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Symmetry Breaking in Non-Abelian Gauge Theories*

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According to the Goldstone theorem, any manifestly covariant broken-symmetry theory must exhibit massless particles. However, it is known from previous work that such particles need not appear in a relativistic theory such as radiation-gauge electrodynamics, which lacks manifest covariance. Higgs has shown how the massless Goldstone particles may be eliminated from a theory with broken $U(1)$ symmetry by coupling in the electromagnetic field. The primary purpose of this paper is to discuss the analogous problem for the case of broken non-Abelian gauge symmetries. In particular, a model is exhibited which shows how the number of massless particles in a theory of this type is determined, and the possibility of having a broken non-Abelian gauge symmetry with no massless particles whatever is established. A secondary purpose is to investigate the relationship between the radiation-gauge and Lorentz-gauge formalisms. The Abelian-gauge case is reexamined, in order to show that, contrary to some previous assertions, the Lorentz-gauge formalism, properly handled, is perfectly consistent, and leads to physical conclusions identical with those reached using the radiation gauge.

I. INTRODUCTION

THEORIES with spontaneous symmetry breaking (in which the Hamiltonian but not the ground state is symmetric) have played an important role in our understanding of nonrelativistic phenomena like superconductivity and ferromagnetism. Many authors, beginning with Nambu,[1] have discussed the possibility that some at least of the observed approximate symmetries of relativistic particle physics might be interpreted in a similar way. The most serious obstacle has been the appearance in such theories of unwanted massless particles, as predicted by the Goldstone theorem.[2]

In nonrelativistic theories such as the BCS model, the corresponding zero-energy-gap excitation modes may be eliminated by the introduction of long-range forces. The first indication of a similar effect in relativistic theories was provided by the work of Anderson,[3] who showed that the introduction of a long-range field, like the electromagnetic field, might serve to eliminate massless particles from the theory. More recently, Higgs[4] has exhibited a model which shows explicitly how the massless Goldstone bosons are eliminated by coupling the current associated with the broken symmetry to a gauge field. The reasons for the breakdown of the Goldstone theorem in this case have been analyzed by Guralnik, Hagen, and Kibble.[5] The situation is identical with that in the nonrelativistic domain. In either case the theorem is inapplicable in the presence of long-range forces, essentially because the continuity equation $\partial_\mu j^\mu=0$ no longer implies the time independence of expressions like $\int d^3x\, [j^0(x),\phi(0)]$, since the relevant surface integrals do not vanish in the limit of infinite volume. (In the relativistic case, the theorem does apply if we use the Lorentz gauge; but then it tells us nothing about whether the massless particles are physical.) It should be noted that the extension or corollary of the Goldstone theorem discussed by Streater[6] also fails in precisely this case. If long-range fields are introduced, spontaneous symmetry breaking can lead to mass splitting.

As has been emphasized recently by Higgs,[7] it thus appears that the only way of reconciling spontaneous symmetry breaking in relativistic theories with the absence of massless particles is to couple in gauge fields. Another possibility is that Goldstone bosons may turn out to be completely uncoupled and therefore physically irrelevant. In this case, however, the Hilbert space decomposes into the direct product of a physical Hilbert space and a free-particle Fock space for the Goldstone bosons. The broken symmetry appears only in the latter, and no trace of it remains in any physical quantities. In most simple cases, the symmetry transformations leave the physical Hilbert space completely invariant; and in any case they act unitarily on it. Such theories cannot therefore explain observed approximate symmetries. This decoupling of Goldstone modes does occur in the Lorentz-gauge treatment of models like that discussed by Higgs, in which in fact no trace of the original $U(1)$ symmetry remains in the physical states. However it does not occur in corresponding non-Abelian gauge theories, to which the conventional (i.e., Gupta-Bleuler) Lorentz-gauge formation is inapplicable.

It has been suggested by Fuchs[8] that in the case of non-Abelian gauges the massless particles may persist

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[1] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960). Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); M. Baker and S. L. Glashow, *ibid.* **128**, 2462 (1962); S. L. Glashow, *ibid.* **130**, 2132 (1962).

[2] J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[3] P. W. Anderson, Phys. Rev. **130**, 439 (1963).

[4] P. W. Higgs, Phys. Letters **12**, 132 (1964).

[5] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, Phys. Rev. Letters **13**, 585 (1964). See also T. W. B. Kibble, in *Proceedings of the Oxford International Conference on Elementary Particles, 1965* (Rutherford High Energy Laboratory, Harwell, England, 1966), p. 19.

[6] R. F. Streater, Phys. Rev. Letters **15**, 475 (1965).

[7] P. W. Higgs, Phys. Rev. **145**, 1156 (1966).

[8] N. Fuchs, Phys. Rev. **140**, B911, (1965).

even after the introduction of gauge fields. His argument is based on the use of the Lorentz gauge, and Schwinger's extended-operator formalism.[9] His conclusions disagree with those reached by Higgs[4], using the radiation gauge. However, Fuchs has already remarked that his method leads to considerable difficulties of interpretation inasmuch as the energy spectrum is not bounded below. This is not the only difficulty which the method encounters.

In order to bring out clearly the relationships between the Lorentz-gauge and radiation-gauge treatments, we shall begin by re-examining, in Secs. II and III, the simple Abelian-gauge case. The Coulomb-gauge treatment given in Sec. II contains a summary of some of Higgs's results, re-expressed in a form appropriate to the comparison with the Lorentz-gauge treatment given in Sec. III. In particular, we aim to show that the correct treatment of $U(1)$ symmetry breaking in Schwinger's extended-operator formalism does not involve any alteration in the "subsidiary condition" which selects the gauge-invariant physical states. (This condition is changed in the method proposed by Fuchs.)

