Pair Production in Photon-Photon Collisions*

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General formulas for the absorption probability from the process $\gamma+\gamma'\rightarrow e^{+}+e^{-}$ are given for high-energy photons traversing isotropic photon gases having various spectra. Asymptotic formulas are derived and general results are given in graphical form for photon spectra having blackbody and power-law form. The results are applied in a following paper to calculate the absorption probability for high-energy photons traversing cosmic distances.

I. INTRODUCTION

THE process $\gamma+\gamma\rightarrow e^{+}+e^{-}$, pair production in photon-photon collisions, has been shown[1] recently to be of considerable importance for the absorption of high-energy photons traversing cosmic distances. The fundamental process is well understood and its amplitude can be calculated accurately by the general perturbation methods developed for quantum electrodynamics.[2] The existence of the process cannot be questioned, since all detailed predictions of quantum electrodynamics have been verified; moreover the reverse reaction, two-photon electron-positron annihilation is observed. Of course, pair production by photons passing near atomic nuclei and interacting with the virtual photons of the nuclear Coluomb field is also observed. Here we shall be interested in the interaction of a high-energy photon with a "free" (not virtual) photon gas by means of the above process. The process has a threshold, since in the center-of-mass or -momentum (c.m.) frame the total photon energy must be greater than $2mc^2$, where m is the electron mass.

In this paper we shall compute the absorption probability per unit path length for photons traversing photon gases having different-type spectra. The essential results for absorption by a blackbody photon gas have already been published.[1] Here we attempt to derive some useful formulas and present some results graphically which are useful for the calculation of high-energy photon absorption by various cosmic low-energy photon spectra. The results and discussion of these atsrophysical applications will be given in a following paper.

II. ABSORPTION PROBABILITY—GENERAL FORMULAS

Consider the collision between a high-energy photon (energy E) and a low-energy photon (energy ϵ) in the "lab" system in which the high-energy photon is moving along, say, the x axis in the positive direction and the low-energy photon is moving in a direction making an angle θ with the x axis. We are interested in the process $\gamma+\gamma'\rightarrow e^{+}+e^{-}$, for which the total cross section is[2]

$$\sigma=\tfrac{1}{2}\pi r_0^2(1-\beta^2)\left[(3-\beta^4)\ln\frac{1+\beta}{1-\beta}-2\beta(2-\beta^2)\right], \quad (1)$$

where $r_0=e^2/mc^2$ is the classical electron radius and βc is the electron (and positron) velocity in the center-of-mass system. The relationship between β and E, ϵ, and θ is easily obtained, since the (squared) magnitude of the total energy-momentum four vector for the two photons is an invariant; then

$$2\epsilon E(1-\cos\theta)=4E_e^2, \quad (2)$$

where E_e is the electron (also the positron) total energy in the c.m. frame. Following Nikishov[3] we define

$$s=(E_e/mc^2)^2=(\epsilon E/2m^2c^4)(1-\cos\theta), \quad (3)$$

so that

$$\beta=|\mathbf{p}_e|c/E_e=(1-1/s)^{1/2}. \quad (4)$$

Further, we write

$$s=s_0 z, \qquad s_0=\epsilon E/m^2c^4, \quad z=\tfrac{1}{2}(1-\cos\theta). \quad (5)$$

Clearly, for pair production to occur, $s>1$. The threshold condition for a head-on ($\theta=\pi$, $z=1$) photon collision is $\epsilon E=m^2c^4$.

It is desired to calculate the absorption probability per unit time or path length $d\tau_{\text{abs}}/dx$, where τ_{abs} is the absorption "optical depth" for photons of energy E traversing an isotropic photon gas with an energy spectrum $n(\epsilon)$. Here $n(\epsilon)$ is the number of photons per unit volume per unit of energy interval. For an isotropic distribution the fraction of these photons moving in the differential cone at an angle θ and within θ and $\theta+d\theta$ is $\tfrac{1}{2}\sin\theta d\theta$. Then the differential photon density at energy ϵ and angle θ is

$$dn=\tfrac{1}{2}n(\epsilon)\sin\theta d\epsilon d\theta. \quad (6)$$

The relative velocity of the high-energy and the low-energy photons along the direction of the former is $c(1-\cos\theta)$. Then the absorption probability per unit

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[1] R. J. Gould and G. P. Schréder Phys. Rev. Letters **16**, 252 (1966), see also J. V. Jelley, *ibid.* **16**, 479 (1966).

