where F' denotes the derivative of the hypergeometric function with respect to its argument. For $\alpha>0$, $F_{\lambda\lambda'}$ contains some additional terms which are written as

$$\left[\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\right]^2 \frac{\tan\pi\alpha}{(\alpha+\frac{1}{2})(-2z)^{2\alpha+1}}\left[g^{(2)}{}_{\lambda\lambda'}F\left(1+\frac{\alpha}{2},\ \tfrac{1}{2}+\alpha,\ \alpha+\tfrac{3}{2},\ \frac{1}{z^2}\right)+h^{(2)}{}_{\lambda\lambda'}F'\left(1+\frac{\alpha}{2},\ \tfrac{1}{2}+\alpha,\ \alpha+\tfrac{3}{2},\ \frac{1}{z^2}\right)\right]. \quad \text{(A13)}$$

The functions $f_{\lambda\lambda'}$, $g_{\lambda\lambda'}{}^{(1,2)}$, and $h_{\lambda\lambda'}{}^{(1,2)}$ are tabulated for some small values of λ and λ' in Table II.

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Daughter Trajectories and Unequal-Mass Scattering*

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It has recently been demonstrated by Goldberger and Jones (I) and by Freedman and Wang (II) that Regge asymptotic behavior obtains at high energy even in regions in which the crossed-channel $\cos\theta$ variable is constrained by unequal-mass kinematics to remain finite. Approaches I and II differ, however, in other important respects. In this note it is shown that method I can be adapted and used to prove the existence and properties of the Regge daughter trajectories found in II. In this argument, an extra assumption necessary in II is avoided, and the restriction $\alpha(0)<\frac{1}{2}$ found in I is eliminated.

RECENTLY two different arguments have been given to show that the Regge asymptotic behavior $u^{\alpha(s)}$ is maintained in the backward scattering of unequal-mass particles even though the cosine of the u-channel scattering angle remains small.[1,2] In both methods the persistence of the behavior $u^{\alpha(s)}$ is a consequence of the analyticity of the full amplitude at $s=0$, a property not shared by the individual Regge-pole terms.

In I, dispersion relations are used to correct the analyticity of the original Regge pole terms, whereas in II a representation of the scattering amplitude as the Sommerfeld-Watson transform of power series in the Mandelstam variables u and t, called the Khuri representation, is employed. For the asymptotic contribution at $s=0$ of the leading Regge pole $\alpha_0(s)$, both methods find the dominant term $\gamma(0)u^{\alpha_0(0)}$ and the next dominant term $s^{-1}u^{\alpha_0(0)-1}$, which has an s^{-1} singularity not shared by the full amplitude and which must, therefore, be cancelled.

The main difference between I and II lies in the mechanism by which this singularity is cancelled. In I it is argued that the singularity is cancelled by the background term of the Regge representation, and the restriction $\alpha_0(0)<\frac{1}{2}$ is therefore found. In II it is argued that the singularity is cancelled by contributions of other Regge poles, and it is found that to effect this cancellation there must occur daughter trajectories $\alpha_k(s)$, correlated with the leading or parent trajectory by the conditions $\alpha_k(0)=\alpha_0(0)-k$. No restriction on the position of the leading trajectory stronger than that of Froissart [namely, $\alpha_0(0)\leqslant 1$] is found. Mathematically there does not seem to be any *a priori* reason to prefer either mechanism, but it is found in II that the daughter trajectory mechanism is satisfied in all Bethe-Salpeter models which Reggeize, and empirically it is known that the Pomeranchuk trajectory violates the constraint $\alpha(0)<\frac{1}{2}$.

The analyticity of the Khuri power-series coefficients at $s=0$ is important to the argument of II. It was made plausible there but not rigorously proved, and was left as an extra assumption. The purpose of this article is to show that the existence and properties of the first daughter trajectory can be proved without such an extra assumption by using the techniques of I and demanding consistency between the Regge representation and Mandelstam analyticity in the case where there are Regge poles to the right of $\mathrm{Re}\,l=\frac{1}{2}$ for $s=0$. In this way we eliminate the restriction $\alpha(0)<\frac{1}{2}$ and asymptotic fixed powers larger than background (see I).

It is not clear how to take the Regge background integral to the left of $\mathrm{Re}\,l=-\frac{1}{2}$ with this technique because of the threshold accumulation of poles there, and therefore the discussion of lower-lying daughter trajectories from this point of view may be difficult.

In the treatment here we rely heavily on references to I and II. For simplicity we follow I in assuming that

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[1] M. L. Goldberger and C. E. Jones, Phys. Rev. **150**, 1269 (1966); referred to as I. Also Phys. Rev. Letters **17**, 105 (1966).
[2] D. Z. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967); referred to as II. See also Phys. Rev. Letters **17**, 569 (1966).

the amplitude has only the s-u double spectral function. The roles of s and u have been interchanged from those in II. Implicit in this work are the assumptions that cuts in the angular momentum plane are absent and that Regge trajectories do not intersect.