In Sec. IV we go on to discuss the generalization of the model treated by Higgs to an arbitrary non-Abelian group. Our aim here is not to give a complete discussion of this model but mainly to show how the number of massless fields in the theory is determined, in terms of the "canonical number" introduced by Bludman and Klein.[10] Finally, in Sec. V, we exhibit a model involving spontaneously broken $U(2)$ symmetry which is entirely free of massless particles, and moreover, in which the physical states retain clear indications of the underlying symmetry. As in all such theories, the most obvious indication of symmetry breaking is the appearance of an incomplete multiplet of massive scalar particles.

Our results lead to the following conclusion: If all the currents associated with a broken non-Abelian symmetry group are coupled to gauge-vector fields, the number of massless vector bosons remaining in the theory is just the dimensionality of the subgroup of *unbroken* symmetry transformations. In particular, if there are no unbroken components of the symmetry group, then no massless particles remain.

II. COULOMB GAUGE

It will be useful to begin by summarizing in rather different language some of the results obtained by Higgs.[7]

We start with the Goldstone model: a complex scalar field ϕ described by the Lagrangian

$$L=\phi^{\mu *}\partial_\mu\phi+\phi^\mu\partial_\mu\phi^*-\phi^{\mu *}\phi_\mu-V(\phi^*\phi). \qquad (1)$$

This is clearly invariant under the constant gauge transformations

$$\phi(x)\rightarrow e^{ie\lambda}\phi(x). \qquad (2)$$

[9] J. Schwinger, Phys. Rev. **125**, 1043 (1962); **130**, 402 (1963).
[10] S. A. Bludman and A. Klein, Phys. Rev. **131**, 2364 (1963).

Consequently, the current

$$j^\mu=-ie(\phi^{\mu *}\phi-\phi^\mu\phi^*) \qquad (3)$$

satisfies

$$\partial_\mu j^\mu=0. \qquad (4)$$

If $V(\phi^*\phi)$ has a maximum at $\phi^*\phi=0$ and a minimum elsewhere, then we may expect that the expectation value of ϕ in the vacuum (or ground state) is nonzero. From the equations of motion we obtain the consistency condition

$$\langle\partial V/\partial\phi^*\rangle=\langle\phi V'(\phi^*\phi)\rangle=0, \qquad (5)$$

which serves to determine the magnitude of $\langle\phi\rangle$. If $\langle\phi\rangle=\eta$ is a possible solution, then so is $\langle\phi\rangle=\eta e^{i\alpha}$. There is, therefore, an infinitely degenerate set of vacuum states, parametrized by the phase α. Formally, transformations from one to another are generated by the "unitary operator" $e^{i\lambda Q}$ with

$$Q=\int d^3x\, j^0(x). \qquad (6)$$

However, when $\langle\phi\rangle\neq 0$ the integral here is divergent, and the various degenerate vacuums belong to unitarily inequivalent representations.

The Goldstone theorem requires the existence of massless particles in this theory. They may be exhibited by making the polar decomposition

$$\phi=2^{-1/2}\rho e^{i\vartheta}, \qquad (7)$$

introduced by Higgs. (We shall ignore problems of operator ordering.) The canonically conjugate variables are the time components of the vectors

$$\rho^\mu=2^{-1/2}(\phi^{\mu *}e^{i\vartheta}+\phi^\mu e^{-i\vartheta}),$$

and $\qquad (8)$

$$\theta^\mu=2^{-1/2}i\rho(\phi^{\mu *}e^{i\vartheta}-\phi^\mu e^{-i\vartheta})=-e^{-1}j^\mu.$$

It should be noted that ρ, ρ^μ, and ϑ^μ are all invariant under the transformations (2) while ϑ transforms according to

$$\vartheta(x)\rightarrow\vartheta(x)+e\lambda(x). \qquad (9)$$

This shows, incidentally, that there is no fundamental distinction between transformations expressible as rotations or translations of the field variables, since one may be converted into the other by a change of variables.

In terms of the new variables, the Lagrangian becomes

$$L=\rho^\mu\partial_\mu\rho+\vartheta^\mu\partial_\mu\vartheta-\tfrac{1}{2}\rho^\mu\rho_\mu-\vartheta^\mu\vartheta_\mu/2\rho^2-V(\tfrac{1}{2}\rho^2). \quad (10)$$

The broken-symmetry condition is expressed by setting

$$\rho=|\eta|+\rho', \quad \langle\rho'\rangle=0. \qquad (11)$$

Clearly, the ρ' field describes particles whose mass is determined (to lowest order) by the second derivative of V at $\rho=|\eta|$, while the ϑ field describes massless particles.

Now let us consider the coupling of the current (3) to the electromagnetic field. We have[11]

$$L=-F^{\mu\nu}\partial_\nu A_\mu+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\phi^{\mu*}(\partial_\mu+ieA_\mu)\phi+\phi^\mu(\partial_\mu-ieA_\mu)\phi^*-\phi^{\mu*}\phi_\mu-V(\phi^*\phi),\quad(12)$$

which is invariant not only under (2) but also under the gauge transformations

$$\phi(x)\to e^{ie\lambda(x)}\phi(x),\quad A_\mu(x)\to A_\mu(x)-\partial_\mu\lambda(x).\quad(13)$$

In terms of the polar variables introduced in (7) and (8), this Lagrangian becomes

$$L=-F^{\mu\nu}\partial_\nu A_\mu+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\rho^\mu\partial_\mu\rho+\vartheta^\mu\partial_\mu\vartheta-\tfrac{1}{2}\rho^\mu\rho_\mu-\vartheta^\mu\vartheta_\mu/2\rho^2-V(\tfrac{1}{2}\rho^2)+eA_\mu\vartheta^\mu.\quad(14)$$

We now wish to investigate the relationship between the Coulomb-gauge and Lorentz-gauge treatments of this Lagrangian. Let us consider first the Coulomb gauge. To preserve the analogy with our later treatment of the Lorentz gauge, it will be convenient to impose the Coulomb-gauge condition by adding to L a Lagrange multiplier term

$$-C\partial_k A^k.\quad(15)$$

This destroys the invariance under (13), but not that under the constant gauge transformations (2).

