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Inner Bremsstrahlung in Low-Energy Electron-Neutrino (Antineutrino) Scattering*†

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We have calculated the differential and total cross sections for electron-neutrino (antineutrino) scattering accompanied by hard-photon emission. A numerical analysis of these cross sections including the radiative corrections to order α, soft-photon emission, and neutrino (antineutrino) electromagnetic form-factor interference terms calculated by Lee and Sirlin has been performed, and the results are given graphically.

I. INTRODUCTION

THE Feynman–Gell-Mann current-current theory of weak interactions[1] predicts the existence of electron-neutrino (antineutrino) scattering:

$$e^- + \nu_e \to e^- + \nu_e, \quad (1)$$

$$e^- + \bar{\nu}_e \to e^- + \bar{\nu}_e. \quad (2)$$

ν_e ($\bar{\nu}_e$) refers to the neutrino (antineutrino) associated with the electron. These reactions are also expected in a W-meson theory of weak interactions. The cross section for these processes at energies at which reasonable intensities of ν_e and $\bar{\nu}_e$ are available[2] is very small (10^{-43}–10^{-44} cm^2), and it is not surprising that even if they exist they have not been observed. In this paper we have assumed the existence of both reactions (1) and (2).

Assuming weak interactions are mediated through the exchange of a charged vector meson, the neutrino can acquire an electromagnetic current distribution[3] through the virtual transition

$$\nu_e \rightleftharpoons e^- + W^+, \quad (3)$$

and can therefore interact electromagnetically with charged particles. Figure 1(a) shows reaction (1) occurring via a W-meson exchange; Fig. 1(b) shows the reaction occurring through exchange of a photon. It has been pointed out[3,4] that processes (1) and (2) can be used to measure the electromagnetic form factor of the ν_e or $\bar{\nu}_e$. The interference term between the diagrams of Figs. 1(a) and 1(b) is proportional to the electromagnetic form factor of the neutrino and is α^{-1} times smaller than the cross section due to the weak interactions alone ($\alpha \approx 1/137$ is the fine-structure constant).

If it were possible to make accurate enough measurements, one could extract information on the electromagnetic form factor of the neutrino simply by observing this interference term. To do this, one has to know, of course, the contributions of the other radiative corrections which are also of order α. Furthermore, if the experimental setup is such that no inner bremsstrahlung photons are detected, one has to include the cross section for scattering with emission of hard photons:

$$e^- + \nu_e(\text{or } \bar{\nu}_e) \to e^- + \nu_e(\text{or } \bar{\nu}_e) + \gamma. \quad (4)$$

The radiative corrections to elastic electron-neutrino (antineutrino) scattering to order α have been calculated by Lee and Sirlin.[4] In the case of the inner bremsstrahlung correction they limited themselves to the case that only soft photons are undetectable. It is the purpose of this paper to extend their calculation to include hard-photon emission. This correction has to be included in all experiments in which the photon emitted in reaction (4) remains undetected.

Section II is a summary of the formulas derived in Ref. 4 for the cross sections for reactions (1) and (2), including radiative corrections to order α and the soft-photon emission contribution. In Sec. III we calculate the cross section for process (4) for photon energies $\omega \geqslant \epsilon$ (ϵ is the maximum soft-photon energy). This cross section depends on ϵ. Our calculation accordingly proceeds in two steps. We first isolate the ϵ-dependent part[5] and determine its contribution to the cross section. The ϵ-dependent term is found to have the same magnitude as the corresponding term in the soft-photon calculation, but its sign is opposite, so that the total cross section for inner bremsstrahlung with $\omega \geqslant 0$ is independent of ϵ. We next proceed to evaluate the contribution of the ϵ-independent part. The final formulas obtained are fairly elaborate, and a numerical analysis was necessary to extract their content. The

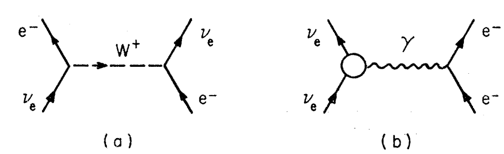

FIG. 1. Feynman diagrams for (a) interaction of electron and neutrino proceeding through the exchange of a W^+ meson, and (b) interaction of an electron and a neutrino via one photon exchange.

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[1] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[2] Reasonable intensities of ν_e are produced by β^+ emitters and in K capture. $\bar{\nu}_e$ are available from β^- decay.

[3] J. Bernstein and T. D. Lee, Phys. Rev. Letters **11**, 512 (1963).

[4] T. D. Lee and A. Sirlin, Rev. Mod. Phys. **36**, 666 (1964).

[5] By ϵ-dependent part, we mean that part of the cross section that diverges logarithmically when one takes the limit $\epsilon \to 0$.

results of the numerical analysis and our conclusions are given in Sec. IV. All our formulas are given in natural units with $\hbar=c=1$.

II. CROSS SECTION FOR ELECTRON-NEUTRINO (ANTINEUTRINO) SCATTERING WITH ONLY SOFT-PHOTON EMISSION

Following Lee and Sirlin[4] let us separate the differential cross section for electron-neutrino (antineutrino) scattering as follows:

$$\frac{d\sigma}{dE}=\frac{d\sigma_0}{dE}+\frac{d\sigma_M}{dE}+\frac{d\sigma_{\rm rad}}{dE}+\frac{d\sigma_\gamma}{dE}. \tag{5}$$

E is the final electron energy in the laboratory system in which the electron is initially at rest. $d\sigma_0/dE$ is the differential cross section for the nonradiative scattering corresponding to the Feynman diagram of Fig. 1(a). $d\sigma_M/dE$ results from the interference of the Feynman diagrams of Figs. 1(a) and 1(b), while $d\sigma_{\rm rad}/dE$ and $d\sigma_\gamma/dE$ are the contributions of the radiative corrections and inner bremsstrahlung (to order α), respectively. $d\sigma/dE$ has been calculated in Ref. 4. In their calculation, Lee and Sirlin limited themselves to the case that only soft photons are undetected. In what follows we summarize their results:

$$d\sigma_0(\nu_e)=(2mG^2/\pi)dE, \tag{6}$$

$$d\sigma_0(\bar\nu_e)=\frac{2mG^2}{\pi}\left(1-\frac{q^2}{2m\nu}\right)^2 dE, \tag{7}$$

$$d\sigma_M(\nu_e)=\frac{2m\alpha G^2}{\pi^2}[\tfrac{1}{4}f(q^2)]\left(1-\frac{q^2}{4\nu^2}\right)dE, \tag{8}$$

$$d\sigma_M(\bar\nu_e)=\frac{2m\alpha G^2}{\pi^2}[\tfrac{1}{4}f(q^2)]\left[\left(1-\frac{q^2}{2m\nu}\right)^2-\frac{q^2}{4\nu^2}\right]dE, \tag{9}$$

$$d\sigma_{\rm rad}(\nu_e)=\frac{2m\alpha G^2}{\pi^2}\left\{I_{\rm rad}-\left[1+\frac{q^2}{4m\nu^2}(m-2\nu)\right]\times\frac{\varphi}{\sinh 2\varphi}\right\}dE, \tag{10}$$

$$d\sigma_{\rm rad}(\bar\nu_e)=\frac{2m\alpha G^2}{\pi^2}\left\{\left(1-\frac{q^2}{2m\nu}\right)^2 I_{\rm rad}-\left[1+\frac{q^2}{4m\nu^2}(m-2\nu)\right]\frac{\varphi}{\sinh 2\varphi}\right\}dE, \tag{11}$$

$$d\sigma_{\text{soft }\gamma}=(\alpha/\pi)I_\gamma d\sigma_0, \tag{12}$$

where

$$f(q^2)\approx-\frac{5}{3}\left(\ln\alpha+\frac{38}{15}\right)-\frac{8}{3}\ln\left(\frac{m_W}{m}\right)+\frac{16}{3}\left(\frac{m^2}{q^2}\right)+\frac{4}{3}\left(1-\frac{2m^2}{q^2}\right)\left[1+\left(\frac{4m^2}{q^2}\right)\right]^{1/2}\times\ln\left[\frac{1+(1+4m^2/q^2)^{1/2}}{-1+(1+4m^2/q^2)^{1/2}}\right], \tag{13}$$

$$I_{\rm rad}=-\varphi\tanh\varphi-\frac{1}{\tanh 2\varphi}\int_0^\varphi 4\alpha\tanh\alpha\, d\alpha-2\left(1-\frac{2\varphi}{\tanh 2\varphi}\right)\left(1+\ln\frac{\Delta}{m}\right), \tag{14}$$

and

$$I_\gamma=\left(2-\frac{4\varphi}{\tanh 2\varphi}\right)\ln\frac{\Delta}{\epsilon}+(1-2\ln 2)+\frac{1}{2\tanh 2\varphi}\times\{4\varphi[1-2\ln(\sinh 2\varphi)]+L(e^{4\varphi})-L(e^{-4\varphi})\}. \tag{15}$$

These results apply to the frame of reference in which the target electron is initially at rest. m and m_W are the electron and W-meson mass, respectively, while q is the 4-momentum transfer from neutrino (antineutrino) to electron. ν is the energy of the incident neutrino (antineutrino) and E, that of the electron in the final state.

$$q^2=2m(E-m)=4m^2\sinh^2\varphi. \tag{16}$$

G is the Fermi μ-decay coupling constant, Δ a fictitious mass associated with the photon for the infrared calculation, and ϵ is the maximum soft-photon energy.

$f(q^2)$ is connected to the neutrino form factor $F(q^2)$ through the relation[3]

$$F(q^2)=-(G^2e/16\pi^2)q^2f(q^2). \tag{17}$$

$(-e)$ is the charge of the electron and $\alpha=e^2/4\pi\approx 1/137$ the fine-structure constant.

The Spence function $L(x)$ is defined as

$$L(x)=\int_0^x\frac{\ln|1-y|}{y}dy. \tag{18}$$

In formula (6) through (13) terms of order $(m/m_W)^2$, $\frac{1}{4}(q^2/m_W^2)$ and $(\nu/m_W)^2$ were neglected. This is justified in all practical applications since the available ν_e and $\bar\nu_e$ sources of reasonable intensity[2] do not give neutrinos (antineutrinos) with energies greater than 20 MeV.

