# Measurement of $G_{En}$ from Elastic e-d Scattering: Relativistic Corrections and Model Dependence*

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(Received 31 October 1966)

Relativistic corrections to elastic *e-d* scattering are evaluated and shown to be large enough to account for the discrepancy between *e-d* scattering and thermal-neutron measurements of the slope of $G_{En}$.

We wish to point out that, recently, the theory of elastic *e-d* scattering has been improved to the point where contributions of order $q^2/M^2$ to the deuteron-charge form factor can be calculated. These additional terms lead to significant changes in the values of the neutron electric form factor $G_{En}$, previously determined from this experiment. In this paper, we will discuss the quantitative importance of (1) alternative models of the nonrelativistic deuteron wave functions, (2) relativistic corrections, and (3) meson-exchange current contributions.

Of special interest are the implications of these considerations for the apparent discrepancy between measurements of $G_{En}$ as obtained by elastic *e-d* scattering at low momentum transfers[1,2] and the slope $(dG_{En}/dq^2)|_{q^2=0}$ determined from atomic scattering of thermal neutrons.[3] These results are shown in Fig. 1. The values of $G_{En}$ obtained from the Drickey and Hand data, using a Partovi wave function[4] and a nonrelativistic impulse approximation [Fig. 1(a)] are consistent with $G_{En}=0$, and in apparent disagreement with the thermal-neutron slope.[5] When these same data are reanalyzed, using a theory incorporating the relativistic corrections discussed below, the extracted values of $G_{En}$ are increased, as shown in Fig. 1(b). These corrected results are more in agreement with the slope and less consistent with $G_{En}=0$. Finally, if the original data are reanalyzed with the deuteron boundary-condition wave function proposed by Feshbach and Lomon,[6] and the relativistic corrections are also included, the data are completely consistent with the slope [Fig. 1(c)]. The point here is that the Feshbach-Lomon model is sufficiently different in percentage *D* state and core radius from the Hamada, Breit, or Partovi models used previously to affect significantly the determination of $G_{En}$, and hence there may be a greater difference between realistic deuteron models than has been previously supposed.

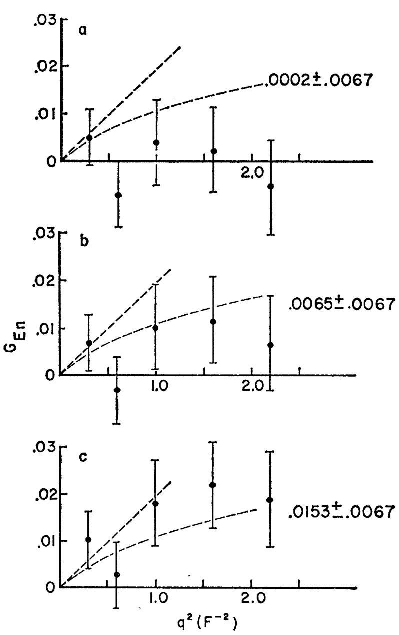


FIG. 1. The data of Drickey and Hand (Ref. 1). The dashed lines are the thermal neutron slope $+0.0193\pm0.0004$ (Ref. 3) and the fit of Chilton and Uhrhane (Ref. 5). The number at the right of each figure is the slope determined from the first three data points in each graph. (a) The data analyzed with the Partovi wave function, (b) the data analyzed with the Partovi wave function including relativistic corrections, and (c) the data analyzed with the Feshbach-Lomon wave function including relativistic corrections.