[2] Cf. J. M. Jauch and F. Rohrlich, *Theory of Photons and Electons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955).

[3] A. I. Nikishov, Zh. Eksperim. i Teor. Fiz. **41**, 549 (1961) [English transl.: Soviet Phys.—JETP **14**, 393 (1962)].

path length is

$$\frac{d\tau_{\text{abs}}}{dx}=\int\int \tfrac{1}{2}\sigma n(\epsilon)(1-\cos\theta)\sin\theta d\epsilon d\theta. \quad (7)$$

Changing variables to an integration over s instead of θ by (3) and (5), we have

$$\frac{d\tau_{\text{abs}}}{dx}=\pi r_0^2\left(\frac{m^2c^4}{E}\right)^2\int_{m^2c^4/E}^{\infty}\epsilon^{-2}n(\epsilon)\bar{\varphi}[s_0(\epsilon)]d\epsilon, \quad (8)$$

where

$$\bar{\varphi}[s_0(\epsilon)]=\int_1^{s_0(\epsilon)}s\bar{\sigma}(s)ds, \quad \bar{\sigma}(s)=\frac{2\sigma(s)}{\pi r_0^2}. \quad (9)$$

This result is identical to Nikishov's, except that we have defined the dimensionless function $\bar{\varphi}$ and dimensionless cross section $\bar{\sigma}$. Knowledge of the function $\bar{\varphi}$ allows calculation of the absorption probability (8) as a function of energy for photons traversing any isotropic radiation field $n(\epsilon)$. In the following sections we shall calculate the probability for absorption by several classes of photon spectra.

The function $\bar{\varphi}[s_0]$ was computed by Nikishov and presented in rough graphical form for $1<s_0<10$. Here we give a more accurate representation and also give some asymptotic expansions for $\bar{\varphi}$ when $s_0-1\ll 1$ and $s_0\gg 1$. Integrating the function (9) using the cross section (1) one finds

$$\bar{\varphi}[s_0]=\frac{1+\beta_0^2}{1-\beta_0^2}\ln w_0-\beta_0^2\ln w_0-\ln^2 w_0-\frac{4\beta_0}{1-\beta_0^2}$$
$$+2\beta_0+4\ln w_0\ln(w_0+1)-L(w_0), \quad (10)$$

where

$$\beta_0^2=1-1/s_0, \quad w_0=(1+\beta_0)/(1-\beta_0),$$
$$L(w_0)\equiv\int_1^{w_0}w^{-1}\ln(w+1)dw. \quad (11)$$

The last integral can be written, since

$$(w+1)=w(1+1/w), \quad L(w_0)=\tfrac{1}{2}\ln^2 w_0+L'(w_0),$$

where

$$L'(w_0)=\int_1^{w_0}w^{-1}\ln\left(1+\frac{1}{w}\right)dw,$$
$$=\frac{\pi^2}{12}-\sum_{n=1}^{\infty}(-1)^{n-1}n^{-2}w_0^{-n}. \quad (12)$$

These exact expressions allow ready calculation of $\bar{\varphi}[s_0]$ to essentially any accuracy for any value of s_0. Some

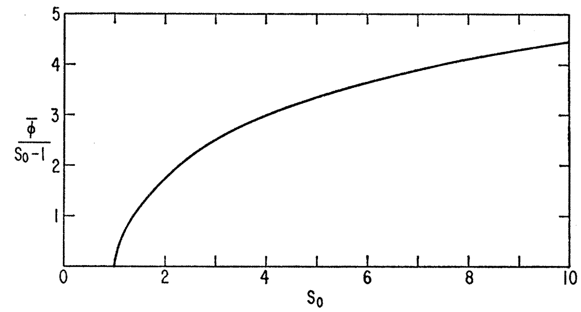

FIG. 1. Graph of the function [see Eq. (9)] $\bar{\varphi}[s_0]/(s_0-1)$.

useful asymptotic formulas are[4]:

$$\bar{\varphi}[s_0]=2s_0(\ln 4s_0-2)+\ln 4s_0(\ln 4s_0-2)-(\pi^2-9)/3$$
$$+s_0^{-1}(\ln 4s_0+9/8)+\cdots(s_0\gg 1); \quad (13a)$$

$$\bar{\varphi}[s_0]=(2/3)(s_0-1)^{3/2}+(5/3)(s_0-1)^{5/2}$$
$$-(1507/420)(s_0-1)^{7/2}+\cdots(s_0-1\ll 1). \quad (13b)$$

The function[5] $\bar{\varphi}[s_0]/(s_0-1)$ is plotted in Fig. 1 for $1<s_0<10$; for larger s_0 it has essentially a logarithmic dependence on s_0.