III. CROSS SECTION FOR HARD-PHOTON EMISSION IN THE REACTION ν_e (or $\bar\nu_e$)$+e^-\rightarrow\nu_e$ (or $\bar\nu_e$)$+e^-+\gamma$

A. The Transition Probability

We want to calculate the transition probability for the inner bremsstrahlung process (4) when the photon is emitted with energy $\omega \geqslant \epsilon$ (ϵ is a small positive nonzero energy). Let us introduce the following 4-momenta[6]:

$k_\nu = (\mathbf{v}, i\nu)$ = initial 4-momentum of neutrino (antineutrino);

$k_\nu' = (\mathbf{v}', i\nu')$ = final 4-momentum of neutrino (antineutrino);

$p = (0, im)$ = initial 4-momentum of electron;

$p' = (\mathbf{l}, iE)$ = final 4-momentum of electron;

$k = (\mathbf{k}, i\omega)$ = 4-momentum of photon.

m is the electron mass, and all momenta are measured in the laboratory system.

We restrict ourselves to the case when $\frac{1}{4}(q^2/m_W^2) \ll 1$ and $(m/m_W)^2 \ll 1$ $(q = k_\nu' - k_\nu)$. In this limit, one can calculate the transition rate for process (4) using the 4-Fermi Lagrangian density interaction[7]

$$\mathcal{L}_I = \frac{G}{\sqrt{2}}[-i\bar{\psi}_{\nu e}(x)\gamma_\lambda(1+\gamma_5)\psi_{\nu e}(x)] \times [-i\bar{\psi}_e(x)\gamma_\lambda(1+\gamma_5)\psi_e(x)]. \quad (19)$$

The lowest-order Feynman diagrams for reaction (4) in the context of the 4-Fermi theory are given in Fig. 2. Using Feynman rules, one can evaluate the transition amplitude corresponding to these diagrams, and from this, the transition probability. After averaging over initial electron helicities and summing over final electron and photon helicities one obtains the following result for the transition probability w per unit volume and time:

$$w = \left[\frac{(2\pi)^2 eG}{2V^2(\omega V)^{1/2}}\right]^2 \delta^4(p' + k_\nu' + k - p - k_\nu) T, \quad (20)$$

where V is the normalization volume of our wave functions. T is given by

$$T = \frac{8}{m^2\omega E} T_1 + \frac{8}{mE(E\omega - \mathbf{l}\cdot\mathbf{k})} T_2 + \frac{8}{E(E\omega - \mathbf{l}\cdot\mathbf{k})^2} T_3, \quad (21)$$

where

$$T_1(\nu_e) = \omega(E - \mathbf{l}\cdot\hat{p}')(1 - \hat{k}\cdot\hat{p}) + m\omega(\hat{k}\cdot\hat{p})(1 - \hat{k}\cdot\hat{p}'), \quad (22a)$$

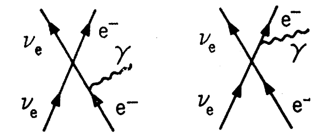
$$T_2(\nu_e) = \omega(1 - \hat{k}\cdot\hat{p}')[m + (\mathbf{l} - E\hat{k})\cdot\hat{p}] - m[(\mathbf{l}\cdot\hat{p}') - (\hat{k}\cdot\mathbf{l})(\hat{k}\cdot\hat{p}')] + [(p'+k)\cdot e^{(\nu')}][\mathbf{l}\cdot\hat{p} - (\hat{k}\cdot\hat{p})(\hat{k}\cdot\mathbf{l})], \quad (22b)$$

$$T_3(\nu_e) = -[l^2 - (\hat{k}\cdot\mathbf{l})^2](p'+k)\cdot e^{(\nu')}. \quad (22c)$$

[6] The fourth component of our 4-vectors are imaginary. If $a = (\mathbf{a}, a_4) = (\mathbf{a}, ia_0)$ and $b = (\mathbf{b}, b_4) = (\mathbf{b}, ib_0)$ are two 4-vectors (a_0 and b_0 are real), then their scalar product is defined as $a\cdot b = \mathbf{a}\cdot\mathbf{b} - a_0 b_0$.

[7] $\psi_e(x)$ and $\psi_{\nu e}(x)$ are the electron and neutrino fields, respectively. $\bar{\psi} = \psi^\dagger\gamma_4$, where $\psi^\dagger$ is the Hermitian conjugate of ψ. The set of γ matrices we use is Hermitian.

FIG. 2. Inner bremsstrahlung in electron-neutrino scattering in context of 4-Fermi theory.



We have used the notation

$$e^{(\nu)} = k_\nu/\nu; \quad e^{(\nu')} = k_\nu'/\nu', \quad (23)$$

$$\hat{p} = \mathbf{v}/\nu; \quad \hat{p}' = \mathbf{v}'/\nu', \quad (24)$$

$$\hat{k} = \mathbf{k}/|\mathbf{k}|. \quad (25)$$

The corresponding expressions $T_i(\bar{\nu}_e)$ for the $\bar{\nu}_e$ case can be obtained from $T_i(\nu_e)$ by making the substitutions $\hat{p} \to \hat{p}'$, $\hat{p}' \to \hat{p}$, and $e^{(\nu')} \to e^{(\nu)}$.[8]

B. The Cross Section

The cross section for hard-photon inner bremsstrahlung is

$$\sigma_{\text{hard }\gamma} = \frac{e^2G^2}{4(2\pi)^5} \int T\delta^4(p' + k_\nu' + k - p - k_\nu) d\mathbf{l} \frac{d\mathbf{k}}{\omega} d\mathbf{v}'. \quad (26)$$

We use the δ function to carry out the $\mathbf{v}'$ and $|\mathbf{k}|$ integrations. This gives

$$\sigma_{\text{hard }\gamma} = \frac{e^2G^2}{4(2\pi)^5} \int T \frac{|\mathbf{k}|}{(1 - \hat{k}\cdot\hat{p}')} d\mathbf{l}\, d\Omega_k, \quad (27)$$

with

$$p + k_\nu = p' + k' + k_\nu'. \quad (28)$$

Equation (28) simply expresses the over-all conservation of energy and momentum.

Define the 4-vector

$$S = (\mathbf{s}, is_0) = k_\nu + p - p'. \quad (29)$$

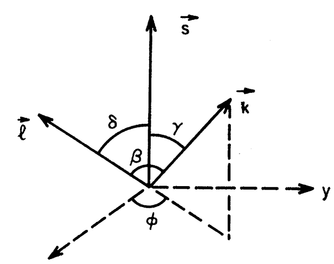
We also introduce the four angles β, γ, δ, and θ defined as follows (see Fig. 3):

$$\hat{k}\cdot\hat{l} = \cos\beta; \quad \hat{k}\cdot\hat{s} = \cos\gamma$$
$$\hat{l}\cdot\hat{s} = \cos\delta; \quad \hat{l}\cdot\hat{p} = \cos\theta, \quad (30)$$

where

$$\hat{s} = \mathbf{s}/s, \quad s = |\mathbf{s}|$$
$$\hat{l} = \mathbf{l}/l, \quad l = |\mathbf{l}|. \quad (31)$$

FIG. 3. Momenta and angles relative to coordinate system chosen.



[8] The relations (21) and (22) were derived by Lee and Sirlin (private communication). I wish to thank Professor Lee and Professor Sirlin for allowing me to use their results.

To perform the integration over $d\Omega_k$ we have chosen the z axis along $\mathbf{s}$, and the xz plane to be the plane defined by $\mathbf{s}$ and $\mathbf{l}$. In this coordinate system, Φ is the azimuthal angle of $\mathbf{k}$. It is easy to see that

$$\cos\beta=\cos\gamma\ \cos\delta+\sin\gamma\ \cos\Phi\ \sin\delta. \tag{32}$$

The integrals involving T_1 and T_2 are independent of ϵ in the sense that they remain finite when the limit $\epsilon\to 0$ is taken. On the other hand, the integral involving T_3 depends on ϵ, diverging logarithmically when $\epsilon\to 0$. We wish to separate this ϵ-dependent part as it requires special treatment. Some formal manipulations give

$$\sigma_{\text{hard }\gamma}=\frac{e^2G^2}{4(2\pi)^5}\int T'\frac{|\mathbf{k}|}{(1-\hat{k}\cdot\hat{p}')}d\mathbf{l}d\Omega_k+I_\epsilon, \tag{33}$$

where

$$T'=\frac{8}{m^2\omega E}T_1-\frac{8}{mE(p'\cdot k)}T_2+\frac{8}{E(p'\cdot k)^2}T_3', \tag{34}$$

with

$$T_3(\nu_e)'=-\frac{1}{\nu'}[l^2-(\hat{k}\cdot\mathbf{l})^2]k\cdot(k_\nu'-p'), \tag{35a}$$

$$T_3(\bar{\nu}_e)'=\frac{1}{\nu}[l^2-(\hat{k}\cdot\mathbf{l})^2]\left[\frac{\omega}{\nu'}(p'\cdot k_\nu)-k\cdot k_\nu\right], \tag{35b}$$

$$I_\epsilon(\nu_e)=\frac{4e^2G^2}{(2\pi)^5}\int\frac{d\mathbf{l}}{E}\left(\frac{p'\cdot S}{S\cdot S}\right)K, \tag{36a}$$

$$I_\epsilon(\bar{\nu}_e)=\frac{4e^2G^2}{(2\pi)^5}\int\frac{d\mathbf{l}}{E}\frac{s_0}{\nu}\left(\frac{p'\cdot k_\nu}{S\cdot S}\right)K, \tag{36b}$$

where

$$K=\int d\Omega_k\left[-1+\frac{2E}{(E-l\cos\beta)}-\frac{m^2}{(E-l\cos\beta)^2}\right]. \tag{37}$$

All the ϵ dependence of $\sigma_{\text{hard }\gamma}$ is contained in I_ϵ.