We write the Regge representation

$$A(s,u)=B(s,u)+\sum_i \gamma_i(s)\nu^{\alpha_i(s)}Q_{-1-\alpha_i(s)}\left(-1+\frac{r^2/s-u}{2\nu}\right), \quad (1)$$

where

$$\gamma_i(s)\nu^{\alpha_i(s)}=[2\alpha_i(s)+1]\beta_i(s)[\cos\pi\alpha_i(s)]^{-1}$$

and

$$\nu=\frac{[s-(m-\mu)^2][s-(m+\mu)^2]}{4s}, \quad r^2=(m^2-\mu^2)^2. \quad (2)$$

The summation index in (1) runs over the finite number of Regge poles that appear in the region $\mathrm{Re}l>-\frac{1}{2}+\epsilon$ for any real energy s, $-a\leqslant s<+\infty$, where a is any small positive number. The background function $B(s,u)$ has the asymptotic behavior $B(s,u)=o(u^{-1/2+\epsilon})$ as $u\to\infty$, for all positive $s>s_0$.

We begin at negative u and express the amplitude as a single-variable dispersion relation

$$A(s,u)=\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}\,\mathrm{Im}B(s',u)+\frac{1}{\pi}\sum_i\int_{s_0}^{\infty}\frac{ds'}{s'-s}\times\mathrm{Im}\left[\gamma_i(s')(\nu')^{\alpha_i}Q_{-1-\alpha_i}\left(-1+\frac{r^2/s'-u}{2\nu'}\right)\right]. \quad (3)$$

From the asymptotic behavior of $B(s,u)$ we expect that the first term on the right side of (3) to behave like $u^{-1/2+\epsilon}$ for all s.[3] As in I the Regge-pole term is expressed as a contour integral and evaluated as

$$A_R(s,u)=\sum_i\left\{\gamma_i(s)\nu^{\alpha_i(s)}Q_{-1-\alpha_i(s)}\left(-1+\frac{r^2/s-u}{2\nu}\right)+\frac{1}{2\pi i}\int_C\frac{ds'}{s'-s}\gamma_i(s')\nu'^{\alpha_i}Q_{-1-\alpha_i}\left(-1+\frac{r^2/s'-u}{2\nu'}\right)\right\}, \quad (4)$$

where the contour C encloses the cut of $Q_{-1-\alpha_i}(-1+(r^2/s'-u)/2\nu')$ from $s=0$ to $s=r^2/u$ in a counterclockwise direction. The suppressed argument of the functions α_i is s'.

In Ref. 1, it was tacitly assumed that the residue functions $\gamma_i(s)$ are analytic at $s=0$ and therefore the contour integral in (4) was collapsed to the cut. We now wish to allow for the possibility that $\gamma_i(s)$ may have poles of arbitrary order at $s=0$.[4] If such poles are present, collapse of the contour C to the cut may not be possible.

The essence of the method here is to demand the asymptotic consistency of Eqs. (1) and (3). This requires that the correction term to the Regge-pole contributions, as expressed by the contour integral in (4) summed over all poles, must be of background size (bounded by $u^{-1/2}$ for $u\to\infty$) in the region $s>s_0$.

Therefore, we examine the asymptotic behavior of the contour integrals in (4). On the contour C the function $Q_{-1-\alpha_i(s)}(-1+(r^2/s'-u)/2\nu')$ can be approximated by

$$(-1)^{\alpha_i(s')}2^{-1-\alpha_i(s')}\frac{\Gamma[-\alpha_i(s')]^2}{\Gamma[-2\alpha_i(s')]}(2\nu')^{-\alpha_i(s')}\times[u^{\alpha_i(s')}+\alpha_i(s')u^{\alpha_i(s')-1}\times s'-2m^2-2\mu^2-r^2/2s')]. \quad (5)$$

Since this approximate expression is uniform, it can be integrated to give the asymptotic behavior of the contour integrals. For residue functions with poles of order n at $s'=0$, we are led to consider integrals of the form

$$\frac{1}{2\pi i}\int_C\frac{f(s')u^{\alpha(s')}}{s'^n(s'-s)}ds', \quad (6)$$

which can be evaluated using residue theorems. For $n=0$ the integral vanishes, and for $n=1$ it is equal to $f(0)s^{-1}u^{\alpha(0)}$. For $n>1$, the most singular term at $s=0$ goes like $s^{-n}u^{\alpha(0)}$, and in addition there are terms involving less singular powers of s multiplied by $u^{\alpha(0)}$, powers of $\ln u$, and derivatives at $s'=0$ of $\alpha(s')$ and $f(s')$.