* Supported in part by the Office of Naval Research.
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[1] D. Drickey and L. Hand, Phys. Rev. Letters **9**, 521 (1962).
[2] For other relevant experimental results see D. Benaksas, D. Drickey, and D. Frerejacque, Phys. Rev. Letters **13**, 353 (1964); Phys. Rev. **148**, 1327 (1966); B. Grossetete and P. Lehmann, Nuovo Cimento **28**, 429 (1963); D. J. Drickey, B. Grossetete, and P. Lehmann, in *Proceedings of the Sienna International Conference on Elementary Particles and High-Energy Physics, 1963*, edited by G. Bernardini and G. P. Puppi (Societa Italiana di Fisica, Bologna, 1963). These other data have been omitted for simplicity. All are consistent with the Drickey-Hand data with the exception of the three points obtained by Grossetete and Lehmann with a $CD_2$ target. These latter points are omitted because they were not obtained by a ratio experiment, and hence are sensitive to the proton data. Furthermore, recent measurements of Benaksas *et al.* at higher $q^2$ tend to confirm the Drickey-Hand results.
[3] For the most recent work, see V. E. Krohn and G. R. Ringo, Phys. Rev. **148**, 1303 (1966); for earlier references see R. Hofstadter, *Nuclear and Nucleon Structure* (W. A. Benjamin, Inc., New York, 1963) for reprints of a number of papers.
[4] We are indebted to E. F. Erickson for providing us with numerical wave functions developed by Partovi from a Hamada potential. They have a core radius of 0.485 F and a *D*-state probability of 7%.
[5] We use the word "apparent," because one can conjecture that $G_{En}$ has enough curvature so that it is zero beyond $q^2$ of 1 $F^{-2}$, and yet still agrees with the slope at $q^2=0$. This is the point of view taken by F. Chilton and F. J. Uhrhane (unpublished). We do not agree that the curvature they obtain is alone sufficient to eliminate the discrepancy.
[6] H. Feshbach and E. Lomon, in Advances in High-Energy Physics, edited by R. E. Marshak (to be published). This model has a *D*-state probability of 4.6% and a core radius of 0.735 F.

The sensitivity of $G_{En}$ to the deuteron model has long been recognized. The cross section can be written[7]

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma}{d\Omega}\bigg|_0\bigg[A(q^2)+\frac{q^2}{6M_d^2}\times\bigg[1+2\bigg(1+\frac{q^2}{4M_d^2}\bigg)\tan^2\frac{\theta}{2}\bigg]G_M^2\bigg], \quad (1)$$

where $q^2$ is the magnitude of the squared 4-momentum transferred by the electron, $(d\sigma/d\Omega)|_0$ is the cross section for a point deuteron, and

$$A(q^2)=[G_{En}(q^2)+G_{Ep}(q^2)]^2F_d^2(q^2)+M(q^2). \quad (2)$$

The functions $F_d(q^2)$ (including effects of the deuteron structure) and $M(q^2)$ (including effects of meson-exchange currents) are determined theoretically. Since $G_{En}$ is much smaller than the proton electric form factor, $G_{Ep}$, $A(q^2)$ is very insensitive to $G_{En}$, and even a small theoretical error in $F_d$ is magnified into a large error in $G_{En}$. The importance of this simple fact is often not sufficiently appreciated, as we shall discuss below.

## RELATIVISTIC CORRECTIONS

In addition to the question of model dependence, there are corrections to $F_d$ of relativistic origin. Normally these would be of little interest, but, because of the extreme sensitivity of $G_{En}$ to the value of $F_d$, they cannot be ignored. These corrections come primarily from relativistic modifications of (a) the deuteron wave functions, and (b) the nucleon current. The latter have been discussed by Gourdin,[8] and also by Gross,[9] and turn out to be small. The modifications of the wave function are the largest effect, however, and we will report on the results of a recent calculation of these.[9]

If we look at the scattering process in the Breit system, the momenta of the incoming and outgoing deuterons are $-\frac{1}{2}\mathbf{q}$ and $+\frac{1}{2}\mathbf{q}$, respectively. Nonrelativistically, for spin-zero nucleons and deuterons, the structure function is[7]

$$F_d(q^2)=\frac{1}{2(2\pi)^4}\int d^3k\,\psi_0[(\mathbf{k}-\tfrac{1}{4}\mathbf{q})^2]\psi_0[(\mathbf{k}+\tfrac{1}{4}\mathbf{q})^2] \quad (3a)$$

$$=\int_0^\infty dr\,u^2(r)j_0\bigg(\frac{qr}{2}\bigg), \quad (3b)$$

where $\psi_0(\mathbf{p})$ and $u(r)/r$ are the nonrelativistic deuteron wave functions in momentum and position space, respectively. The subscript on $\psi_0$ refers to the total momentum of the deuteron, and its value **0** reminds us that $\psi_0$ was determined from nonrelativistic potentials which are applied in the center-of-mass system. If we assume for the moment that correct wave functions can be determined from such potentials, Eq. (3) is still wrong, because the total momenta of the deuterons are not zero. Hence, in the Breit frame, we should write

$$F_d(q)=\frac{1}{2(2\pi)^4}\int d^3k\,\psi_{+(1/2)\mathbf{q}}[(\mathbf{k}-\tfrac{1}{4}\mathbf{q})^2]\times\psi_{-(1/2)\mathbf{q}}[(\mathbf{k}+\tfrac{1}{4}\mathbf{q})^2] \quad (4)$$