C. Evaluation of the ε-Dependent Part

We will describe a convenient and elegant method of evaluating I_ϵ devised by Sirlin.[9] Let

$$I_\epsilon=[2e^2G^2/(2\pi)^5]A, \tag{38}$$

where

$$A(\nu_e)=2\int\frac{d\mathbf{l}}{E}\left(\frac{p'\cdot S}{S\cdot S}\right)K, \tag{39a}$$

$$A(\bar{\nu}_e)=2\int\frac{d\mathbf{l}}{E}\frac{s_0}{\nu}\left(\frac{p'\cdot k_\nu}{S\cdot S}\right)K. \tag{39b}$$

The $|\mathbf{k}|$ integration is restricted to $|\mathbf{k}|\geqslant\epsilon$. In Appendix A, we show that for $|\mathbf{k}|\geqslant\epsilon$, the domain of integration can be divided into two regions such that in region I the kinematic limits are restricted by (A11) while in region II they are restricted by (A14). Let A_{I} and A_{II} denote the contributions to A from regions I and II, respectively. Consider first A_{I}. In region I there is no restriction on the orientation of $\mathbf{k}$. It is convenient to integrate over $d\Omega_k$ first, choosing the polar axis along $\mathbf{l}$. The $d\mathbf{l}$ integration is then quite easily performed by taking the polar axis along $\boldsymbol{\nu}$. The result of these two integrations is

$$A_{\text{I}}(\nu_e)=8\pi^2m\left[\ln\left(\frac{\epsilon}{m}\right)\right]\int_m^{E_{\max}}dE\left(2+\frac{E}{l}\ln\frac{E-l}{E+l}\right)-8\pi^2\int_m^{E_{\max}}dE\left(2+\frac{E}{l}\ln\frac{E-l}{E+l}\right)$$
$$\times\left\{m\ln\left[\left(\frac{E-l}{4ms_0}\right)(t-m)\right]+m\ln\left(\frac{W'-t}{m}\right)-\frac{1}{\nu}[ms_0-\nu(E-l)]\right\}, \tag{40a}$$

$$A_{\text{I}}(\bar{\nu}_e)=8\pi^2m\left[\ln\left(\frac{\epsilon}{m}\right)\right]\int_m^{E_{\max}}dE\left(2+\frac{E}{l}\ln\frac{E-l}{E+l}\right)\frac{s_0^2}{\nu^2}-8\pi^2\int_m^{E_{\max}}dE\left(2+\frac{E}{l}\ln\frac{E-l}{E+l}\right)\frac{s_0}{\nu^2}$$
$$\times\left\{ms_0\ln\left[\left(\frac{E-l}{4ms_0}\right)(t-m)\right]+ms_0\ln\left(\frac{W'-t}{m}\right)-[ms_0-\nu(E-l)]\right\}, \tag{40b}$$

where

$$t=E+l, \tag{41}$$

and

$$W'=m+2\nu. \tag{42}$$

$E_{\max}$ is defined by the relation (A6).

It is important to note that at the upper limit of integration the argument of $\ln[(W'-t)/m]$ vanishes in the limit when $\epsilon\to 0$, so that the differential cross section diverges logarithmically at the end of the electron spectrum. A similar divergence is found when one considers the radiative corrections to μ decay.[10,11] This divergence is a result of the degeneracy between the energy states of a free electron and those of an electron accompanied by soft

[9] A. Sirlin (private communication). I wish to thank Professor Sirlin for communicating his results and allowing me to use them.
[10] R. E. Behrends, R. J. Finkelstein, and A. Sirlin, Phys. Rev. **101**, 866 (1956).
[11] T. Kinoshita and A. Sirlin, Phys. Rev. **113**, 1652 (1959).

photons. Applying the Lee-Nauenberg theorem,[12] one can get finite results by simply summing the transition probability over the degenerate set of states. We can achieve this by averaging the differential cross section over a small energy interval ΔE, corresponding to the experimental energy resolution. If we do this, we find that the divergence is removed.

We now turn over to the evaluation of A_{II}. To do the $d\Omega_k$ integration we choose the polar axis along $\mathbf{s}$. After that, the final integration over $d\mathbf{l}$ is easily performed by taking the polar axis along $\mathbf{v}$. For this last integration it is convenient to replace the variable of integration ($\cos\theta$) by v, where v is given by Eq. (A13). Dropping terms that lead to contributions of $O(\epsilon)$ to A_{II} we can write

$$d(\cos\theta)\approx\frac{S\cdot S}{2\nu l(1-v)}dv. \tag{43}$$

The result of these manipulations is

$$A_{\text{II}}(\nu_e)=-4\pi^2 m\int_m^{E_{\max}} HdE, \tag{44a}$$

$$A_{\text{II}}(\bar{\nu}_e)=-4\pi^2 m\int_m^{E_{\max}}\frac{s_0^2}{\nu^2}HdE, \tag{44b}$$

where

$$H=\int_{-1}^{1}\frac{dv}{(1-v)}\left\{(1-v)-\frac{2E}{l}\ln\frac{(E+l)(1-\cos\delta)}{lv-E\cos\delta+[m^2\sin^2\delta+(lv-E\cos\delta)^2]^{1/2}}+1-\frac{(Ev-l\cos\delta)}{[m^2\sin^2\delta+(lv-E\cos\delta)^2]^{1/2}}\right\}.$$

In the above

$$\cos\theta\approx u$$

and

$$\cos\delta\approx(1/s_0)(u\nu-l),$$

where u is defined by the relation (A5). If we substitute $y=lv-E\cos\delta$, H reduces to an integral of the form M_2 discussed in Appendix B. The final result is

$$H=2-2\frac{E}{l}\left\{\left[\ln\left(\frac{E-l}{E+l}\,\frac{1+f}{1-f}\right)\right]\left[\ln\frac{m\nu}{s_0(E-l)}\right]+\frac{1}{2}\left(\ln\frac{E+l}{E-l}\right)^2+L\left[\frac{2(E-lf)}{(E-l)(1+f)}\right]\right.$$
$$\left.-L\left(\frac{2}{1+f}\right)-L\left(\frac{2l}{E+l}\right)\right\}+\frac{E}{l}\ln\left(\frac{E+l}{E-l}\right)+2\ln\frac{s_0}{\nu}, \tag{45}$$

where

$$f=(1/s_0 l)(Es_0-m\nu), \tag{46}$$

and $L(x)$ is the Spence function.

Combining $(I_\epsilon)_{\text{I}}$ and $(I_\epsilon)_{\text{II}}$ we finally obtain

$$I_\epsilon(\nu_e)=\frac{2\alpha G^2}{\pi^2}m\left(\ln\frac{\epsilon}{m}\right)\int_m^{E_{\max}}dE\left(2+\frac{E}{l}\ln\frac{E-l}{E+l}\right)-\frac{2\alpha G^2}{\pi^2}\int_m^{E_{\max}}dE\left(2+\frac{E}{l}\ln\frac{E-l}{E+l}\right)\left\{m\ln\left[\left(\frac{E-l}{4ms_0}\right)(t-m)\right]\right.$$
$$\left.+m\ln\left(\frac{W'-t}{m}\right)-\frac{1}{\nu}[ms_0-\nu(E-l)]\right\}-\frac{\alpha G^2 m}{\pi^2}\int_m^{E_{\max}}HdE, \tag{47a}$$

$$I_\epsilon(\bar{\nu}_e)=\frac{2\alpha G^2}{\pi^2}m\left(\ln\frac{\epsilon}{m}\right)\int_m^{E_{\max}}dE\left(2+\frac{E}{l}\ln\frac{E-l}{E+l}\right)\frac{s_0^2}{\nu^2}-\frac{2\alpha G^2}{\pi^2}\int_m^{E_{\max}}dE\left(2+\frac{E}{l}\ln\frac{E-l}{E+l}\right)\frac{s_0}{\nu^2}$$
$$\times\left\{ms_0\ln\left[\left(\frac{E-l}{4ms_0}\right)(t-m)\right]+ms_0\ln\left(\frac{W'-t}{m}\right)-[ms_0-\nu(E-l)]\right\}-\frac{\alpha G^2 m}{\pi^2}\int_m^{E_{\max}}\frac{s_0^2}{\nu^2}HdE. \tag{47b}$$

In Appendix D it is shown that the term proportional to $\ln(\epsilon/m)$ in I_ϵ is of the same magnitude but of opposite sign to the term proportional to $\ln(\epsilon/m)$ in $\sigma_{\text{soft}\,\gamma}$. Thus, upon adding $\sigma_{\text{soft}\,\gamma}$ and $\sigma_{\text{hard}\,\gamma}$ to give σ_γ (the total bremsstrahlung cross section without any restriction on k apart from those imposed by energy-momentum conservation), the ϵ dependence completely drops out.

[12] T. D. Lee and M. Nauenberg, Phys. Rev. **133**, B1549 (1964).

D. Evaluation of Non-ε-Dependent Part

Consider

$$\sigma_{\text{hard }\gamma}-I_\epsilon=\frac{e^2G^2}{4(2\pi)^5}\int T'\frac{|\mathbf{k}|}{(1-\hat{k}\cdot\hat{p}')}d\mathbf{l}d\Omega_k. \tag{48}$$

This integral has no ϵ dependence,[5] so that in computing it we can simply take ϵ to be zero. [This is equivalent to neglecting terms of $O(\epsilon)$.] The ranges of integration for the case $\epsilon=0$ have been determined in Appendix A and are given by (A7).

