3p_1}{p_1^0}\frac{d^3p_2}{p_2^0}\,, \tag{12}$$

[6] Mathematically, Omnes's error (see the opening paragraph of his Sec. 8) comes from assuming that if two functions $f(x)$ and $g(x,y)$ satisfy

$$f(x) = \int g(x,y)k(y)dy,$$

and if $f(x)$ is analytic in a certain region, then $g(x,y)$ is analytic in x in the same region for fixed y. This is certainly wrong since adding $s(x)t(y)$ to $g(x,y)$ where $s(x)$ is arbitrary and

$$\int t(y)k(y)dy = 0$$

leaves $f(x)$ unchanged, but modifies the x dependence of $g(x,y)$ in an almost arbitrary manner.

where $\psi_i(\mathbf{p}_1,\mathbf{p}_2)$ is the incident wave, and $T_\alpha(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1'\cdots\mathbf{p}_n')$ is the appropriate T-matrix element. We take the incident wave to be

$$\psi_i(\mathbf{p}_1,\mathbf{p}_2)=(A^2/\pi)^{3/4}e^{-\mathbf{P}^2A^2/2}\phi(\mathbf{p}), \tag{13}$$

where

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2, \tag{14}$$

$$\mathbf{p}=\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2), \tag{15}$$

and

$$\phi(\mathbf{p})=(\lambda a/\pi)^{3/4}e^{-(\mathbf{p}-\mathbf{k})^2\lambda a/2}e^{-i\mathbf{p}\cdot\mathbf{a}}, \tag{16}$$

with

$$\mathbf{k}\cdot\mathbf{a}=0.$$

A will be taken large enough so that $|\psi_i(\mathbf{p}_1,\mathbf{p}_2)|^2$ is essentially a delta function of P.

The short-range hypothesis is formulated as

$$\int|\psi_\alpha(\mathbf{p}_1'\cdots\mathbf{p}_n')|^2\delta(E-p_1'^0\cdots-p_n'^0)\times\frac{d^3p_1'}{p_1'^0}\cdots\frac{d^3p_n'}{p_n'^0}<C_1e^{-2\sigma a}, \tag{17}$$

where σ is given in (6), and we take μ independent of the channel α.

We define of the absorptive part of the two-body scattering amplitude due to the intermediate state α,

$$A_\alpha(E,\cos\theta)=\int T_\alpha(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1'\cdots\mathbf{p}_n')T_\alpha^*(\mathbf{p}_1'',\mathbf{p}_2'';\mathbf{p}_1'\cdots\mathbf{p}_n')\times\delta^4(p_1+p_2-p_1'\cdots-p_n')\frac{d^3p_1'}{p_1'^0}\cdots\frac{d^3p_n'}{p_n'^0}, \tag{18}$$

with

$$E=p_1^0+p_2^0, \tag{19}$$

and where θ is the angle $\mathbf{p}$ and $\mathbf{p}''$ defined in (15). Then using (12), (17) can be written as

$$\int A_\alpha(E,\hat{p}\cdot\hat{p}'')\phi(\mathbf{p})\phi^*(\mathbf{p}'')d\Omega_p d\Omega_{p}''<C_3e^{-2\sigma a}, \tag{20}$$

where C_3 contains come unimportant energy-dependent terms and $d\Omega_p$ is the element of solid angle in the center-of-mass system. So far we have followed Omnes almost verbatim.

From (16) it is clear that unless $|p|=|k|$, the left side of (20) will decrease exponentially with a. Thus (20) is most significant if $|p|=|k|$ and we shall assume this from now on.