It is now convenient to introduce new variables

$$B_\mu=A_\mu+e^{-1}\partial_\mu\vartheta.\quad(16)$$

Then, using the equation

$$\vartheta_\mu=e\rho^2B_\mu$$

to eliminate ϑ_μ from the Lagrangian,[12] we obtain

$$L=-F^{\mu\nu}\partial_\nu B_\mu+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}e^2\rho^2B_\mu B^\mu+\rho^\mu\partial_\mu\rho-\tfrac{1}{2}\rho^\mu\rho_\mu-V(\tfrac{1}{2}\rho^2)-C\partial_k(B^k-e^{-1}\partial^k\vartheta).\quad(17)$$

This Lagrangian is still invariant under (2), or its equivalent

$$\vartheta(x)\to\vartheta(x)+e\lambda,\quad(18)$$

but in a completely trivial way. In fact, ϑ is determined by

$$\nabla^2\vartheta=-e\partial_kB^k,\quad(19)$$

and (18) represents merely the arbitrariness in the solution of this equation. (Explicit dependence on the coordinates is ruled out by the requirement of translational invariance.) It may be noted that the equation obtained by variation of ϑ, namely

$$\nabla^2C=0,\quad(20)$$

shows similarly that C is at most a constant.

From the structure of (17) we see that no massless particles remain in the theory. The mass of the scalar particles described by ρ' is as before determined to lowest order by the second derivative of V at $\rho=|\eta|$. Now, however, the massless particles described by ϑ have become the longitudinal modes of the vector field described by B_μ, whose mass is to a first approximation $e|\eta|$.

It may be worth recalling the reasons for the failure of the Goldstone theorem to apply in this case. The essential point is that the continuity equation (4) no longer implies the time independence of the commutator

$$\int d^3x\langle[j^0(x),\phi(0)]\rangle\quad(21)$$

because the relevant surface integral fails to vanish in the limit of infinite volume. Although the operator (6) does not exist, its commutators do exist in a formal sense provided we perform the space integration after the evaluation of the commutator. In the absence of long-range fields Q is time-independent (in the sense that these commutators are so), but, as was pointed out by Guralnik, Hagen, and Kibble,[5] this is no longer true when gauge fields are present. This is easy to verify for our particular model. We have

$$j^\mu=-e^2\rho^2B^\mu,\quad(22)$$

whence

$$\langle[j^\mu(x),\vartheta(0)]\rangle=-e^3(1/\nabla^2)\partial_k\langle[\rho^2B^\mu(x),B^k(0)]\rangle.\quad(23)$$

This form clearly exhibits the nonlocal structure of $\langle[j^l,\vartheta]\rangle$. Inserting a Lehmann spectral form on the right-hand side of this equation, it is easy to see that $\langle[j^0,\vartheta]\rangle$ is causal but that its space integral is not time-independent. Indeed in lowest order it is $-e\cos(e|\eta|x^0)$.

III. LORENTZ GAUGE

Now let us turn to the description of this model in terms of the Lorentz gauge. Following Schwinger,[9] we may impose the Lorentz-gauge condition by adding to L a Lagrange multiplier term

$$-G\partial_\mu A^\mu+\tfrac{1}{2}\alpha GG,\quad(24)$$

where α is an arbitrary constant introduced to allow direct comparison both with Schwinger's formalism ($\alpha=0$) and with the more conventional formalism ($\alpha=1$) adopted by Fuchs. Note that in second-order form (24) is equivalent to

$$-\tfrac{1}{2}\alpha^{-1}(\partial_\mu A^\mu)^2,\quad(25)$$

so that $\alpha=1$ corresponds to the usual Fermi Lagrangian.

[11] This model, or a closely related one, has been discussed in Refs. 4, 5, and 7 and also by F. Englert and R. Brout, Phys. Rev. Letters, **13**, 321 (1964).

[12] This is permissible since it is an algebraic equation for ϑ_μ. One is not allowed to solve an equation of motion and substitute the results in the Lagrangian, but one is allowed to substitute explicit solutions obtained without integration. See, for example, T. W. B. Kibble and J. C. Polkinghorne, Nuovo Cimento **8**, 74 (1958).

The advantage of the first-order form (24) lies precisely in the possibility of taking $\alpha=0$.

The generator of the gauge transformations (13) is

$$G(\lambda)=\int d^3x\,[\lambda(x)\partial^0G(x)-G(x)\partial^0\lambda(x)]$$

$$=\int d^3x\,[\lambda(j^0-\partial_kF^{0k})-G\partial^0\lambda].$$

In Schwinger's extended-operator formalism the physical states are distinguished by the gauge-invariance requirement

$$G(\lambda)\Psi=0, \tag{26}$$

or equivalently,

$$G\Psi=0,\quad (j^0-\partial_kF^{0k})\Psi=0. \tag{27}$$

On the other hand, in the more familiar Gupta-Bleuler formalism,[13] only the positive frequency components of G are required to annihilate the physical states. Both these formalisms will be considered in the sequel.

The important difference between the Coulomb and Lorentz gauges lies of course in the number of degrees of freedom. Since (15) involves no time derivatives, C is not a dynamical variable. However, in the Lorentz gauge, the canonically conjugate pairs of variables are (A_i,F^{0i}), (G,A^0), (ρ,ρ^0), and (ϑ,ϑ^0). The ρ-field excitations are essentially irrelevant to our discussion. So for simplicity we shall make the approximation of replacing ρ by $|\eta|$. This should be a good first approximation if the mass determined by the second derivative of V is large. Thus we have to consider the Lagrangian

$$L=-F^{\mu\nu}\partial_\nu A_\mu+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\vartheta^\mu\partial_\mu\vartheta -\vartheta^\mu\vartheta_\mu/2|\eta|^2+eA_\mu\vartheta^\mu-G\partial_\mu A^\mu+\tfrac{1}{2}\alpha GG. \tag{28}$$

Since this Lagrangian is only quadratic in the field variables, the resulting theory is exactly soluble.