The integrals in relation (48) can all be brought to the following form:

$$\int\frac{d\mathbf{l}}{E}d\Omega_k\frac{D(E,\cos\theta)}{(s_0-s\cos\gamma)^n(E-l\cos\beta)^m},$$

where n and m are positive integers, including zero. It is convenient to perform first the integration over $d\Omega_k$. For n and/or m equal to zero the integration is trivial. If both are different from zero the integral is slightly more involved and a method for its evaluation can be found in Appendix B. After this is done, one is left with the integration over $d\mathbf{l}$. Most of the terms can be trivially integrated. In some cases it is convenient to replace ($\cos\theta$) by s as variable of integration. The more difficult integrals can all be brought to either the form M_1 or M_2 which are discussed in Appendix B. The final result after integration is

$$\sigma_{\text{hard }\gamma}=\frac{2m^2\alpha G^2}{\pi^2}\int_m^{E_{\max}}\sum_{i=1}^{4}J_i\frac{dE}{m}+I_\epsilon, \tag{49}$$

where

$$J_1(\nu_e)=\frac{1}{4m^2\nu^2}\Bigg\{[ms_0-\nu(E-l)]\bigg[Es_0+\tfrac{1}{2}(\nu^2-l^2)+\tfrac{1}{2}s_0^2-6m\nu+\frac{E^2s_0}{m}-\frac{1}{4}\frac{s_0\nu^2}{m}+\frac{3}{4}\frac{s_0l^2}{m}-\frac{s_0^3}{8m}\bigg]+\frac{1}{2}\left(\frac{E}{m}-\tfrac{1}{2}+\frac{3}{8}\frac{s_0}{m}\right)$$
$$\times[ms_0-\nu(E-l)][s_0^2+(\nu-l)^2]+\frac{3}{16m}s_0^2(s_0-4m)(\nu+l)^2(1-r^2)\Bigg\}, \tag{50a}$$

$$J_2(\nu_e)=\frac{1}{4m^2\nu^2}\Bigg\{s_0\bigg[m\nu s_0+2E^2s_0+2m\nu E+\frac{s_0^3}{4}\left(1-\frac{s_0}{4m}\right)+E\left(1-\frac{s_0}{2m}\right)(s_0^2-\nu^2+l^2)-\tfrac{1}{2}s_0\left(1-\frac{s_0}{2m}\right)(\nu^2+l^2)-\frac{1}{16m}(\nu^2-l^2)^2\bigg]$$
$$\times[-2\ln2+(1-r)\ln(1-r)+(1+r)\ln(1+r)]-\frac{s_0^3}{3}\bigg[\tfrac{1}{4}s_0+E+\frac{Es_0}{2m}-\frac{1}{8m}(\nu^2+l^2+s_0^2)\bigg][1+2\ln2-r^2$$
$$-(1-r^3)\ln(1-r)-(1+r^3)\ln(1+r)]-\frac{3}{80}\frac{s_0^5}{m}[\tfrac{3}{2}+2\ln2-r^2-\tfrac{1}{2}r^4-(1-r^5)\ln(1-r)-(1+r^5)\ln(1+r)]$$
$$+\tfrac{1}{2}\bigg[(\nu^2-l^2)\left(2Es_0+2m\nu-\frac{Es_0^2}{m}\right)+\tfrac{1}{2}(\nu^2-l^2)^2\left(1-\frac{s_0}{2m}\right)+s_0^2(\nu^2+l^2)\left(1-\frac{s_0}{4m}\right)\bigg]\bigg[2\ln2+\left(\frac{1-r}{r}\right)\ln(1-r)$$
$$-\left(\frac{1+r}{r}\right)\ln(1+r)\bigg]-\tfrac{1}{4}s_0^2(\nu+l)^2\left(1-\frac{s_0}{4m}\right)\bigg[(1+2\ln2)r^2-1+\left(\frac{1-r^3}{r}\right)\ln(1-r)-\left(\frac{1+r^3}{r}\right)\ln(1+r)\bigg]\Bigg\}, \tag{50b}$$

$$J_3(\nu_e)=\frac{1}{4m^2}(1-u)^2El\ln\left(\frac{E+l}{E-l}\right), \tag{50c}$$

$$J_4(\nu_e)=\frac{m^2}{4\nu^2}\Bigg\{\frac{13}{8}+\frac{2\nu}{m}+\left(2\frac{W^2}{m^2}-1\right)[L(z_2)-L(z_1)+L(1)-L(z_2/z_1)]-\frac{2\nu}{m}\left(1-\frac{\nu}{m}\right)\ln\frac{2\nu}{m}+\frac{1}{4}\bigg[\tfrac{1}{2}(z_1-z_2)^2+\frac{1}{2}\left(\frac{z_2}{z_1}\right)^2$$
$$-3\left(z_1+\frac{z_2}{z_1}\right)-\frac{z_2^2}{z_1}\bigg]+\left[B(z_1,z_2)+\frac{\nu}{m}\left(1-\frac{\nu}{m}\right)\right]\ln(z_1-1)+\left[-B(z_1,z_2)+\frac{\nu}{m}\left(1-\frac{\nu}{m}\right)\right]\ln\left(\frac{z_2}{z_1}-1\right)\Bigg\}. \tag{50d}$$

$$J_1(\bar{\nu}_e)=\frac{l}{2m^2\nu}(1-u)[-m^2+\tfrac{1}{2}ms_0+\nu(E-l)], \tag{51a}$$

$$J_2(\bar{\nu}_e)=\frac{s_0^2}{2m^2\nu^2}\left[\left\{2m(s_0-m)\left(1-\frac{s_0}{4m}\right)+\frac{s_0^2}{4m}(s_0-2m)\right\}\{-2\ln2+(1-r)\ln(1-r)+(1+r)\ln(1+r)\}+\frac{s_0}{6m}(s_0-m)^2\right.$$
$$\times\{1+2\ln2-r^2-(1-r^3)\ln(1-r)-(1+r^3)\ln(1+r)\}-\frac{s_0^3}{20m}$$
$$\left.\times\{\tfrac{3}{2}+2\ln2-r^2-\tfrac{1}{2}r^4-(1-r^5)\ln(1-r)-(1+r^5)\ln(1+r)\}\right], \tag{51b}$$

$$J_3(\bar{\nu}_e)=\frac{1}{4m^2}(1-u)^2El\ln\left(\frac{E+l}{E-l}\right), \tag{51c}$$

$$J_4(\bar{\nu}_e)=\tfrac{1}{2}(U_1+U_2+U_3), \tag{51d}$$

where

$$U_1=\frac{m^2}{2\nu^2}\left[\tfrac{1}{2}h_1\left\{L(1)-L\left(\frac{z_2}{z_1}\right)+L(z_2)-L(z_1)\right\}+\frac{13}{8}+\frac{2\nu}{m}-\frac{2\nu}{m}\left(1-\frac{\nu}{m}\right)\ln\frac{2\nu}{m}+\frac{1}{4}\left\{\tfrac{1}{2}(z_1-z_2)^2+\frac{1}{2}\left(\frac{z_2}{z_1}\right)^2-3\left(z_1+\frac{z_2}{z_1}\right)-\frac{z_2^2}{z_1}\right\}\right.$$
$$\left.+\left\{B(z_1,z_2)+\frac{\nu}{m}\left(1-\frac{\nu}{m}\right)\right\}\ln(z_1-1)+\left\{-B(z_1,z_2)+\frac{\nu}{m}\left(1-\frac{\nu}{m}\right)\right\}\ln\left(\frac{z_2}{z_1}-1\right)\right]$$
$$+\frac{mE}{\nu^2}\left[\frac{2W}{m}-\frac{4\nu}{m}\ln\frac{2\nu}{m}-\left(z_1+\frac{z_2}{z_1}\right)+(1+z_1)\left(\frac{z_2}{z_1}-1\right)\ln\left(\frac{z_2}{z_1}-1\right)+\left(1+\frac{z_2}{z_1}\right)(z_1-1)\ln(z_1-1)\right], \tag{52a}$$

$$U_2=\frac{1}{m^5\nu}\left[m^4a_2\left\{L(z_2)-L(z_1)+L(1)-L\left(\frac{z_2}{z_1}\right)\right\}+\frac{2}{(\eta-1)}\left(\frac{a}{\eta}+m^2a_1\right)\{\ln(\eta-1)-\ln2\}+\frac{2}{(\eta+1)}\left(\frac{a}{\eta}-m^2a_1\right)\ln(\eta+1)\right.$$
$$-\frac{m}{\eta\nu}(a+m^2z_2a_1)\{\ln z_1-2\ln(z_1+\eta)\}-\frac{m}{\nu}\left\{\left(1+\frac{1}{z_2}\right)a+2m^2a_1\right\}\ln\frac{2\nu}{m}+\frac{m}{\nu}\{(z_1+1)a+m^2(z_1+z_2)a_1\}\frac{(z_1-1)}{2z_1X}$$
$$\left.\times\ln(z_1-1)-\frac{m}{\nu}\left\{\left(\frac{z_2}{z_1}+1\right)a+m^2\left(\frac{z_2}{z_1}+z_2\right)a_1\right\}\frac{(z_2-z_1)}{2z_2X}\ln\left(\frac{z_2}{z_1}-1\right)\right]$$
$$+\frac{2b_0}{m^3\nu\eta}\left\{\ln\left(\frac{W}{m}+\eta\right)-\ln\frac{\nu}{m}-\ln\left(\frac{s_1'}{m^2}+\eta\right)\right\}-\frac{2b_1}{m\nu}\ln\frac{\nu}{m}, \tag{52b}$$

$$U_3=U_3^{(1)}+U_3^{(2)}+U_3^{(3)}, \tag{52c}$$

$$U_3^{(1)}=\frac{1}{4\nu^2}\left[\frac{2}{m^2}\left(\frac{\alpha_0}{m^2}+\alpha_1\right)\ln\left(\frac{z_2}{z_1}\right)-\frac{m}{\eta\nu}\left(\frac{\alpha_0}{m^4}+z_2\frac{\alpha_1}{m^2}+z_2\alpha_2\right)\left\{2\ln\left(\frac{\eta+1}{z_1+\eta}\right)-\ln\frac{\nu}{m}+\ln z_1\right\}+\frac{m}{\nu}\left(\frac{\alpha_0}{m^4}+\frac{\alpha_1}{m^2}+z_2\alpha_2\right)\right.$$
$$\left.\times\left\{\ln\frac{z_2}{z_1}+\ln\frac{\nu}{m}\right\}-\frac{m}{\nu}\left\{\left(\frac{\alpha_0}{m^4}+\frac{\alpha_1}{m^2}+z_2\alpha_2\right)+\frac{1}{\eta}\left(\frac{\alpha_0}{m^4}+\frac{z_2\alpha_1}{m^2}+z_2\alpha_2\right)\right\}\ln X\right], \tag{53a}$$

$$U_3^{(2)}=\frac{1}{2\nu^2}\left[(mW-s_1')-\frac{1}{2m^2\nu^2}(d_1+m^4z_2d_3)-\frac{W}{2m\nu^2}\left(\frac{1}{z_2}\frac{d_0}{m^4}+d_2+m^4z_2\right)+d_3\ln\frac{\nu}{m}+\frac{1}{2m^2\eta}\left(\frac{1}{z_2}\frac{d_0}{m^4}-d_2-3m^4z_2\right)\right.$$
$$\times\left\{\ln\left(\frac{W}{m}+\eta\right)-\ln\frac{\nu}{m}-\ln\left(\frac{s_1'}{m^2}+\eta\right)\right\}+\frac{1}{2m^4}\left\{(d_1+m^4z_2d_3)+\left(\frac{1}{z_2}\frac{d_0}{m^4}+d_2+m^4z_2\right)s_1'\right\}$$
$$\left.\times\frac{1}{X^2}+\frac{1}{2m^2\eta}\left(\frac{1}{z_2}\frac{d_0}{m^4}-d_2-2m^2\eta d_3-3m^4z_2\right)\ln X\right], \tag{53b}$$