Let us now make a partial-wave expansion of $A_\alpha(E,\hat{p}\cdot\hat{p}'')$ and of $\phi(\mathbf{p})$. That is,

$$A_\alpha(E,\hat{p}\cdot\hat{p}'')=\sum_l A_l^\alpha(E)P_l(\hat{p}\cdot\hat{p}'')(2l+1)=4\pi\sum_{lm}A_l^\alpha(E)Y_l^{m*}(\hat{p})Y_l^m(\hat{p}''), \tag{21}$$

$$\phi(\mathbf{p})=\sum_{lm}d_{lm}(k,a)Y_l^m(\hat{p}). \tag{22}$$

Defining

$$f_l(k,a)=\sum_{m=-l}^{l}|d_{lm}(k,a)|^2, \tag{23}$$

(20) becomes

$$4\pi\sum_l A_l^\alpha(E)f_l(k,a)<C_3e^{-2\sigma a}. \tag{24}$$

By unitarity, each term on the left side of (24) is positive and so the inequality is strengthened if we replace the sum by a single term. Since we are interested in bounds on A_l, it would be useless to pick a term where f_l are very small. The partial-wave expansion of the incident packet given in the Appendix proves that for large l, f_l is maximized when $a\approx l/k$ and is given by

$$f_l\simeq\text{const}l/k^3. \tag{25}$$

The fact that this value of the impact parameter makes the corresponding partial wave large, confirms the semi-classical consideration given in the introduction. Substituting (25) into (24), we obtain

$$A_l^\alpha(E)<C_4e^{-2\sigma a}, \tag{26}$$

that is,

$$A_l^\alpha(E)<C_4e^{-2\sigma l/k}. \tag{27}$$

We have thus shown simply that the absorptive part of the partial-wave scattering amplitude decreases exponentially for large l.

This bound allows us to conclude that the Legendre expansion of the absorptive part (21) converges to an analytic function in the $\cos\theta$ plane in an ellipse with foci at ±1, and with semimajor axis r where[7]

$$r=\tfrac{1}{2}(e^{2\sigma/k}+e^{-2\sigma/k}). \tag{28}$$

Recalling that

$$t=2k^2(\cos\theta-1), \tag{29}$$

the absorptive part is analytic in t in an ellipse with apex

$$t_{\max}=k^2(e^{2\sigma/k}+e^{-2\sigma/k}-2), \tag{30}$$

and foci at 0 and $-4k^2$.

From (6), one can see that σ is maximized for

[7] From (27) it is clear that $\sum A_l^\alpha(E)x^l$ converges for $|x|<e^{2\sigma/k}$. The rest follows from the relation between power series and Legendre expansions. See for example E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1952), 4th ed., p. 323.

minimum λ. Choosing λ minimum subject to (5), that is

$$\lambda = 1/k, \tag{31}$$

one sees that for large k, σ approaches μ, and $t_{\max}$ approaches $4\mu^2$.

What we have done is to compute the analyticity domain of that portion of the absorptive part which comes from the intermediate channel α. At a given energy, only a finite number of channels are open, each having at least this analyticity domain, and therefore the full absorptive part will have this analyticity domain.

Thus the assumption of short-range forces in the form (17) has led to a significant part of the analyticity domain which Martin[2] had obtained from axiomatic field theory.

For small k,

$$2\sigma/k \approx 1, \tag{32}$$

and $t_{\max}$ approaches 0. If σ were independent of k for small k, the analyticity domain would extend to

$$t_{\max} \sim k^2 e^{2\sigma/k}, \tag{33}$$

which approaches infinity much too quickly to be reasonable. For example, analyticity in so large a domain does not hold for scattering by Yukawa potentials.

IV. CONCLUSIONS

We have shown that a slightly strengthened form of Omnes's short-range-force condition leads easily and in a physically transparent way to analyticity of the absorptive part of the two-body scattering amplitude.