As before, it is convenient to introduce new variables. We write

$$m^2=e^2|\eta|^2 \tag{29}$$

and make a canonical transformation to the pairs (B_i,F^{0i}), (G,G^0), and (ψ,ψ^0), where

$$\begin{aligned} B_i&=A_i+\frac{1}{e}\partial_i\vartheta+\frac{1}{m^2}\partial_iG,\\ G^0&=A^0+\frac{1}{m^2}\partial_kF^{0k},\\ \psi&=|\eta|\vartheta, \end{aligned} \tag{30}$$

and

$$\psi^0=\frac{1}{|\eta|}\vartheta^0+\frac{1}{m}\partial_kF^{0k}.$$

[13] See, for example, J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), Chap. 2.

To exhibit the covariance of the Lagrangian it is convenient to introduce also new dependent field variables B_0, G^i, and ψ^i, so that we may write

$$L=-F^{\mu\nu}\partial_\nu B_\mu+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}m^2B^\mu B_\mu +G^\mu\partial_\mu G+\psi^\mu\partial_\mu\psi-\tfrac{1}{2}\psi^\mu\psi_\mu+m\psi^\mu G_\mu+\tfrac{1}{2}\alpha GG. \tag{31}$$

Clearly, we have a vector field describing particles of mass m, and two scalar fields (in addition to field ρ' corresponding to the suppressed modes). The vector particles are, of course, precisely those found earlier in the Coulomb gauge.

To discuss the physical significance of the scalar fields in (31), we have to be more precise about the conditions on physical states. Let us first consider Schwinger's extended-operator formalism. Here the fields are not to be regarded as operators in a Hilbert space; they are extended operators acting on a more general space of functionals in which no scalar product is defined. The states are labeled by the eigenvalues of, for example, ψ, G^0, and B_i, all at one time t_0:

$$\Psi=\langle\psi',G^{0\prime},B_i'|\rangle. \tag{32}$$

The canonically conjugate variables ψ^0, G, and F^{0i} are represented as functional derivatives with respect to the appropriate variables. Then the functionals representing physical states are distinguished by the gauge-invariance requirement (26) or (27), which may now be written

$$\begin{aligned} G\Psi&=i(\delta/\delta G^{0\prime})\Psi=0,\\ -\psi^0\Psi&=i(\delta/\delta\psi')\Psi=0. \end{aligned} \tag{33}$$

It follows that Ψ is actually independent of the two scalar fields, and may be represented by a functional of B_i' alone. This is, of course, in conformity with the conclusion reached in the Coulomb-gauge treatment, that only vector particles appear in the physical states.

It should be remarked that the symmetry breaking corresponds to having a nonzero value of η and therefore of the vector particle mass m, and has nothing whatsoever to do with the conditions (33) which should be imposed whether or not the symmetry is broken. The physical states are still gauge invariant in the sense described by (33) even when the original symmetry (2) is broken. Indeed, in this formalism, the local-gauge transformations (13) do not act on the physical states at all, although of course the global transformations (2) do. This is associated with the well-known fact that while the one-parameter group of gauge transformations (2) yields a conservation law for the electric charge, the infinite-parameter group of local transformations (13) does not yield an infinite set of physical conservation laws.

The symmetry-breaking discussed by Fuchs[8] is the breaking of the conditions (33). This means that more states than usual are admitted as physical. This procedure has a number of grave disadvantages, notably the fact that when (33) is relaxed, the energy spectrum is no longer bounded below. It should also be noted that (33) provides the only guarantee that the equations of motion agree, for physical states, with those obtained from the original gauge-invariant Lagrangian. To relax this condition is not merely to break a symmetry but to change the physical equations of motion. It may be that symmetry breaking of this type has some physical relevance, despite these difficulties. What we are concerned to show here, however, is that it is not the method which is the true Lorentz-gauge analog of the Coulomb-gauge formalism described in the preceding section. To achieve agreement between the two approaches it is necessary (and sufficient) to break the symmetry under the transformations (2) while maintaining the conditions (33) intact.

It is interesting to examine in more detail the unphysical fields G and ψ, which satisfy the field equations

$$\begin{aligned} \partial_\mu G &= -m\psi_\mu\,, \quad \partial_\mu \psi = \psi_\mu - mG_\mu\,, \\ \partial_\mu G^\mu &= \alpha G\,, \quad \partial_\mu \psi^\mu = 0\,, \end{aligned} \tag{34}$$

whence

$$\Box G = 0\,, \quad \Box\psi = -\alpha m G. \tag{35}$$

From these equations and the canonical commutation rules, we can easily derive covariant commutation relations. The consistency of the conditions (33) for different times is assured by the relation

$$[G(x),G(0)] = 0. \tag{36}$$

We also find

$$[G(x),\psi(0)] = -imD(x) = \frac{im}{2\pi}\epsilon(x^0)\delta(x^2)\,, \tag{37}$$

and finally,

$$\begin{aligned} [\psi(x),\psi(0)] &= iD(x) - i\alpha m^2\left[\frac{\partial}{\partial\kappa^2}\Delta(x,\kappa^2)\right]_{\kappa^2=0} \\ &= -\frac{i}{2\pi}\epsilon(x^0)[\delta(x^2) + \tfrac{1}{4}\alpha m^2 \vartheta(x^2)]. \end{aligned} \tag{38}$$

In the extended-operator formalism, these relations are not required to possess a representation in a Hilbert space, and pose no particular problem.