instead of (3a). Nonrelativistically, of course, $\psi_d(\mathbf{p})=\psi_0(\mathbf{p})$, because the internal wave function depends only on $\mathbf{p}$. Relativistically, this is no longer true, because $\psi$ depends on the 4-momentum $p$ (i.e., the relative energy as well as the relative momentum). Hence, one expects $\psi_d(\mathbf{p})$ to be distorted by a Lorentz-contraction factor and by the fact that the equal-time wave function in the rest frame [from which one obtains $\psi_0(\mathbf{p})$] is not the same as the equal-time wave function in the moving frame [from which one obtains $\psi_d(\mathbf{p})$]. Careful consideration of the problem using Bethe-Salpeter wave functions shows that, to order $(v/c)^2=q^2/4M^2$, the correct replacement is

$$\psi_d(\mathbf{p})=\bigg(1-\frac{\mathbf{d}^2}{16M^2}\bigg)\psi_0[\mathbf{p}^2-\Delta(\mathbf{p},\mathbf{d})], \quad (5)$$

where

$$\Delta(\mathbf{p},\mathbf{d})=\tfrac{1}{4}M^{-2}[(\mathbf{p}\cdot\mathbf{d})^2-2(\mathbf{p}\cdot\mathbf{d})(\mathbf{p}^2+\alpha^2)]. \quad (6)$$

Substituting (5) into (4), one obtains an expression similar to (3a), but with the arguments of the wave functions shifted by a small amount of order $M^{-2}$.

It is now a straightforward matter to expand the arguments of the wave functions in the Taylor series[10] and obtain workable expressions for the corrections in terms of $u$ and $w$, the $S$- and $D$-state wave functions of the deuteron. For details with spin, the reader is referred to Ref. 9. The complete results for the deuteron-charge form factor are

$$G_C=(G_{Ep}+G_{En})D_C, \quad (7)$$

where

$$D_C=\bigg(1-\frac{q^2}{32M^2}\bigg)\int_0^\infty (u^2+w^2)j_0(\tau)dx - \frac{q^2}{24M^2}\int_0^\infty (u^2-uxu'-u\kappa u + 7w^2-wxw'-w\kappa w)[j_0(\tau)+j_2(\tau)]dx + \frac{q^2}{16M^2}\int_0^\infty (uxu'+wxw')j_0(\tau)dx. \quad (8)$$

[7] For a discussion of the nonrelativistic theory and a derivation of the cross section see V. Z. Jankus [Phys. Rev. **102**, 1586 (1956)] and M. Gourdin [Nuovo Cimento **28**, 533 (1963); **32**, 493 (1964)].

[8] M. Gourdin, Nuovo Cimento **35**, 1105 (1965).

[9] F. Gross, Phys. Rev. **140**, B410 (1965); **142**, 1025 (1966). See, however, the Erratum to the second paper [Phys. Rev. **152**, 1517 (1966)].

[10] To avoid difficulties with the violent behavior of the wave functions at the core, we integrate only over $r>r_c$.

In this expression, the prime refers to differentiation with respect to $x$, $\kappa = x^2[(d^2/dx^2-\alpha^2)]$, and $\tau=\frac{1}{2}qx$, where $q$ is the magnitude of the 4-momentum transfer. The first integral in $D_C$ is the usual Jankus[7] nonrelativistic result. There is a similar expression for the quadrupole moment, which we can call $D_Q$.[9,11] The complete result for $F_d$ is then

$$F_d^2=D_C^2+(q^4/18M_d^4)D_Q^2. \qquad (9)$$

The correction introduced into the structure function $F_d$ by the small terms in Eq. (8) is called $\Delta F_d$. In computing this correction, it is a good approximation to treat the quadrupole form factor $D_Q$ nonrelativistically, since, for $q^2<5F^{-2}$, this term is small compared to $D_C$. Then, since the experimental numbers $A(q^2)$ are unchanged, the resulting change in $G_{En}$ introduced by $\Delta F_d$ is

$$\Delta G_{En}=-(G_{Ep}+G_{En})\Delta F_d\cong -G_{Ep}\Delta F_d\,, \qquad (10)$$

where the value of $G_{En}$ on the right-hand side of Eq. (10) is the value of $G_{En}$ calculated previously with the uncorrected theory. Since it is much less than the proton form factor $G_{Ep}$, it can be neglected.