$$
\begin{aligned}
U_3{}^{(3)}=&\frac{R_1}{6\eta(\eta-1)}\left[-\frac{(z_2+1)}{(\eta-1)^2}+\frac{(z_1{}^2+z_2)}{(z_1-\eta)^2}\right]+\frac{R_2}{6\eta(\eta+1)}\left[\frac{(z_2+1)}{(\eta+1)^2}-\frac{(z_2+z_1{}^2)}{(z_1+\eta)^2}\right]+\left[R_1C(z_2)+\frac{R_3}{(\eta-1)}\right]\\
&\times[\ln(z_1-1)+2\ln(\eta-1)+\ln z_1]+\left[R_2D(z_2)+\frac{R_4}{(\eta+1)}\right][-2\ln(\eta+1)+2\ln(z_1+\eta)-\ln z_1]\\
&+R_5\left[L(z_2)-L(z_1)+L(1)-L\left(\frac{z_2}{z_1}\right)\right]+\left[R_1\left\{-\frac{1}{3}\frac{1}{z_2(\eta-1)^3}-\frac{1}{2}\frac{1}{z_2(\eta-1)^2}-C(z_2)\right\}\right.\\
&\left.+R_2\left\{-\frac{1}{3}\frac{1}{z_2(\eta+1)^3}+\frac{1}{2}\frac{1}{z_2(\eta+1)^2}+D(z_2)\right\}-\frac{1}{\eta}R_3\frac{(\eta+1)}{(\eta-1)}+\frac{1}{\eta}\left(\frac{\eta-1}{\eta+1}\right)R_4\right]\ln\frac{2\nu}{m}\\
&+\left[R_2\left\{\frac{1}{3}\frac{\eta}{(z_1+\eta)^3}-\frac{1}{2}\frac{1}{(z_1+\eta)^2}-D(z_2)\right\}-R_4\frac{(z_1-1)}{(\eta+1)(z_1+\eta)}\right]\ln(z_1-1)+\left[R_2\left\{\frac{1}{3}\frac{\eta}{(z_2/z_1+\eta)^3}\right.\right.\\
&\left.\left.-\frac{1}{2}\frac{1}{(z_2/z_1+\eta)^2}-D(z_2)\right\}-R_4\frac{(z_2-z_1)}{\eta(\eta+1)(z_1+\eta)}\right]\ln\left(\frac{z_2}{z_1}-1\right)+R_1\left[\frac{\eta}{3}\left\{\frac{1}{(z_1-\eta)^3}\ln(z_1-1)+\frac{1}{(z_2/z_1-\eta)^3}\ln\left(\frac{z_2}{z_1}-1\right)\right\}\right.\\
&\left.+\frac{1}{2}\left\{\frac{1}{(z_1-\eta)^2}\ln(z_1-1)+\frac{1}{(z_2/z_1-\eta)^2}\ln\left(\frac{z_2}{z_1}-1\right)\right\}+C(z_2)\ln\left(\frac{z_2}{z_1}-1\right)\right]\\
&+R_3\left[\left\{\frac{1}{(z_1-\eta)}\ln(z_1-1)+\frac{1}{(z_2/z_1-\eta)}\ln\left(\frac{z_2}{z_1}-1\right)\right\}+\frac{1}{(\eta-1)}\ln\left(\frac{z_2}{z_1}-1\right)\right]\\
&-2R_1C(z_2)\ln(z_1-\eta)-\frac{2R_3}{(\eta-1)}\ln(z_1-\eta).\qquad(53c)
\end{aligned}
$$

a, a_i, b_i, d_i, α_i, and R_i are functions of ν and E and are given in Appendix C. We have used the following notation:

$$W=m+\nu\,,\qquad(54)$$

$$r=(\nu-l)/s_0\,,\qquad(55)$$

$$z_1=(1/m^2)\{EW-l\nu+|lW-E\nu|\}\,,\qquad(56)$$

$$z_2=1+2\nu/m\,,\qquad(57)$$

$$\eta=\sqrt{z_2}\,,\qquad(58)$$

$$X=(1/m^2)|lW-E\nu|\,,\qquad(59)$$

$$s_1'=EW-l\nu\,,\qquad(60)$$

$$B(z_1,z_2)=\left(z_1-\frac{z_2}{z_1}\right)\left\{1-\frac{1}{4}\left(z_1+\frac{z_2}{z_1}\right)\right\}\,,\qquad(61)$$

$$C(z_2)=\frac{1}{3}\frac{\eta}{(\eta-1)^3}-\frac{1}{2}\frac{1}{(\eta-1)^2}\,,\qquad(62)$$

$$D(z_2)=\frac{1}{3}\frac{\eta}{(\eta+1)^3}-\frac{1}{2}\frac{1}{(\eta+1)^2}\,,\qquad(63)$$

$$h_1=2z_2+\frac{4}{m^2}(2E-\nu)(2E-W)\,.\qquad(64)$$

u is defined by the relation (A5).

It is interesting to note that

$$\sum_{i=1}^{4}J_1=0\,,$$

for $l=0$ and $l=l_{\max}$.

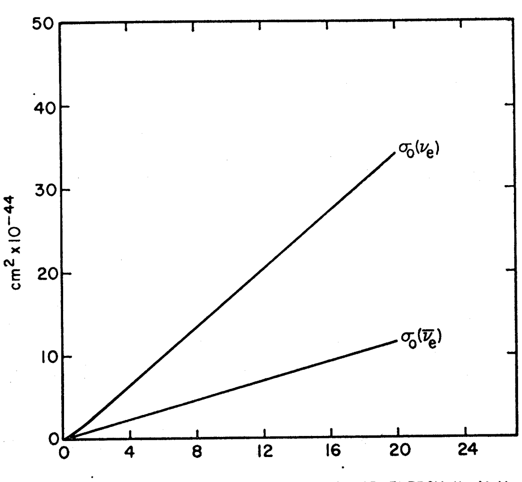


FIG. 4. Total cross section for radiationless elastic electron-neutrino scattering $\sigma_0(\nu_e)$, and elastic electron-antineutrino scattering $\sigma_0(\bar{\nu}_e)$.

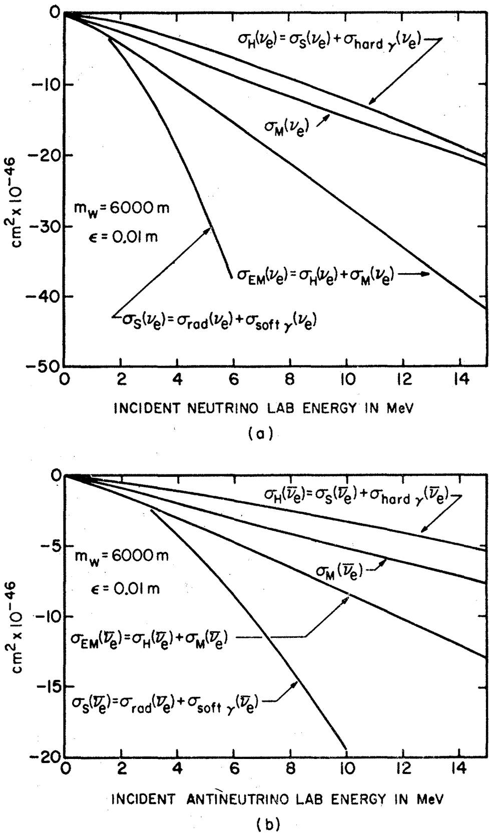


FIG. 5. Electromagnetic corrections to (a) $\sigma_0(\nu_e)$ and (b)$\sigma_0(\bar{\nu}_e)$ assuming $m_W=6000m$ and $\epsilon=0.01m$.

IV. NUMERICAL ANALYSIS OF RESULTS AND CONCLUSIONS

The results we have obtained are fairly elaborate and a numerical analysis was necessary to clarify their content. We computed total and differential cross sections for incident neutrino (antineutrino) energies of 5, 10, and 15 MeV. We also considered the case when the incident neutrino has an energy of 1.4 MeV, as this corresponds to the energy of the neutrinos produced in the K capture of Zn^{65}, which has been suggested[4] as a convenient source of neutrinos for observing reaction (1). The graphs given in Figs. 4 through 11 show the results of this numerical analysis assuming $m_W=6000m$. The value of G we used was the one determined from μ decay[13] (including radiative corrections), namely,

$$G=(1.4350\pm0.0011)\times10^{-49}\ \text{erg cm}^3.$$

The following cross sections and radiative corrections were plotted in these graphs:

$d\sigma_0/dE$—differential cross section for nonradiative electron-neutrino (antineutrino) scattering [see relations (6) and (7)].

$d\sigma_s/dE$—This is the radiative correction to processes (1) and (2) to order α, assuming that photons with energy less than ϵ cannot be detected (the contribution coming from the electromagnetic form factor of the neutrino is not included).

$$\frac{d\sigma_s}{dE}=\frac{d\sigma_{\text{rad}}}{dE}+\frac{d\sigma_{\text{soft}\ \gamma}}{dE}. \tag{65}$$

In computing $d\sigma_s/dE$ we assume that[14] $\epsilon=0.01m$.

$d\sigma_H/dE$—This is the radiative correction to processes (1) and (2) to order α, assuming that the experimental setup is such that any photons emitted in the scattering process are undetectable. It does not include the con-

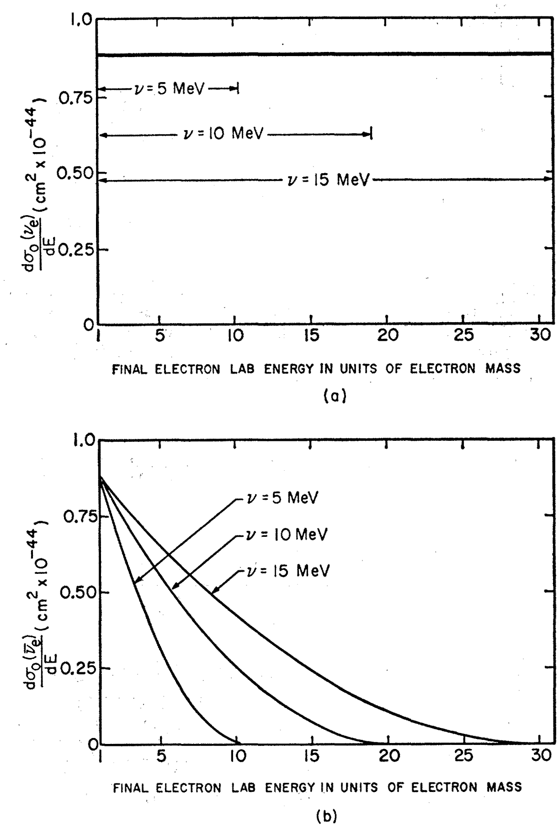


FIG. 6. (a) Differential cross section $d\sigma_0(\nu_e)/dE$ for radiationless elastic electron-neutrino scattering. The cross section is a constant independent of the incident neutrino energy. (b) Differential cross section $d\sigma_0(\bar{\nu}_e)/dE$ for radiationless elastic electron-antineutrino scattering, at incident antineutrino energies ν of 5, 10, and 15 MeV.