However, let us now examine the formalism of Gupta and Bleuler, in which the fields are operators in a Hilbert space of indefinite metric, and the physical states are selected by the condition

$$G^{(+)}|\,\rangle = 0. \tag{39}$$

Then, taking the vacuum expectation value of the commutator (38), and denoting the diagonal elements (± 1) of the metric operator by ρ_n, we find

$$\begin{aligned} &\sum_n \rho_n |\langle n|\psi(0)|0\rangle|^2 (2\pi)^4 [\delta_4(p_n - k) - \delta_4(p_n + k)] \\ &= \epsilon(k^0)2\pi\delta(k^2) - \alpha m^2\left[\frac{\partial}{\partial\kappa^2}\{\epsilon(k^0)2\pi\delta(k^2-\kappa^2)\}\right]_{\kappa^2=0}. \end{aligned}$$

For $k\neq 0$, we can drop the second term in the square brackets on the left-hand side of this equation, and replace $\epsilon(k^0)$ on the right by $\theta(k^0)$. However, in the neighborhood of $k=0$ this is *not* correct. For, although $(\partial/\partial\kappa^2)[\epsilon(k^0)2\pi\delta(k^2-\kappa^2)]$ yields a well-defined distribution in the limit $\kappa^2 \to 0$, the corresponding symmetric expression $(\partial/\partial\kappa^2)[2\pi\delta(k^2-\kappa^2)]$ does not.[14] In fact, integrating it over a small volume around $k=0$, one finds a result which diverges like $\ln\kappa^2$ as $\kappa^2 \to 0$. To obtain a well-defined distribution, one must subtract off this infinite term, and consider for example the expression

$$\frac{\partial}{\partial\kappa^2}[2\pi\delta(k^2-\kappa^2)] - 2\pi^2 \ln\frac{\kappa^2}{M^2}\delta_4(k)\,,$$

for any constant mass M, which does possess a well-defined limit as $\kappa^2 \to 0$. Thus, we obtain

$$\begin{aligned} &\sum_n \rho_n |\langle n|\psi(0)|0\rangle|^2 (2\pi)^4 \delta_4(p_n - k) \\ &= \theta(k^0)2\pi\delta(k^2) - \alpha m^2\left[\frac{\partial}{\partial\kappa^2}\{\theta(k^0)2\pi\delta(k^2-\kappa^2)\} - \pi^2 \ln\frac{\kappa^2}{M^2}\delta_4(k)\right]_{\kappa^2=0}. \end{aligned} \tag{40}$$

The arbitrariness in the constant M arises from the fact that division of a distribution equation by $\epsilon(k^0)$ yields a result arbitrary to the extent of a multiple of $\delta_4(k)$. This arbitrary constant may also be regarded as a manifestation of the arbitrary additive constant in the field ψ, since a field translation of ψ would clearly change the left-hand side of (40) by a multiple of $\delta_4(k)$.

It is evident that unless $\alpha=0$, the matrix elements of ψ must be extremely singular. The basic reason for this may be seen from the equations of motion (35). Since the field G has a singularity at $k^2=0$, like $1/k^2$, we see that ψ must have an even worse singularity, like $(1/k^2)^2$. The structure in the right side of (40) is itself well defined[14] (for given M) but can only be obtained by a cancellation between infinite positive and negative terms on the left. It is in this sense that the conventional Lorentz-gauge treatment (which implies $\alpha=1$) is inconsistent.

It is very interesting that the particular case $\alpha=0$ does not encounter these problems. This case is interesting from another point of view also. Since this theory is manifestly covariant, the Goldstone theorem is certainly applicable. In general, the massless bosons it predicts are described by the field G [since $\partial_\mu G$ is the conserved current; note that the proof of the theorem rests on the commutator function (37)]. However, in the special case $\alpha=0$, there are two independent massless fields corresponding to the fact that in that case the Lagrangian is invariant (up to a divergence) not only under the transformations (2) but also under $G(x) \to G(x)+\lambda$.

[14] J. Gårding and S. Lions, Nuovo Cimento Suppl. 14, 9 (1959).

We note that Schwinger's extended-operator formalism is more flexible, even in the simple Abelian-gauge case, than that of Gupta and Bleuler, which works properly only for $\alpha=0$. In the case of non-Abelian gauges, the Gupta-Bleuler formalism is wholly inapplicable, because the analog to G no longer satisfies the free wave equation, and cannot therefore be resolved covariantly into positive and negative frequency components, so that (39) becomes meaningless.

IV. NON-ABELIAN GAUGE MODELS

Let us now consider a model of an n-component real scalar field ϕ, which transforms according to a given n-dimensional representation of a compact Lie group G of dimension g,

$$\phi(x)\to e^{\lambda\cdot T}\phi(x)\,,$$

and

$$\lambda\cdot T=\lambda^A T_A\,. \tag{41}$$

Here the λ^A are g real parameters and the T_A are g real antisymmetric $n\times n$ matrices obeying the commutation relations of the associated Lie algebra

$$[T_A,T_B]=T_C t^C{}_{AB}\,. \tag{42}$$

These relations are satisfied in particular by the matrices $t_A=(t^C{}_{AB})$ of the adjoint representation.

The Lagrangian density is taken to be

$$L=\phi^\mu\partial_\mu\phi-\tfrac{1}{2}\phi^\mu\phi_\mu-V(\phi)\,, \tag{43}$$

where ϕ_μ transforms like ϕ, and $V(\phi)$ is invariant, under (41). (The notation implies the use of an invariant scalar product in the n-dimensional space of the variables ϕ.) From the invariance of the Lagrangian we may infer the existence of conserved currents,

$$j_A{}^\mu=-\phi^\mu T_A\phi\,. \tag{44}$$

In any finite space-time volume, the transformation (41) is generated by the operators

$$\lambda^A\int d^3x\, j_A{}^0(x)\,. \tag{45}$$

However if ϕ has a nonvanishing expectation value so that the symmetry is broken, then as usual the integrals over all space do not exist, and the transformation (41) is not unitarily implementable.