III. ABSORPTION BY A BLACKBODY PHOTON GAS

Here

$$n(\epsilon)=(\hbar c)^{-3}(\epsilon/\pi)^2(e^{\epsilon/kT}-1)^{-1}. \quad (14)$$

Then, by (8), writing $r_0=\alpha\Lambda$, where $\alpha=1/137$ and $\Lambda(=\hbar/mc=3.86\times10^{-11}$ cm) is the (rationalized) electron Compton wavelength,

$$\frac{d\tau_{\text{abs}}}{dx}=\frac{\alpha^2}{\Lambda\pi}\left(\frac{mc^2}{E}\right)^2\int_{m^2c^4/E}^{\infty}(e^{\epsilon/kT}-1)^{-1}\bar{\varphi}[s_0(\epsilon)]d\epsilon. \quad (15)$$

This expression can be written in convenient form by defining the nondimensional energies

$$\epsilon=\epsilon/kT, \quad \nu=m^2c^4/EkT. \quad (16)$$

Then $s_0=\epsilon/\nu$, and the absorption probability (15) can be written in a form convenient for the calculation of the absorption as a function of energy by a blackbody photon gas at any temperature:

$$\frac{d\tau_{\text{abs}}}{dx}=\frac{\alpha^2}{\pi\Lambda}\left(\frac{kT}{mc^2}\right)^3 f(\nu), \quad (17)$$

$$f(\nu)=\nu^2\int_\nu^\infty(e^\epsilon-1)^{-1}\bar{\varphi}(\epsilon/\nu)d\epsilon. \quad (18)$$

The function $f(\nu)$ as calculated from (18) is plotted in

[4] An expression of form similar to (13a) given by Nikishov is apparently accurate only near $s_0=10$.

[5] Here we plot the transcendental function in a form whereby it is *slowly varying*. We have adopted this procedure of plotting "modified" functions throughout this paper so that the graphs contain the maximum amount of information.

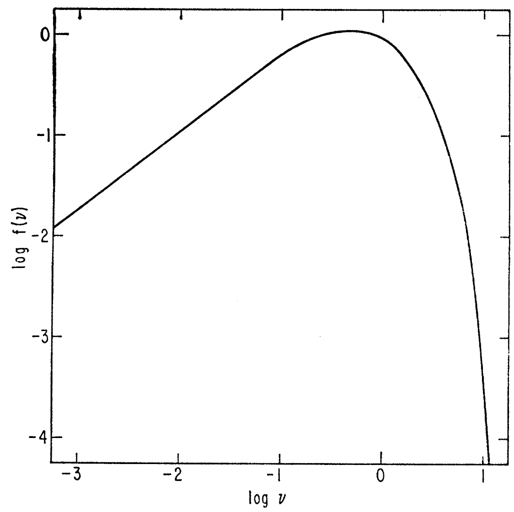

FIG. 2. Graph of the function [Eq. (18)] $f(\nu)$.

Fig. 2; it has a maximum value ≈ 1 at $\nu \approx 1$. Useful asymptotic forms for $f(\nu)$ may be derived by employing the asymptotic forms (13a) and (13b) for $\bar{\varphi}$:

$$f(\nu) \to (\pi^2/3)\nu \ln(0.117/\nu), \quad \nu \ll 1 \tag{19a}$$

$$f(\nu) \to (\pi\nu/4)^{1/2} e^{-\nu}(1+75/8\nu+\cdots), \quad \nu \gg 1. \tag{19b}$$

Clearly the absorption probability is a maximum for $E \sim m^2c^4/kT$ corresponding to $\nu \sim 1$. Because of the exponential photon spectral distribution and the existence of a threshold energy for the absorption process, for $E \ll m^2c^4/kT$ ($\nu \ll 1$) the absorption probability is small; for $E \gg m^2c^4/kT$ ($\nu \ll 1$) the absorption probability goes as $1/E$.