[13] C. S. Wu, Rev. Mod. Phys. **36**, 618 (1964).

[14] For such a value of ϵ it is essentially sufficient to consider only one photon bremsstrahlung.

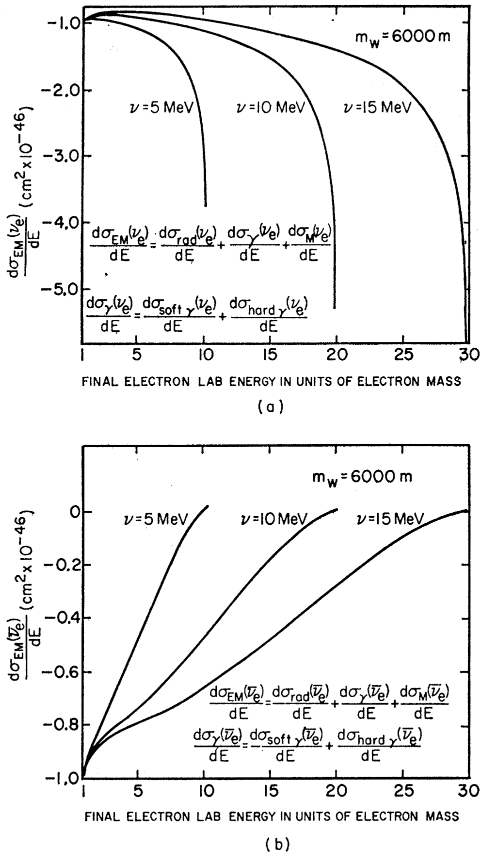


FIG. 7. Electromagnetic corrections to (a) $d\sigma_0(\nu_e)/dE$ and (b) $d\sigma_0(\bar{\nu}_e)/dE$ to order α, including soft- and hard-photon emission and neutrino (antineutrino) form-factor interference term. The curves correspond to incident neutrino (antineutrino) energies ν of 5, 10, and 15 MeV, and $m_W=6000m$.

tribution coming from the electromagnetic form factor of the neutrino.

$$\frac{d\sigma_H}{dE}=\frac{d\sigma_{\text{rad}}}{dE}+\frac{d\sigma_{\text{soft }\gamma}}{dE}+\frac{d\sigma_{\text{hard }\gamma}}{dE}. \tag{66}$$

As discussed in Sec. III, $d\sigma_{\text{hard }\gamma}/dE$ diverges logarithmically at the end point of the electron spectrum. This divergence was removed by averaging $d\sigma_{\text{hard }\gamma}/dE$ over a small energy interval ΔE corresponding to the experimental electron energy resolution. We chose ΔE to be 100 keV.

As it is, to get the total electromagnetic correction to order α, one also has to include the correction due to the electromagnetic form factor of the neutrino, $d\sigma_M/dE$. We accordingly evaluated and plotted

$$\frac{d\sigma_{\text{EM}}}{dE}=\frac{d\sigma_H}{dE}+\frac{d\sigma_M}{dE}. \tag{67}$$

This is the total electromagnetic correction to reactions (1) and (2) to order α. It is the electromagnetic correction one would measure, as there is no way to measure $d\sigma_H/dE$ and $d\sigma_M/dE$ separately. To be more accurate, one should really say that what one measures is

$$\frac{d\sigma}{dE}=\frac{d\sigma_0}{dE}+\frac{d\sigma_{\text{EM}}}{dE}. \tag{68}$$

By subtracting the theoretical value of $d\sigma_0/dE$ from this, one can infer what $d\sigma_{\text{EM}}/dE$ is. The reason we also plotted $d\sigma_H/dE$ and $d\sigma_M/dE$ separately was to give an idea of their relative contribution to the total electromagnetic correction. By measuring $d\sigma_{\text{EM}}/dE$ as described above, and subtracting from it the theoretical value of $d\sigma_H/dE$, one can compare the result with the theoretical $d\sigma_M/dE$ and thereby verify the theory that was used in deriving the relation (13). We did not plot $d\sigma/dE$ as $d\sigma_{\text{EM}}/dE$ is only of the order of 1% $d\sigma_0/dE$, and on the scale we are using would not change $d\sigma_0/dE$ significantly.

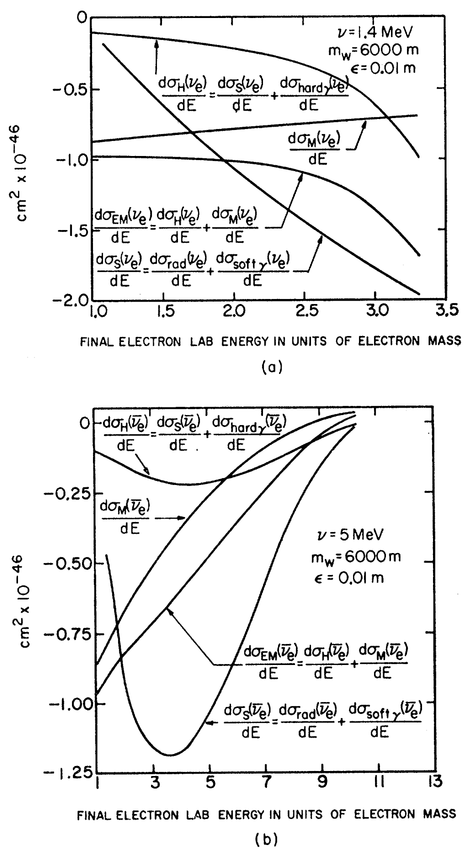


FIG. 8. (a) Electromagnetic corrections to $d\sigma_0(\nu_e)/dE$ to order α, for an incident neutrino energy ν of 1.4 MeV, $m_W=6000m$ and $\epsilon=0.01m$. (b) Electromagnetic corrections to $d\sigma_0(\bar{\nu}_e)/dE$ to order α, for an incident antineutrino energy ν of 5 MeV, $m_W=6000m$ and $\epsilon=0.01m$.

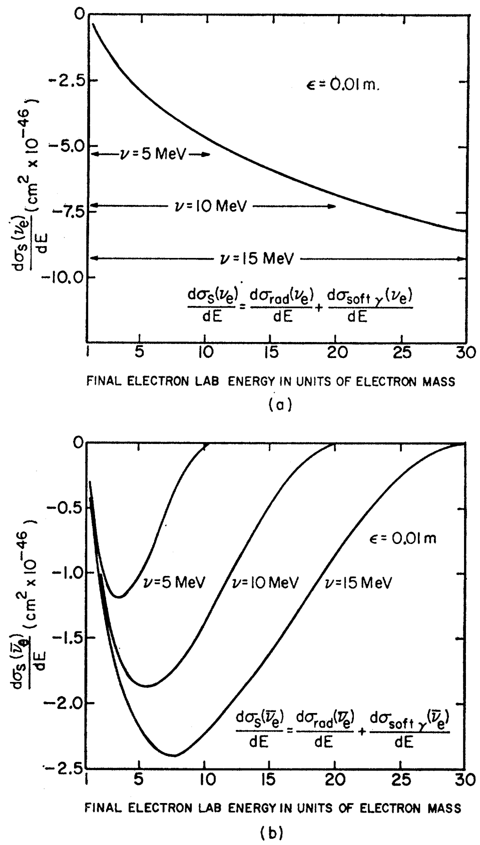


FIG. 9. (a) Radiative corrections to $d\sigma_0(\nu_e)/dE$ to order α, including soft photon emission but excluding neutrino form factor interference term. This correction is very insensitive to the incident neutrino energy ν and on the scale used the curves for $\nu=5$, 10, and 15 MeV coincided. We assumed $\epsilon=0.01m$. (b) Radiative corrections to $d\sigma_0(\bar{\nu}_e)/dE$ to order α, including soft photon emission but excluding the antineutrino form factor interference term. The curves correspond to incident antineutrino energies ν of 5, 10, and 15 MeV, and $\epsilon=0.01m$.

As the results show, the cross section for elastic electron-neutrino scattering is about 3 times that for elastic electron-antineutrino scattering, showing that the detection of process (1) is somewhat easier than observing process (2). We also see that for $\nu \gg m$, σ_0 increases linearly with ν.

σ_{EM} is negative for both ν_e and $\bar{\nu}_e$, but the magnitude of $\sigma_{EM}(\nu_e)$ is about three times that of $\sigma_{EM}(\bar{\nu}_e)$. Furthermore, as ν increases so does $|\sigma_{EM}|$. In fact, we see that $|\sigma_{EM}(\nu_e)|$ increases from 1.35% of $\sigma_0(\nu_e)$ to 1.65% as the incident neutrino energy increases from 3 to 15 MeV. Correspondingly, $|\sigma_{EM}(\bar{\nu}_e)|$ increases from 1% of $\sigma_0(\bar{\nu}_e)$, to 1.5%. It is important to note that more than half the contribution to σ_{EM} comes from the form factor term σ_M.

For any given incident neutrino (antineutrino) energy we found that

$$d\sigma_0 + d\sigma_s < d\sigma_0 + d\sigma_H .$$

TABLE I. Neutrino and antineutrino form factor interference terms for different W-meson masses, and an incident neutrino (antineutrino) energy of 5 MeV.

m_W (electron mass units)	$\sigma_M(\nu_e)$ ($cm^2 \times 10^{-46}$)	$\sigma_M(\bar{\nu}_e)$ ($cm^2 \times 10^{-46}$)
6000	−7.315	−2.524
10 000	−7.948	−2.736
15 000	−8.451	−2.904

This is exactly what we expected, since by also including the emission of hard photons we obviously increase the cross section.