The expectation value $\langle\phi\rangle$ in a translationally invariant (vacuum) state is restricted by the consistency condition

$$\langle\partial V/\partial\phi\rangle=0\,. \tag{46}$$

If $\langle\phi\rangle=\eta$ is a consistent broken-symmetry solution then so also is $\langle\phi\rangle=e^{\lambda\cdot T}\eta$, for any λ. If we choose a particular η, then all other physically equivalent solutions may be expressed in this form. [There may, of course, be other physically inequivalent disjoint solutions of (46)]. However, not all these will be independent in general, since there may be a subgroup G_η of G which leave η invariant (the *isotropy group* of G at η[15]). This is the subgroup corresponding to symmetries which are *not* broken. Let ν—the "canonical number" of Bludman and Klein[10]—be the number of algebraically independent invariants constructible from η, or equivalently the number of algebraically independent invariant "Hartree conditions" (46). We assume that η can be brought to a canonical form in which only ν of its components are nonzero. Further, we assume that none of these components is accidentally zero. Then the set of equivalent solutions $\langle\phi\rangle$ forms a manifold of dimension $r=n-\nu$. It is clear that this manifold may be identified with the factor space G/G_η (not in general a group). In fact, we may write the representative of each element of G in the form

$$e^{\mu\cdot T}e^{\nu\cdot T}\,, \tag{47}$$

where $e^{\nu\cdot T}$ is an element of the $(g-r)$-dimensional subgroup G_η, and the remaining r parameters μ serve to parametrize the solutions $\langle\phi\rangle$ by the identification

$$\langle\phi\rangle=e^{\mu\cdot T}\eta\,. \tag{48}$$

Since these solutions are physically equivalent (though, of course, unitarily inequivalent) there is no essential loss of generality in choosing

$$\langle\phi\rangle=\eta\,. \tag{49}$$

It will be convenient to adopt a set of coordinates in which the first r elements of η are zero, while the last ν elements are not. It is then useful to make a corresponding "polar decomposition" of ϕ, analogous to (7). We write

$$\phi=e^{\vartheta\cdot T}\rho=e^{\vartheta\cdot T}(\eta+\rho')\,, \tag{50}$$

where ϑ, like μ, has r components, while the first r components of ρ are zero. We shall distinguish these components by using indices $a, b, \cdots=1, \cdots, r$ and $\alpha, \beta, \cdots=r+1, \cdots, n$.

Consider the action of the generators T_A on ρ. We note that $T_A\rho=0$ for those indices A belonging to elements of the Lie algebra of G_η. A nonzero result occurs only for T_a, $a=1, \cdots, r$. Moreover, consider the matrix

$$X_a{}^b=(T_a\rho)^b\,, \tag{51}$$

which consists of the components of these vectors in the subspace in which ρ itself vanishes. We assert that this matrix is nonsingular. For, if not, we can find some linear combination of the generators T_a which gives zero acting on all vectors ρ. But then this linear combination should be an element of the Lie algebra of G_η, which it is not.

[15] See, for example, R. Hermann, *Lie Groups for Physicists* (W. A. Benjamin, Inc., New York, 1966), p. 3.

The canonically conjugate variables to ϑ and ρ are, as in (8), the time components of the 4-vectors

$$\vartheta_a{}^\mu = \phi^\mu e^{\vartheta \cdot T} T_b \rho \Lambda^b{}_a, \tag{52}$$

$$\rho_\alpha{}^\mu = (\phi^\mu e^{\vartheta \cdot T})_\alpha, \tag{53}$$

where

$$\Lambda^b{}_a = [(1-e^{-\vartheta \cdot t})/\vartheta \cdot t]^b{}_a. \tag{54}$$

This follows from the relation

$$e^{-\lambda \cdot T} \frac{\partial}{\partial \lambda_A} e^{\lambda \cdot T} = T_B \left(\frac{1-e^{-\lambda \cdot t}}{\lambda \cdot t} \right)^B{}_A. \tag{55}$$

Note, however, that we have defined Λ as an $r \times r$ submatrix of the $g \times g$ matrix appearing in (55). This is permissible because, as we have seen, $T_B \rho = 0$ unless B is one of the first r indices. Thus, in place of $T_B \rho \Lambda^B{}_a$, we can write $T_b \rho \Lambda^b{}_a$.

The currents (44) may all be expressed in terms of the canonical variables ϑ and ϑ^μ. In fact

$$j^\mu{}_A = -\vartheta_a{}^\mu (\Lambda^{-1})^a{}_b (e^{-\vartheta \cdot t})^b{}_A. \tag{56}$$

In terms of the new variables the Lagrangian (43) may be written

$$L = \rho^\mu \partial_\mu \rho - \tfrac{1}{2} \rho^\mu \rho_\mu - V(\rho) + \vartheta_a{}^\mu \partial_\mu \vartheta^a - \tfrac{1}{2} [\{ \vartheta_a{}^\mu (\Lambda^{-1})^a{}_b - \rho^\mu T_b \rho \} (X^{-1})^b{}_c]^2. \tag{57}$$

Here we have used explicitly the fact that X, as defined by (51), is nonsingular. This Lagrangian, of course, retains the invariance under (41). Clearly, ρ and ρ^μ are invariant. The effect on ϑ of an infinitesimal transformation may be written as

$$\delta \vartheta^a = (\Lambda^{-1})^a{}_b (e^{-\vartheta \cdot t})^b{}_A \delta \lambda^A, \tag{58}$$

while the effect on ϑ^μ may most easily be expressed in the form

$$\delta \vartheta_a{}^\mu = -\vartheta_b{}^\mu \frac{\partial}{\partial \vartheta^a} [(\Lambda^{-1})^b{}_c (e^{-\vartheta \cdot t})^c{}_A] \delta \lambda^A. \tag{59}$$

The masses of the particles described by the fields ρ' would be principally determined, if the interaction is weak, by the second derivatives of V near the point $\rho = \eta$. Normally, these will all be positive. The absence from (57) of any terms involving ϑ, but not ϑ^μ, is indicative of the fact that the fields ϑ contain the massless excitations required by the Goldstone theorem.