To extend this proof to the full partial-wave scattering amplitude T_l one has only to consider unitarity. Since unitarity implies

$$A_l(E) \geqslant |T_l(E)|^2, \tag{34}$$

our bound on the absorptive part is also a bound on the full elastic amplitude. Using the same proof as in Sec. III it follows that the full elastic scattering amplitude is analytic in the $\cos\theta$ plane in our ellipse with foci at ± 1 and a semimajor axis given by

$$Y = \tfrac{1}{2}(e^{\sigma/k} + e^{-\sigma/k}). \tag{35}$$

For large energy this corresponds to analyticity in the t plane in an ellipse with apex at $t = \mu^2$. Analyticity in any larger ellipse would exclude the possibility of a pole at $t = \mu^2$, and is therefore undesirable.

The proof we have given can be reversed. From the analyticity of $A(E, \cos\theta)$ or $T(E, \cos\theta)$, one can prove[8] that $A_l(E)$ or $T_l(E)$ decreases exponentially with l, for large enough l. Substituting this into our formalism one obtains an exponential falloff of the scattered wave for large impact parameters. The fall off coefficient may, however, differ from σ.

[8] A. Martin, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorhouse (Oliver and Boyd, London, 1964).

In comparing the presented physical approach with that of axiomatic field theory, it should be pointed out that the formulation of the short-range condition in Eq. (17), especially the form of σ given in Eq. (6), is somewhat arbitrary. The form of σ we have used was obtained from crude estimates in potential theory which can be improved. Any form of σ, however, will be an *ad hoc* assumption in relativistic theory. In axiomatic field theory the short range of forces is built in as a condition on the allowed mass spectrum which is not an *ad hoc* assumption.

The choice of σ in Eq. (6) has the virtue of being rather simple but it leads to an analyticity domain which is smaller than the Lehmann ellipse for small energies. One could, however, choose σ so as to reproduce the axiomatic field theory results. Such a complicated choice for σ would run counter to the main purpose of this approach which is to provide a simple but intuitive basis from which analyticity properties can be understood.

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APPENDIX

Consider the wave packet

$$\phi(\mathbf{p}) = (\lambda a/\pi)^{3/4} e^{-(p-k)^2 \lambda a/2} e^{-i\mathbf{p}\cdot\mathbf{a}}, \tag{A1}$$

where $|p| = |k|$ and $\mathbf{k}\cdot\mathbf{a} = 0$. We expand ϕ into partial waves

$$\phi(\mathbf{p}) = \sum_{lm} d_{lm}(k,a) Y_l^m(\hat{p}). \tag{A2}$$

We wish to show that, for large a,

$$f_l(k,a) = \sum_{m=-l}^{l} |d_{lm}(k,a)|^2 \tag{A3}$$

is maximized for $l \approx ka$, and then f_l is of order l. From (A2),

$$d_{lm}(k,a) = \int \phi(\mathbf{p}) Y_l^{m*}(\hat{p}) d\Omega p. \tag{A4}$$

To evaluate (A4), we choose $\mathbf{k}$ along the z axis, $\mathbf{a}$ along the x axis, and take $\lambda = 1/k$. The last choice is a matter of convenience and the same results can be obtained without it.

We now use the following formulas[9]:

$$Y_l^m(\hat{p})=\left(\frac{2l+1}{4\pi}\right)^{1/2}\left(\frac{(l-m)!}{(l+m)!}\right)^{1/2} \times(-1)^{l+m}e^{im\phi}P_l^m(\cos\theta)\,, \quad \text{(A5)}$$

$$J_m(ka\sin\theta)=\frac{1}{2\pi}\int_0^{2\pi}e^{+im\phi}e^{-ika\sin\theta\sin\phi}d\phi\,, \quad \text{(A6)}$$

$$e^{ka\cos\theta}J_m(ka\sin\theta) =(-1)^m\sum_{n=0}^{\infty}\frac{(ka)^{m+n}}{(2m+n)!}P_{n+m}^m(\cos\theta)\,, \quad \text{(A7)}$$

and one finds that

$$|d_{lm}(k,a)|^2=4\pi\left(\frac{a}{\pi k}\right)^{3/2}\frac{(ka)^{2l}}{(l+m)!(l-m)!}e^{-2ka}. \quad \text{(A8)}$$

[9] For (A6) and (A7) see G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1952), 2nd ed., pp. 31, 149, and the *Batemen Manuscript Project* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II, p. 182.