In Fig. 2, we have plotted $\Delta G_{En}$, calculated from Eqs. (8)–(10) with the Partovi wave functions.[12] The value of the correction for the Feshbach-Lomon wave function is slightly smaller at higher $q^2$, but, to a good approximation, the results are model-independent. For comparison, we have shown on the same graph the change in $G_{En}$ which results from use of Feshbach-Lomon wave functions instead of Partovi wave functions in the nonrelativistic expression for $F_d$. We have also included on the same graph the corrections to $G_{En}$ arising from meson-exchange currents which we will discuss below. The numerical results shown in Fig. 2 were used in our analysis of the data in Fig. 1.

At this point, we can explain why a number of previous analyses of relativistic corrections have given results which differ from ours. The principal reason for this is that these analyses all make specific assumptions about the relativistic structure of the deuteron wave functions.[13–15] The results obtained are equivalent to using wave functions with essentially no hard core, and hence they all give values of $F_d$ too large. But the size of the core is a model-dependent question, and should not be confused with the relativistic corrections to $e$-$d$ scattering. In some of the previous calculations, attempts were made to avoid this difficulty and separate out this model-dependent effect by comparing their relativistic results with a "corresponding" nonrelativistic model.[14,15] However, in every case, the nonrelativistic model chosen for comparison is *not* the same as the relativistic model evaluated in the deuteron rest frame. [That is, Eq. (5) is not true.] This latter requirement seems essential to us, since one can only assume that a good nonrelativistic model already simulates all relativistic effects in the *rest frame*, and is only wrong in trying to use the rest-frame expression for a moving deuteron. The advantage of the approach described here is that it picks up those corrections which do not depend on the deuteron model and which are characteristic of $e$-$d$ scattering only.

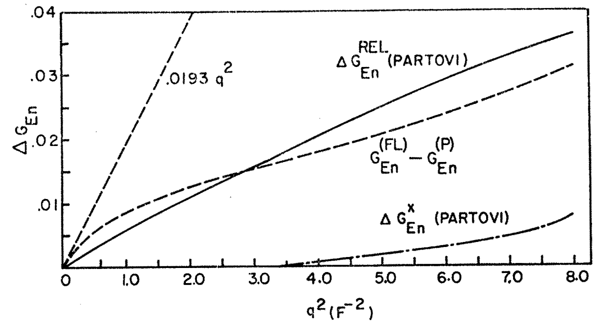


FIG. 2. The change in $G_{En}$ as determined by Eq. (10) from several effects discussed in the text. The thermal neutron slope is given for reference. The relativistic correction is denoted by $\Delta G_{En}^{\text{rel}}$, the difference between the Feshbach-Lomon model and the Partovi model is labeled $G_{En}^{\text{FL}}-G_{En}^{\text{P}}$, and the meson-exchange current contributions are labeled $\Delta G_{En}^{X}$.

## MESON-EXCHANGE EFFECTS

The meson-exchange current contributions shown in Fig. 2 were calculated assuming a value[16,17] of $g_{\rho\pi\gamma}=0.36$. This is the value necessary to put the magnetic moment into agreement with experiment, and corresponds to a decay width of about 0.5 MeV. It is different from that obtained by Adler and Drell, because the meson-exchange contribution must compensate for an additional relativistic correction to the magnetic moment which emerges from the analysis sketched above.[9,11] For the Partovi model, this additional correction is

$$\mu_{d\ \text{rel}}=-0.0063$$

in nuclear magnetons. For the Feshbach-Lomon model, the correction is about one-third as large. In any case, it is clear from Fig. 1 that the meson-exchange contributions are negligible below $q^2$ of 10 $F^{-2}$.

Recently, Dietz and Month[18] have arrived at the

[11] For quantitative results for the magnetic and quadrupole form factors see C. Buchanan and E. Erickson (to be published).

[12] In our calculations, we neglected the factor $[1-(q^2/32M^2)]$ in the first term of Eq. (8) which gives a small positive contribution.

[13] J. Tran Thanh Van, Nuovo Cimento **30**, 1100 (1963); F. Gross, Phys. Rev. **134**, B405 (1964); **136**, B140 (1964).

[14] H. F. Jones, Nuovo Cimento **26**, 790 (1962); R. J. Adler and E. F. Erickson, *ibid.* **40**, 236 (1965).

[15] Y. Renard and T. Tran Thanh Van, Orsay report TH/166 (unpublished). This interesting work agrees qualitatively with our work, but differs quantitatively, for the reasons mentioned in the text.

[16] R. J. Adler and S. D. Drell, Phys. Rev. Letters **13**, 349 (1964); R. J. Adler, Phys. Rev. **141**, 1499 (1966).