Now let us consider the coupling of the currents (44) or (56) to a set of g gauge fields $A^A{}_\mu$. Thus, we now take

$$L = -\tfrac{1}{2} F_A{}^{\mu\nu} (\partial_\nu A^A{}_\mu - \partial_\mu A^A{}_\nu - t^A{}_{BC} A^B{}_\mu A^C{}_\nu) + \tfrac{1}{4} F_A{}^{\mu\nu} F^A{}_{\mu\nu} + \rho^\mu \partial_\mu \rho - \tfrac{1}{2} \rho^\mu \rho_\mu - V(\rho) + \vartheta_a{}^\mu \partial_\mu \vartheta^a - \tfrac{1}{2} [\{ \vartheta_a{}^\mu (\Lambda^{-1})^a{}_b - \rho^\mu T_b \rho \} (X^{-1})^b{}_c]^2 + \vartheta_a{}^\mu (\Lambda^{-1})^a{}_b (e^{-\vartheta \cdot t})^b{}_A A^A{}_\mu. \tag{60}$$

We shall work implicitly in the Coulomb gauge, but will not explicitly indicate the Lagrange multiplier term analogous to (15). It is now convenient to introduce new variables

$$B^A{}_\mu = (e^{-\vartheta \cdot t})^A{}_B A^B{}_\mu + [(1-e^{-\vartheta \cdot t})/\vartheta \cdot t]^A{}_b \partial_\mu \vartheta^b$$

and

$$G_A{}^{\mu\nu} = F_B{}^{\mu\nu} (e^{\vartheta \cdot t})^B{}_A. \tag{61}$$

Then, elimination of the variables ϑ^μ yields the Lagrangian

$$L = -\tfrac{1}{2} G_A{}^{\mu\nu} (\partial_\nu B^A{}_\mu - \partial_\mu B^A{}_\nu - t^A{}_{BC} B^B{}_\mu B^C{}_\nu) + \tfrac{1}{4} G_A{}^{\mu\nu} G^A{}_{\mu\nu} + \rho^\mu \partial_\mu \rho - \tfrac{1}{2} \rho^\mu \rho_\mu - V(\rho) + \rho^\mu T_a \rho B^a{}_\mu + \tfrac{1}{2} (X^a{}_b B^b{}_\mu)^2. \tag{62}$$

It is clear from this form that $g-r$ of the g vector fields have zero mass and do not interact directly with the fields ρ, while the remaining r have masses given in lowest order by the mass matrix

$$M_{ab} = -(\eta T_a)_c (T_b \eta)^c, \tag{63}$$

which is positive definite because of the antisymmetry of the matrices T_a.

We may summarize the situation as follows. Before introducing the gauge-vector fields, we have r massless scalar fields which may be placed in one-to-one correspondence with the *broken* components of the symmetry group, and $n-r=\nu$ massive scalar fields. When the vector fields are coupled in, the ν massive scalar fields remain, but the massless scalar fields combine with r of the vector fields to yield r massive vector fields. We are left finally with $g-r$ massless vector fields corresponding to the unbroken components of the symmetry group. Thus, in order to avoid the appearance of *any* massless particles, it is necessary to choose a representation for which $r=g$; or, in other words, for which the subgroup G_η of elements of G leaving η invariant is trivial or at most a discrete group.

V. A SIMPLE MODEL

As an illustration of the discussion in the preceding section, we shall consider here a simple model of broken $U(2)$ symmetry in which no massless particles remain.

The model contains a complex three-component field $\boldsymbol{\phi} = (\phi_i)$ and four vector fields A_μ and $\mathbf{A}_\mu = (A_{i\mu})$. It is described by the Lagrangian

$$L = -\tfrac{1}{2} F^{\mu\nu} (\partial_\nu A_\mu - \partial_\mu A_\nu) + \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} - \tfrac{1}{2} \mathbf{F}^{\mu\nu} \cdot (\partial_\nu \mathbf{A}_\mu - \partial_\mu \mathbf{A}_\nu - e \mathbf{A}_\mu \times \mathbf{A}_\nu) + \tfrac{1}{4} \mathbf{F}^{\mu\nu} \cdot \mathbf{F}_{\mu\nu} + \boldsymbol{\phi}^{\mu *} \cdot (\partial_\mu \boldsymbol{\phi} + e \mathbf{A}_\mu \times \boldsymbol{\phi} + ie A_\mu \boldsymbol{\phi}) + \boldsymbol{\phi}^\mu \cdot (\partial_\mu \boldsymbol{\phi}^* + e \mathbf{A}_\mu \times \boldsymbol{\phi}^* - ie A_\mu \boldsymbol{\phi}^*) - \boldsymbol{\phi}^{\mu *} \cdot \boldsymbol{\phi}_\mu - V(\boldsymbol{\phi}^* \cdot \boldsymbol{\phi}; |\boldsymbol{\phi}^2|). \tag{64}$$

This is invariant under infinitesimal transformations of the form

$$\delta \boldsymbol{\phi} = ie\delta\lambda \boldsymbol{\phi} + e\delta\boldsymbol{\omega} \times \boldsymbol{\phi}, \quad \delta \boldsymbol{\phi}_\mu = ie\delta\lambda \boldsymbol{\phi}_\mu + e\delta\boldsymbol{\omega} \times \boldsymbol{\phi}_\mu,$$
$$\delta A_\mu = -\partial_\mu \delta\lambda, \quad \delta F_{\mu\nu} = 0,$$
$$\delta \mathbf{A}_\mu = e\delta\boldsymbol{\omega} \times \mathbf{A}_\mu - \partial_\mu \delta\boldsymbol{\omega}, \quad \delta \mathbf{F}_{\mu\nu} = e\delta\boldsymbol{\omega} \times \mathbf{F}_{\mu\nu},$$

which belong to the group $U(1)\times 0(3)\simeq U(1)\times SU(2) = U(2)$.