Since

$$\sum_{m=-l}^{l}\binom{2l}{l+m}=2^{2l}\,, \quad \text{(A9)}$$

we have

$$f_l(k,a)=4\pi\left(\frac{a}{\pi k}\right)^{3/2}\frac{(2ka)^{2l}}{(2l)!}e^{-2ka}. \quad \text{(A10)}$$

For large l, Stirling's formula yields the estimate

$$f_l(k,a)\approx4\pi\left(\frac{a}{\pi k}\right)^{3/2}\frac{(2ka)^{2l}}{(2l)^{2l}e^{-2l}(2\pi 2l)^{1/2}}e^{-2ka}. \quad \text{(A11)}$$

For fixed l and k, f_l is maximized when

$$a=(l+\tfrac{3}{4})/k\,, \quad \text{(A12)}$$

and then

$$f_l\approx \text{const}\,l/k^3\,, \quad \text{(A13)}$$

which completes the proof.

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Interpretation of the N^* Effect in Deuteron Compton Scattering*

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Calculations of deuteron Compton scattering based on impulse methods demonstrate a peaking of the energy distributions due to the N^* pole in the factored nucleon amplitude. It is suggested here that the factorization procedure is questionable when a pole exists in the factored amplitude, as is evidenced, for example, by the failure of the procedure near threshold where the nucleon pole term is of importance. This difficulty is obviated in this paper by correctly treating the N^* as an intermediate state. It is shown that there exists a singularity which extends into the so-called anomalous region, very close to the physical scattering domain. This Landau singularity, manifested in a diagram having four propagators, has the effect of simulating a resonance-like behavior just above the N^*-nucleon threshold. However, this "resonance" has the interesting properties that as the deuteron momentum transfer increases, its effective width enlarges, while the peak height substantially diminishes. Using the dominance of the above-mentioned singularity as the basis for a computation, an expression for the deuteron Compton differential cross section was derived. To avoid ambiguities inherent in the spin case, scalar particles were used. A comparison with the limited experimental data available above the photopion threshold produced very encouraging results. However, to further clarify the manner in which the N^* manifests itself, it is suggested that attempts be made to extend the experiments (1) to a photon lab momentum of at least 350 MeV/c (the expected peak value) and (2) to the center-of-mass forward hemisphere, where the cross sections are anticipated to be both appreciably increased and more sharply peaked in the vicinity of the "N^*."

I. INTRODUCTION

THEORETICAL treatments of deuteron Compton scattering have been limited to impulse-approximation calculations.[1,2] In practice the deuteron amplitude is written as the product of the nucleon amplitude and a "sticking factor."[3] The manner in which this factorization is to be carried out is, however, still uncertain. Ambiguity related to the choice of nucleon momentum is just one of the difficulties. In any case, the process of factorization does not appear to be justifiable when the nucleon amplitude is dominated by

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[1] R. H. Capps, Phys. Rev. **106**, 1031 (1957), and references contained therein; R. H. Capps, *ibid.* **108**, 1032 (1957); M. Jacob and J. Mathews, *ibid.* **117**, 854 (1960); V. K. Fedyanin, Zh. Eksperim. i Teor. Phys. **42**, 1038 (1962) [English transl.: Soviet Phys.—JETP **15**, 720 (1961)].

[2] J. D. Fox, Ph.D. thesis, Washington University, 1964 (unpublished).

[3] G. F. Chew, Phys. Rev. **84**, 710 (1951); R. E. Cutkosky, in *Proceedings of the Tenth Annual International Conference on High-Energy Physics at Rochester, 1960*, edited by E. C. G. Sudarshan, J. H. Tincot, and A. C. Melissions (Interscience Publishers, Inc., New York, 1961), p. 236.