Of course, the probability of absorption by a *diluted* blackbody photon gas could be computed from (17) by simply multiplying by the dilution factor of the radiation field. Thus, if the energy density of the radiation field is ρ_r and its blackbody spectral shape is characterized by a temperature T_r, the dilution factor is $W = \rho_r/aT_r^4$, where $a = 7.57 \times 10^{-15}$ cgs units.

IV. ABSORPTION BY A POWER-LAW PHOTON SPECTRUM

Cosmic photon spectra of the form $n(\epsilon) \propto \epsilon^m$ are common; for example, the spectra of radio sources and the radio background spectrum are of this type. Here we consider the absorption by such spectra with the following characteristics: (a) a low-energy cutoff and $m<0$, (b) a high-energy cutoff and $m>0$, (c) a low- and a high-energy cutoff and $m \gtrless 0$. Again we shall assume *isotropic* spectra.

(a) Photon Spectrum with a Low-Energy Cutoff

We assume a spectrum of the form

$$\begin{aligned} n(\epsilon) &= 0, \quad \epsilon < \epsilon_0 \\ &= C\epsilon^{-\alpha}, \quad \epsilon > \epsilon_0, \quad \alpha > 0, \end{aligned} \tag{20}$$

where C and α are constants. Then, writing (8) as an integration over s_0 ($=\epsilon E/m^2c^4$), we have

$$\frac{d\tau_{\mathrm{abs}}}{dx} = \pi r_0^2 C\left(\frac{m^2c^4}{E}\right)^{1-\alpha} \times \begin{cases} F_\alpha(1), & E<E_0 \\ F_\alpha(\sigma_0), & E>E_0, \end{cases} \tag{21}$$

where

$$\sigma_0 = E/E_0 = \epsilon_0 E/m^2c^4, \tag{22}$$

$$F_\alpha(\sigma_0) = \int_{\sigma_0}^{\infty} s_0^{-(\alpha+2)} \bar{\varphi}[s_0] ds_0. \tag{23}$$

With the help of the asymptotic forms (13a), (13b) for $\bar{\varphi}[s_0]$ one readily obtains the following asymptotic forms for $F_\alpha(\sigma_0)$:

$$F_\alpha(\sigma_0) \to 2\alpha^{-1}\sigma_0^{-\alpha}(\ln 4\sigma_0 + \alpha^{-1} - 2), \quad \sigma_0 \gg 1 \tag{24a}$$

$$F_\alpha(\sigma_0) \to F_\alpha(1) - (4/15)(\sigma_0-1)^{5/2} + [2(2\alpha-1)/21](\sigma_0-1)^{7/2} + \cdots, \quad \sigma_0 - 1 \ll 1. \tag{24b}$$

$F_\alpha(1)$ has the values 1.579, 0.6373, 0.3275, and 0.1932 for $\alpha = 1.0$, 1.5, 2.0, and 2.5, respectively. $\alpha\sigma_0^\alpha F_\alpha(\sigma_0)$ is plotted in Fig. 3 for these values of α.

(b) Photon Spectrum with a High-Energy Cutoff

Consider now a spectrum of the form

$$\begin{aligned} n(\epsilon) &= D\epsilon^\beta, \quad \epsilon < \epsilon_m, \quad \beta \geqslant 0 \\ &= 0, \quad \epsilon > \epsilon_m. \end{aligned} \tag{25}$$

For this spectrum we have

$$\frac{d\tau_{\mathrm{abs}}}{dx} = \pi r_0^2 D\left(\frac{m^2c^4}{E}\right)^{1+\beta} \times \begin{cases} 0, & E<E_m \\ F_\beta(\sigma_m), & E>E_m, \end{cases} \tag{26}$$

where

$$\sigma_m = E/E_m = \epsilon_m E/m^2c^4, \tag{27}$$

$$F_\beta(\sigma_m) = \int_1^{\sigma_m} s_0^{\beta-2} \bar{\varphi}[s_0] ds_0. \tag{28}$$

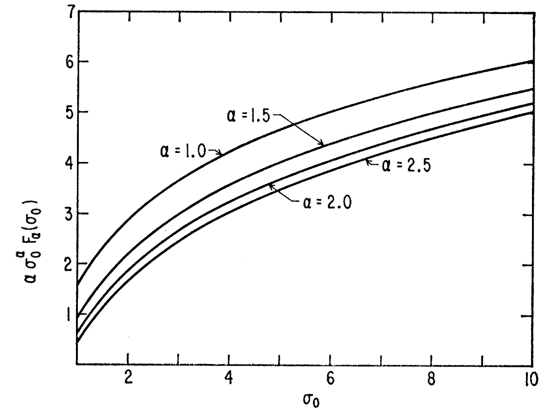

FIG. 3. Graph of the function [Eq. (23)] $\alpha\sigma_0^\alpha F_\alpha(\sigma_0)$ for various values of α.