We have also evaluated σ_M for $\nu=5$ MeV and $m_W=6000m$, $10\,000m$, and $15\,000m$. As the results given in Table I show, σ_M is not too sensitive to the exact value of m_W {the reason why this is so is clear when one observes that $d\sigma_M$ depends on m_W only through a term proportional to $\ln(m_W/m)$ [see expression (13) which is taken from Ref. 3]}.

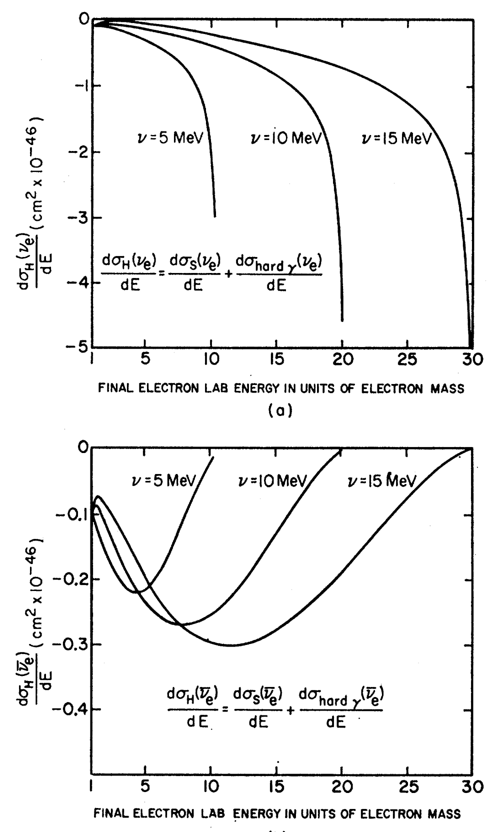


FIG. 10. Radiative corrections to (a) $d\sigma_0(\nu_e)/dE$ and (b) $d\sigma_0(\bar{\nu}_e)/dE$ to order α, including the emission of soft and hard photons but excluding the neutrino (antineutrino) form-factor interference term. The curves correspond to incident neutrino (antineutrino) energies ν of 5, 10, and 15 MeV.

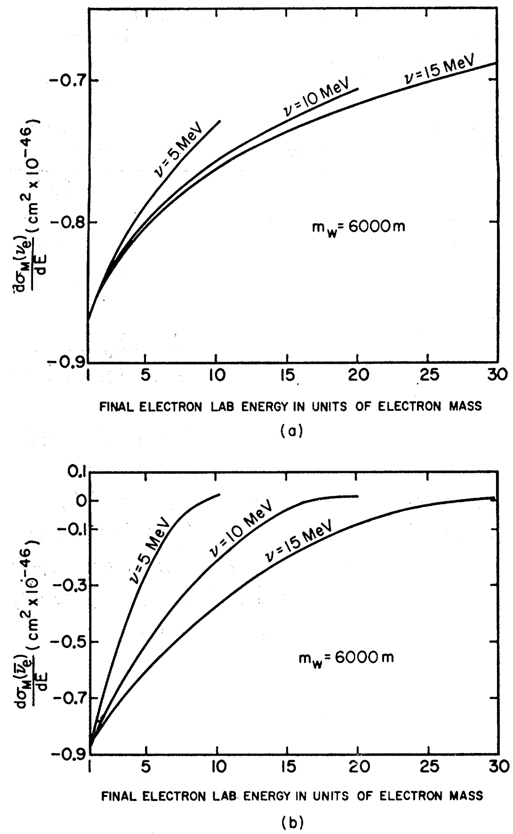


FIG. 11. (a) Neutrino form factor interference term and (b) antineutrino form factor interference term, for incident neutrino (antineutrino) energies ν of 5, 10, and 15 MeV and $m_W=6000m$.

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APPENDIX A

We shall determine the kinematic limits for the reaction (4). Two cases will be of interest:

(a) $\omega\geqslant 0$.

(b) $\omega\geqslant\epsilon$.

ω is the photon energy and ϵ is a very small positive energy.

Case (a) $\omega\geqslant 0$

We choose a frame of reference in which the electron is initially at rest. In terms of the 4-vector $S=(\mathbf{s},is_0)$ defined by Eq. (29), we can express the energy-momentum conservation relation (28) as follows

$$\mathbf{s}=\boldsymbol{\nu}'+\mathbf{k} \tag{A1a}$$

and

$$s_0=\nu'+\omega. \tag{A1b}$$

The kinematic limits we shall determine for this case will be applied to integrals with no infrared divergence, so that we can take $\Delta=0$ (Δ is the fictitious photon mass), and write $\omega=|\mathbf{k}|$.

Eliminating ν' from Eqs. (A1a) and (A1b), we find that

$$|\mathbf{k}|=-\frac{S\cdot S}{2(s_0-s\cos\gamma)}, \tag{A2}$$

where $\cos\gamma=\hat{k}\cdot\hat{s}$ and $S\cdot S=s^2-s_0^2$.

Using Eq. (28) and the definition of S, we find that

$$S\cdot S=(k_\nu'+k)^2=2\nu'|\mathbf{k}|(\hat{\nu}'\cdot\hat{k}-1)\leqslant 0. \tag{A3}$$

Therefore,

$$s_0\geqslant s. \tag{A4}$$

We require that $|\mathbf{k}|=\omega\geqslant 0$. Because of (A3) and (A4) it is easy to see that (A2) satisfies this condition for all $\cos\gamma$, i.e.,

$$-1\leqslant\cos\gamma\leqslant 1.$$

Using (A4) we find that

$$0\leqslant u\leqslant\cos\theta\leqslant 1,$$

where

$$u=(-ms_0+\nu E)/\nu l. \tag{A5}$$

The restriction $u\leqslant 1$ determines the maximum value of E for any given ν. We find that

$$E_{\max}=(W^2+\nu^2)/(W+\nu), \tag{A6}$$

where W is defined by the relation (54). The allowed range of E is therefore

$$m\leqslant E\leqslant E_{\max}.$$

To summarize, the kinematic limits for the case $\omega=|\mathbf{k}|\geqslant 0$ are

$$\begin{aligned}-1\leqslant\cos\gamma&\leqslant 1,\\ u\leqslant\cos&\leqslant 1,\\ m\leqslant E&\leqslant E_{\max}.\end{aligned} \tag{A7}$$

Case (b) $\omega\geqslant\epsilon$

To treat this case we use a method due to Sirlin.[9] As in case (a), we find, using 4-momentum conservation, the restriction (A2) on $|\mathbf{k}|$. We have again taken $\Delta=0$ since we are interested in $\omega\geqslant\epsilon$ and no infrared divergence is encountered. We now impose the condition $\omega=|\mathbf{k}|\geqslant\epsilon$. Together with (A2), this implies that

$$\cos\gamma\geqslant\frac{1}{s}\left\{s_0-\frac{(s_0-s)}{2\epsilon}(s_0+s)\right\}. \tag{A8}$$

Since $s_0 \geqslant s$ [see (A4)], the domain of integration can be divided into two regions, as follows:

$$\text{In region I:} \quad s_0 - s \geqslant 2\epsilon, \tag{A9a}$$

$$\text{In region II:} \quad s_0 - s \leqslant 2\epsilon. \tag{A9b}$$

Consider first region I. Applying the condition (A9a) to (A8), we see that $\cos\gamma \geqslant -1$. Since there is no other restriction on $\cos\gamma$, we must have

$$-1 \leqslant \cos\gamma \leqslant 1.$$

In region I, $s_0 - s \geqslant 2\epsilon$. This gives the following lower limit for $\cos\theta$:

$$\cos\theta \geqslant u^{(\epsilon)},$$

where

$$u^{(\epsilon)} = \frac{2\epsilon s_0 - m s_0 + \nu E}{\nu l} > 0. \tag{A10}$$

Therefore,

$$0 < u^{(\epsilon)} \leqslant \cos\theta \leqslant 1.$$

Since $u^{(\epsilon)} \leqslant 1$, it follows that $E \leqslant E_{\max} + O(\epsilon)$, where $E_{\max}$ is given by the relation (A6). We have therefore found the following kinematic limits in region I:

$$\begin{aligned} &-1 \leqslant \cos\gamma \leqslant 1, \\ &u^{(\epsilon)} \leqslant \cos\theta \leqslant 1, \\ &m \leqslant E \leqslant E_{\max} + O(\epsilon), \end{aligned} \tag{A11}$$

We now deal with region II. In this region

$$0 \leqslant s_0 - s \leqslant 2\epsilon. \tag{A12}$$

This, together with (A8), implies that

$$v \leqslant \cos\gamma \leqslant 1,$$

where

$$v = \frac{1}{s}\left(s_0 + \frac{S \cdot S}{2\epsilon}\right). \tag{A13}$$

The condition (A12) imposes the following limitations on $\cos\theta$:

$$u \leqslant \cos\theta \leqslant u^{(\epsilon)}.$$

Furthermore, the requirement that $u^{(\epsilon)} \leqslant 1$ gives the following upper limit for E:

$$E \leqslant E_{\max} + O(\epsilon).$$

The kinetmatic limits in region II are therefore

$$\begin{aligned} &v \leqslant \cos\gamma \leqslant 1, \\ &u \leqslant \cos\theta \leqslant u^{(\epsilon)}, \\ &m \leqslant E \leqslant E_{\max} + O(\epsilon). \end{aligned} \tag{A14}$$

To summarize, we have found that when $\omega = |\mathbf{k}| \geqslant \epsilon$ the domain of integration can be divided into two regions, such that in region I the kinematic limits are determined by the restrictions (A11), while in region II they are determined by (A14).

APPENDIX B

In this appendix we shall discuss some of the integrals encountered in our calculation.