[17] B. M. Casper, Ph.D. thesis, Cornell University, 1966 (unpublished). In this calculation, the coefficient multiplying $g_{\rho\pi\gamma}$ is $\frac{1}{2}$ as large as that found in Ref. 1. We have used these results in our discussion above.

[18] K. Dietz and M. Month, Phys. Rev. Letters **17**, 546 (1966).

opposite conclusions about the meson-exchange effects. They found that a large "correction" was necessary to reconcile the data with the theory. However, the model of the deuteron wave function which they employ has essentially no hard core, and hence the large correction they need to fit the data is only another indication of the desirability of a core. Since the "meson-exchange contributions" refer to the case in which the photon interacts *directly* with exchanged mesons, it is incorrect to regard the influence of the core as a meson-exchange effect (even though the core may be due to the exchange of heavy mesons).

Finally, recent measurements[19] make it increasingly difficult to believe that $g_{\rho\pi\gamma}$ is as large as is needed to fit the magnetic moment. If this is so, then there is still a discrepancy in the deuteron magnetic moment. Since the relativistic corrections to the magnetic moment came primarily from the expansion of the nucleon current and *not* from the distortion of the deuteron wave function,[9] a realistic calculation of the effect of the off-mass-shell contributions to the nucleon current might resolve the dilemma. The Ward identity guarantees that corrections of this type will be small for the charge, but there is no such restriction for the magnetic moment.[9]

## CONCLUSIONS

Our principal conclusions can be summarized as follows:

1. We have evaluated the relativistic corrections to the impulse approximation of elastic *e-d* scattering using what we believe to be a consistent theory. We find that the corrections to $G_{En}$ are additive and approximately equal to $q^2/8M^2$. The terms omitted from our analysis are of order $2(q^2/4\mu M)^2$, and hence the linearity of the corrections is reliable to $q^2$ of about 5 $F^{-2}$.[20] They are reasonably model-independent.

2. The difference between the use of the Feshbach-Lomon model and the Partovi model in the basic nonrelativistic impulse approximation is as great as in the relativistic corrections. If one assumes that both models are realistic, then there is more model dependence in the nonrelativistic theory than previously realized.

3. The meson-exchange current corrections are model-dependent,[17] but give, in any case, a negligible contribution to the charge form factor below $q^2$ of 10 $F^{-2}$.

4. Using the Partovi model, the relativistic corrections tend to reduce the discrepancy between the data and the thermal neutron slope or, alternatively, require a smaller curvature of $G_{En}$ to fit the slope. The curvature required is in agreement with Ref. 5, and hence is not unreasonable. The relativistic corrections are of the order of a standard deviation at $q^2=2.0$ $F^{-2}$, and would be increasingly important, the better the experiment, and the higher the momentum transfer.

5. Using the Feshbach-Lomon model, the relativistic corrections are sufficient to make the results completely consistent with the neutron slope.

6. Taking points 4 and 5 together, we conclude that there is no longer any reason to believe in a discrepancy, for a $G_{En}$ of reasonable curvature can be found to fit the slope for both models.

7. Aside from the question of model dependence, we believe that the low-$q^2$ theory is now more accurate than the experiments. If the experimental uncertainties were reduced by a factor of 2 or better, this experiment could be added to the large group of experiments which must be explained by any deuteron model (i.e., the deuteron model would have to give the correct slope). Until the model dependence can be completely cleared away, however, it will be difficult to obtain any better information about $G_{En}$ from this experiment than we already have.

## ACKNOWLEDGMENTS

We wish to thank Professor Lomon for supplying us with his wave functions, and for permission to use his model prior to publication. We are grateful to Professor Gourdin for drawing our attention to Ref. 15, and for several stimulating conversations. We are also indebted to Dr. E. Erickson for supplying us with the Partovi wave functions and for interesting conversations and correspondence. We thank K. Berkelman, C. Buchanan, D. Cassel, H. Goldberg, L. Hand, F. von Hippel, and D. R. Yennie for helpful conversations and comments on the manuscript.

[19] Results of the DESY bubble-chamber group, reported at the 13th International Conference on High-Energy Physics, Berkeley, 1966 (unpublished) give a width of $\Gamma_{\rho\pi\gamma}=.05\pm.03$ MeV, which is about 10 times smaller than the 0.5 MeV used in Ref. 16.

[20] Around $q^2$ of 5 $F^{-2}$, the corrections to $D_Q$ can be expected to become important.