We may now write

$$\boldsymbol{\phi}=e^{ie\lambda}e^{\boldsymbol{\vartheta}\cdot\mathbf{t}}\boldsymbol{\varrho}, \tag{65}$$

where

$$t_j=(t^i{}_{jk})=(e\epsilon_{ijk})$$

and

$$\boldsymbol{\varrho}=2^{-1/2}(\boldsymbol{\varrho}_1+i\boldsymbol{\varrho}_2),$$

and we may choose, for example, to set

$$\boldsymbol{\varrho}_1=\begin{pmatrix}\rho_1\\0\\0\end{pmatrix}, \quad \text{and} \quad \boldsymbol{\varrho}_2=\begin{pmatrix}0\\\rho_2\\0\end{pmatrix}.$$

Then the transformation (61) leads to the equivalent Lagrangian

$$\begin{aligned}L=&-\tfrac{1}{2}G^{\mu\nu}(\partial_\nu B_\mu-\partial_\mu B_\nu)+\tfrac{1}{4}G^{\mu\nu}G_{\mu\nu}\\&-\tfrac{1}{2}\mathbf{G}^{\mu\nu}\cdot(\partial_\nu\mathbf{B}_\mu-\partial_\mu\mathbf{B}_\nu-e\mathbf{B}_\mu\times\mathbf{B}_\nu)+\tfrac{1}{4}\mathbf{G}^{\mu\nu}\cdot\mathbf{G}_{\mu\nu}\\&+\rho_1{}^\mu\partial_\mu\rho_1+\rho_2{}^\mu\partial_\mu\rho_2-\tfrac{1}{2}\rho_1{}^\mu\rho_{1\mu}-\tfrac{1}{2}\rho_2{}^\mu\rho_{2\mu}\\&-V(\tfrac{1}{2}(\rho_1{}^2+\rho_2{}^2),\tfrac{1}{2}|\rho_1{}^2-\rho_2{}^2|)\\&+\tfrac{1}{2}e^2[\rho_1{}^2B_2{}^\mu B_{2\mu}+\rho_2{}^2B_1{}^\mu B_{1\mu}\\&+\tfrac{1}{2}(\rho_1+\rho_2)^2(B^\mu-B_3{}^\mu)(B_\mu-B_{3\mu})\\&\qquad+\tfrac{1}{2}(\rho_1-\rho_2)^2(B^\mu+B_3{}^\mu)(B_\mu+B_{3\mu})].\end{aligned} \tag{66}$$

If

$$\langle\rho_{1,2}\rangle=\eta_{1,2},$$

then we have in lowest-order four vector particles of masses

$$\eta_1{}^2,\eta_2{}^2, \quad \text{and} \quad (\eta_1\pm\eta_2)^2,$$

in addition to the two scalar particles whose masses are determined by the second derivatives of V. Provided that we choose the form of V so that none of these quantities vanish, no zero-mass particles remain in the theory.

It should be remarked that this model is in no sense unusual. Suppose we construct such a model for any group G, with g vector fields transforming according to the adjoint representation, and n scalar fields ϕ transforming according to some other specified representation. Then, except for a few representations corresponding to small values of n, the number of invariants constructible from ϕ will be exactly $\nu=n-g$. In that case the model will be completely free of massless particles. For example, for $SU(2)$, the *only* two irreducible representations for which G_η is not trivial are the one-dimensional identity representation and the three-dimensional adjoint representation. [The two-dimensional (fundamental) representation must be complex. It therefore has four real components, ($n=4$), and a single invariant ($r=1$).]

We note certain characteristic features of our model. It is perfectly possible to describe it without ever introducing the notion of symmetry breaking, merely by writing down the Lagrangian (66). Indeed if the physical world were really described by this model, it is (66) rather than (64) to which we should be led by experiment. The only advantage of (64) is that it is easier to understand the appearance of an exact symmetry than of an approximate one. Experimentally, we would discover the existence of a set of four vector bosons with different masses but whose interactions exhibited a remarkable degree of symmetry. We would also discover a pair of scalar particles forming an apparently incomplete multiplet under the group describing this symmetry. In such circumstances it would surely be regarded as a considerable advance if we could recast the theory into a form described by the symmetric Lagrangian (64).

VI. CONCLUSIONS

In this paper we have tried to establish two main points: Firstly, that it is possible to handle the problem of symmetry-breaking consistently in the Lorentz gauge as well as in the Coulomb gauge, and to reach identical conclusions; and, secondly, that in the case of non-Abelian gauge groups (as well as in the Abelian case) the introduction of gauge-vector fields coupled to currents associated with a broken symmetry can serve to eliminate massless particles completely from the theory. The condition that there be no massless particles is also the condition that no components of the symmetry remain unbroken. For each unbroken component there remains a massless vector field. This is of course precisely the physical situation in regard to groups like $SU(3)$. There is just one unbroken component, generated by the electric charge, and one known massless vector boson, the photon.

Considerable difficulties still face any theory of this type. In particular it is not so easy to give the vector bosons a reasonable mass as to give them some nonzero mass. However, it does at least seem worthy of further study.

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