(1) Integrals of the form

$$\int_0^{4\pi} \frac{d\Omega_k}{(s_0 - s\cos\gamma)^n (E - l\cos\beta)^m}.$$

To evaluate these integrals for arbitrary integer (nonzero) n and m, we use the identity

$$\int_0^{4\pi} \frac{d\Omega_k}{(as_0 - s\cos\gamma)^n (bE - l\cos\beta)^m} = \frac{(-1)^{m+n}}{(n-1)!(m-1)!s_0^{n-1}E^{m-1}} \frac{d^{n-1}}{da^{n-1}} \frac{d^{m-1}}{db^{m-1}} \times \int_0^{4\pi} \frac{d\Omega_k}{(as_0 - s\cos\gamma)(bE - l\cos\beta)}, \tag{B1}$$

where a and b are two numerical parameters which are put equal to 1 after the differentiations are performed. It is therefore sufficient to evaluate

$$\mathcal{J} = \int_0^{4\pi} \frac{d\Omega_k}{(as_0 - s\cos\gamma)(bE - l\cos\beta)}.$$

Substituting the relation (32) for $\cos\beta$, and carrying out the azimuthal integration, we obtain

$$\mathcal{J} = 2\pi \int_{-1}^{1} \frac{d(\cos\gamma)}{(as_0 - s\cos\gamma)\{c^2 + (l\cos\gamma - bE\cos\delta)^2\}^{1/2}},$$

where $c^2 = (b^2E^2 - l^2)\sin^2\delta$.

We now carry out two transformations. First

$$x = (l\cos\gamma - bE\cos\delta),$$

followed by

$$y = x + (c^2 + x^2)^{1/2}.$$

This gives

$$\int_0^{4\pi} \frac{d\Omega_k}{(as_0 - s\cos\gamma)(bE - l\cos\beta)} = -\frac{4\pi}{s} N, \tag{B2}$$

where

$$N = \int_{y_1}^{y_2} \frac{dy}{(y^2 - Ry - c^2)} = \frac{1}{(R^2 + 4c^2)^{1/2}} \times \ln\left[\frac{(y_2 - r_1)(y_1 - r_2)}{(y_1 - r_1)(y_2 - r_2)}\right], \tag{B3}$$

and

$$R = 2\left(al\frac{s_0}{s} - bE\cos\delta\right), \tag{B4}$$

$$y_1=(bE-l)(1-\cos\delta);\quad y_2=(bE+l)(1-\cos\delta), \tag{B5}$$

$$r_1=\tfrac{1}{2}[R+(R^2+4c^2)^{1/2}];\quad r_2=\tfrac{1}{2}[R-(R^2+4c^2)^{1/2}]. \tag{B6}$$

(2) Integrals of the form M_1

$$=\int_{s_1'}^{s_2'} f(s')\ln\left[\frac{(s'-m^2)-(s'^2-d^2)^{1/2}}{(s'-m^2)+(s'^2-d^2)^{1/2}}\right]\frac{ds'}{(s'^2-d^2)^{1/2}}.$$

We first substiture

$$x=s'+(s'^2-d^2)^{1/2}.$$

This gives

$$M_1=\int_{x_1}^{x_2} f\left(\frac{x}{2}+\frac{d^2}{2x}\right)\ln\left(\frac{d^2}{m^2x}-1\right)\frac{dx}{x} -\int_{x_1}^{x_2} f\left(\frac{x}{2}+\frac{d^2}{2x}\right)\ln\left(\frac{x}{m^2}-1\right)\frac{dx}{x},$$

where

$$x_1=s_1'+(s_1'^2-d^2)^{1/2};\quad x_2=s_2'+(s_2'^2-d^2)^{1/2}.$$

Introduce the new integration variable $y=d^2/x$ in the first integral and change the name of the variable in the second integral from x to y. Finally, introducing the dimensionless variable of integration $\xi=y/m^2$, gives

$$M_1=Y(\xi_1,\xi_2)+Y\left(\frac{d'^2}{\xi_1},\frac{d'^2}{\xi_2}\right),$$

where

$$Y(\alpha,\beta)=-\int_\alpha^\beta f\left(\frac{m^2}{2}y+\frac{m^2d'^2}{2y}\right)\ln(y-1)\frac{dy}{y},$$

and

$$\xi_1=x_1/m^2;\quad \xi_2=x_2/m^2.$$
$$d'^2=d^2/m^4.$$

(3) Integrals of the form M_2

$$=\int_{s_1'}^{s_2'} f(s',(s'^2-d^2)^{1/2})\frac{ds'}{(s'^2-d^2)^{1/2}}.$$

By making the substitution $x=s'+(s'^2-d^2)^{1/2}$, M_2 is brought to the following more convenient form:

$$M_2=\int_{x_1}^{x_2} f\left(\frac{x}{2}+\frac{d^2}{2x},\frac{x}{2}-\frac{d^2}{2x}\right)\frac{dx}{x}.$$

APPENDIX C

$$a_0=(m^4/\nu)[2s_0(\nu W-EW')-W^2(m+E)],$$

$$a_1=(m^2/\nu)[W(m+E)^2-2\nu Es_0],$$

$$a_2=(m/\nu)(W'-3E)E,$$

$$a=a_0+m^4z_2a_2,$$

$$b_0=-\frac{m^2}{2\nu}[(W'-3E)(mW+W'E)-4mWs_0],$$

$$b_1=\frac{m}{2\nu}\{(W'-3E)(W+E)-4Es_0\}.$$

$$c_0=-\frac{W}{m^3}\{2m^3W-m\nu^2(W+2\nu)+4mWW'E+W'(m-\nu)E^2\},$$

$$c_1=\frac{2}{m^2}\{m^2(2W^2+\nu^2)+WE(5mW-m\nu-3\nu^2)+W'E^2(2W+m)\},$$

$$c_2=-\frac{1}{m^2}\{m^2(2W^2+m^2)+4mWE(2m-\nu)+(7mW+3m\nu-2\nu^2)E^2\},$$

$$c_3=\frac{2}{m}(m^3+mWE-2\nu E^2),$$

$$c_4=E^2+l^2,$$

$$d_0=-m^4W\{m(2m^2+\nu^2)+2W'(m-\nu)E+3W'E^2\},$$

$$d_1=m^2\{m(2m-\nu)(2W^2+\nu^2)+6mW(W+m)E+W'(4m-3\nu)E^2\},$$

$$d_2=m\{-m(5m^2+7m\nu+\nu^2)+2W(2\nu-5m)E-3mE^2\},$$

$$d_3=m(2m-\nu)+2E^2,$$

$$\alpha_0=-m^3(mW^2-2\nu WE+3W'E^2),$$

$$\alpha_1=2mE[W(2m-\nu)+(2W+\nu)E],$$

$$\alpha_2=W^2-4WE-E^2.$$

$$R_1=\frac{1}{2\nu^2z_2}[(c_0+z_2c_2+z_2^2c_4)+\eta(c_1+z_2c_3)] =-\frac{3}{2m^2\nu^2}(2mW-\nu^2-2\eta m^2)\left(E-\frac{W}{\eta}\right)^2,$$

$$R_2=-\frac{1}{2\nu^2z_2}\{(c_0+z_2c_2+z_2^2c_4)-\eta(c_1+z_2c_3)\} =\frac{3}{2m^2\nu^2}(2mW-\nu^2+2\eta m^2)\left(E+\frac{W}{\eta}\right)^2,$$

$$R_3=-\frac{1}{4\nu^2\eta^3}\{(c_0-z_2c_2-3z_2^2c_4)-2\eta^3c_3\},$$

$$R_4=\frac{1}{4\nu^2\eta^3}\{(c_0-z_2c_2-3z_2^2c_4)+2\eta^3c_3\},$$

$$R_5=(1/2\nu^2)c_4.$$

W, r, z_1, z_2, η, X, s_1', $B(z_1,z_2)$, $C(z_2)$, and $D(z_2)$ have already been defined by the relations (54) through (63).

APPENDIX D

From the relations (10), (11), (12), (14), and (15) we see that $d\sigma_{\text{rad}}+d\sigma_{\text{soft}\,\gamma}$ has an ϵ-dependent term

$$Y_1=\frac{2m\alpha G^2}{\pi^2}\left(1-\frac{2\varphi}{\tanh 2\varphi}\right)\times\left(2\ln\frac{m}{\epsilon}\right)\begin{Bmatrix}1\\(1-q^2/2m\nu)^2\end{Bmatrix}dE. \quad \text{(D1)}$$

The upper row applies to ν_e, the lower one to $\bar{\nu}_e$. Using the relations (16) and (29) one can easily show that

$$\left(1-\frac{q^2}{2m\nu}\right)=\frac{s_0}{\nu}. \quad \text{(D2)}$$

Therefore,

$$Y_1=\frac{2m\alpha G^2}{\pi^2}\left(1-\frac{2\varphi}{\tan 2\varphi}\right)\left(2\ln\frac{m}{\epsilon}\right)\begin{Bmatrix}1\\s_0{}^2/\nu^2\end{Bmatrix}dE. \quad \text{(D3)}$$

$d\sigma_{\text{hard}\,\gamma}$ also has an ϵ-dependent term. This is [see expressions (47a) and (47b)]

$$Y_2=\frac{2m\alpha G^2}{\pi^2}\frac{1}{2}\left(2+\frac{E}{l}\ln\frac{E-1}{E+l}\right)\times\left(2\ln\frac{\epsilon}{m}\right)\begin{Bmatrix}1\\s_0{}^2/\nu^2\end{Bmatrix}dE. \quad \text{(D4)}$$

To prove that Y_1 and Y_2 are of equal magnitude but have opposite signs it is sufficient to show that

$$\left(1-\frac{2\varphi}{\tanh 2\varphi}\right) \quad \text{and} \quad \frac{1}{2}\left(2+\frac{E}{l}\ln\frac{E-l}{E+l}\right)$$

are equal.

From the relation (16), it follows that

$$E=m\cosh 2\varphi \quad \text{(D5)}$$

and

$$l=m\sinh 2\varphi. \quad \text{(D6)}$$

Therefore

$$E/l=1/\tanh 2\varphi. \quad \text{(D7)}$$

Using the identity

$$\tfrac{1}{2}\ln\left[\frac{x+(x^2-a^2)^{1/2}}{x-(x^2-a^2)^{1/2}}\right]=\cosh^{-1}\left|\frac{x}{a}\right|, \quad \text{(D8)}$$

with $x=E$ and $a=m$, together with (D5), we find that

$$\tfrac{1}{2}\ln\left(\frac{E+l}{E-l}\right)=2\varphi. \quad \text{(D9)}$$

Combining (D7) and (D9) gives

$$\left(1-\frac{2\varphi}{\tanh 2\varphi}\right)=\frac{1}{2}\left[2+\frac{E}{l}\ln\left(\frac{E-l}{E+l}\right)\right].$$

Therefore

$$Y_1=-Y_2.$$