Again by (13a), (13b), one obtains the asymptotic forms

$$\beta=0: F_\beta(\sigma_m) \to A_\beta+\ln^2\sigma_m-4\ \ln\sigma_m+\cdots,$$

$$\beta\neq0: F_\beta(\sigma_m) \to A_\beta+2\beta^{-1}\sigma_m{}^\beta(\ln 4\sigma_m-\beta^{-1}-2)+\cdots, \quad \sigma_m>10 \quad (29a)$$

$$\text{all } \beta: F_\beta(\sigma_m) \to (4/15)(\sigma_m-1)^{5/2}+[2(2\beta+1)/21](\sigma_m-1)^{7/2}+\cdots, \quad \sigma_m-1\ll1. \quad (29b)$$

$\sigma_m{}^{-\beta}F_\beta(\sigma_m)$ is plotted[6] in Fig. 4 for $\beta=0$, 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0. A_β essentially represents the contribution in the integral of the region [1,10], where the use of the asymptotic form for $\bar{\varphi}[s_0]$ is not legitimate. It has values $A_\beta=8.111$, 13.53, 9.489, 15.675, 34.54, 85.29, and 222.9 for $\beta=0$, 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0, respectively.

(c) Photon Spectrum with Two Cutoffs

We consider two cases corresponding to a spectrum with a negative and positive index:

$$n(\epsilon)=0, \quad \epsilon<\epsilon_0$$
$$=C\epsilon^{-\alpha} \quad \text{or} \quad D\epsilon^\beta, \quad \epsilon_0<\epsilon<\epsilon_m \qquad (30)$$
$$=0, \quad \epsilon>\epsilon_m.$$

[6] To include the case $\beta=0$ it was necessary to plot the function in this form instead of with the factor β.

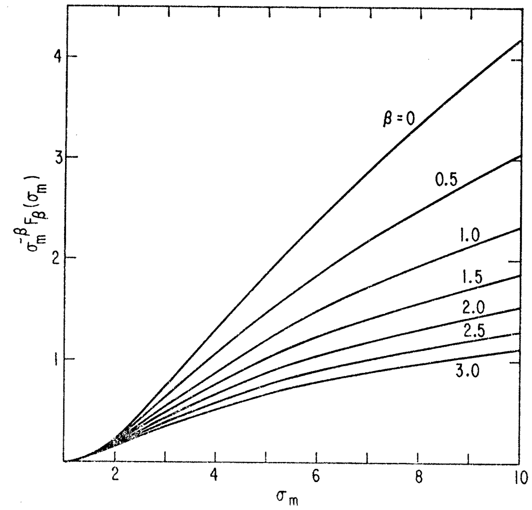

FIG. 4. Graph of the function [Eq. (28)] $\sigma_m{}^{-\beta}F_\beta(\sigma_m)$ for various values of β.

Then one finds

$$(d\tau_{\text{abs}}/dx)_\alpha=\pi r_0{}^2C(m^2c^4/E)^{1-\alpha}\times\begin{cases}0, & E<E_m\\ [F_\alpha(1)-F_\alpha(\sigma_m)], & E_m<E<E_0\\ [F_\alpha(\sigma_0)-F_\alpha(\sigma_m)], & E>E_0;\end{cases} \quad (31a)$$

$$(d\tau_{\text{abs}}/dx)_\beta=\pi r_0{}^2D(m^2c^4/E)^{1+\beta}\times\begin{cases}0, & E<E_m\\ F_\beta(\sigma_m), & E_m<E<E_0\\ [F_\beta(\sigma_m)-F_\beta(\sigma_0)], & E>E_0.\end{cases} \quad (31b)$$

Asymptotic formulas for this case may be found from the formulas for F_α and F_β.