With these remarks in mind, we consider the correction term as defined by the contour integral (4) of the leading Regge trajectory $\alpha_0(s)$. Using the asymptotic expansion (5), we conclude that the residue $\gamma_0(s)$ must be analytic at $s=0$ if $\alpha_0(0)>-\frac{1}{2}$. Otherwise from the first term in (5) there would be a power $u^{\alpha_0(0)}$ larger than background which cannot be cancelled by lower-lying trajectories. If $\alpha_0(0)>\frac{1}{2}$, the second term in (5) then contributes the asymptotic power $r^2s^{-1}u^{\alpha_0(0)-1}$, which is larger than the background, plus terms which are of background order since $\alpha_0(0)$ is restricted by the Froissart bound to be less than 1. Since the asymptotic power $u^{\alpha_0(0)-1}$ is larger than background, it must be cancelled by other Regge-pole contributions. This cancellation can occur only if there is a second Regge trajectory $\alpha_1(s)$ satisfying $\alpha_1(0)=\alpha_0(0)-1$, and which, by the discussion in the preceding paragraph, must have residue $\gamma_1(s)$ with an s^{-1} singularity at $s=0$ (that is, with $n=1$). The desired cancellation requires that the coefficient of this singularity must have the value indicated in Eq. (46) of II. We have thus proved with the techniques of I that each Regge trajectory with $\alpha(0)>\frac{1}{2}$ must be accompanied by a daughter trajectory with exactly the properties found in II.

[3] Although it is not necessarily true that the infinite integral of an asymptotic expansion has the same behavior as its integrand, we assume that it is true in this case.

[4] It is proved in II that the $\gamma_i(s)$ cannot have branch points at $s=0$, but may have multiple poles.

Since the residues of parent trajectories are analytic at $s'=0$, their contour integral contribution in (4) can be collapsed to the cut and the form (3.5) of I obtained. The first daughter residues have poles at $s'=0$, but since they lie one integer below the parents the factor $\nu'^{\alpha_i(s')}$ makes it possible to collapse again the contour integral to the cut and the form (3.5) of I may be used. The rest of the program of I can then be carried through without change and the Regge asymptotic behavior $u^{\alpha(s)}$ established for the scattering amplitude throughout the backward region. In the present version, the restriction $\alpha(0)<\frac{1}{2}$ found in I has been removed.

The advantage of this method is that the extra assumption made in II about the analyticity of Khuri amplitudes at $s=0$ is unnecessary here. In fact the present techniques can be used to prove this assumption for ν (the Khuri variable in II) in the region $\mathrm{Re}\nu>-\frac{1}{2}$. The disadvantage of the present method is that it is not clear how to move the background contour to the left of $\mathrm{Re}l=-\frac{1}{2}$ and establish the existence of lower-lying daughter trajectories.

The asymptotic contribution of a parent and its first daughter trajectory to the full amplitude is given explicitly in Eq. (47) of II. At $s=0$ this contribution takes the form

$$A(0,u)=au^{\alpha(0)}+b(m^2-\mu^2)^2u^{\alpha(0)-1}\ln u+cu^{\alpha(0)-1}+\cdots. \quad (7)$$

The logarithmic term is peculiar to the unequal-mass case and may be significant when accurate fits to high-energy data are possible.

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Current Algebras and Configuration Mixing*

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Assigning mixed configurations to hadron states implies an auxiliary space of "one-particle states" that contains more states than does the space of observable particles and resonances. An approximate model Hamiltonian that has all these states as one-particle eigenstates implies a corresponding modification of the unitary representations of the Lorentz transformations. This in turn requires a modification of the weak-interaction current operators if their tensor transformation properties are to be maintained. This paper describes the formal construction of such models and examines the relations between the exact and the approximate quantities.

In a recent paper[1] the infinite-momentum limit of current algebras was discussed. The restriction of the current operators to a one-particle subspace was a key feature in that discussion, and the formal analysis made use of the Lorentz invariance of that subspace as well as the tensor transformation properties of the currents. The previous paper pointed out that for practical purposes the subspace of the stable particles is certainly too small to allow the commutation rules (20) of I. The purpose of the present article is to present a detailed discussion of procedures for enlarging the subspace sufficiently for an empirical justification of Eq. (20) or (21) of I.

Recent work on representation mixing[2] in hadron states implies such an enlarged "one-particle space," but we should not assume that all states in that space correspond to observable particles or resonances. In the shell-model theory of nuclear structure, the analogous problem is well known. One introduces an approximate shell-model Hamiltonian with a discrete energy spectrum. In diagonalizing a finite submatrix, one finds approximations to some low-lying states. The higher eigenstates of the submatrix do not usually correspond to physical levels.

In a relativistic theory, *the* Hamiltonian H of the system is determined if *the* unitary representation $U(a,\Lambda)$ of the Poincaré group is known. The introduction of a different model Hamiltonian H_0 implies also a modified unitary representation $U_0(a,\Lambda)$ of the Lorentz transformations on the same space of states. Weak interactions are superimposed as a perturbation on the strong-interaction dynamics implied by the representation $U(a,\Lambda)$. The weak-interaction currents $\mathfrak{F}^\alpha(x)$ must have tensor transformation properties under Lorentz transformations. The introduction of an approximate model $U\to U_0$ necessitates also the modification of the current density operator $\mathfrak{F}^\alpha(x)\to\mathfrak{F}_0{}^\alpha(x)$

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