Quantum Optics. I. Excitation of a Damped Radiation Mode by Weakly Coupled Sources

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The isolation usually encountered in optics between the part of a field that is of interest and its source motivates the consideration of a radiation mode weakly coupled to a quantum-mechanical source. After the introduction of some refinements into the quantum mechanics of a damped radiation mode, the field is expressed as the sum of two parts, one due to the source (the source field) and the other due to the loss mechanism (the "vacuum" field). The characteristic function for the field is calculated up to second order in perturbation theory. This function is then compared with the characteristic function for the field in the presence of a classical source. A method is exhibited by which a classical source can be found such that the two characteristic functions are identical when averaged over a half cycle. In particular, the two sources yield the same expectation values for the instantaneous amplitude and energy of the field. The description of the equivalent classical source must be given in statistical terms, in general, and requires only a knowledge of $\langle S^{(0)}(t)\rangle$ and $\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle$, where $S^{(0)}$ is the dipole-moment operator of the quantum-mechanical source unperturbed by the mode under consideration (but otherwise arbitrarily complex, with the possibility of strong coupling to other modes). The theory is illustrated by a consideration of several simple sources—a two-level system, a harmonic oscillator, and a blackbody—for which equivalent classical sources are found. The two-time correlation functions for the field obtained with the two types of sources are compared and are shown to be the same up to first order in $\xi\tau$, where τ is the difference between the two times and ξ is the inverse of the field relaxation time; the physical meaning of the second-order difference in the correlation functions is discussed. A limiting process, in which both the coupling to the source and the damping become small, is suggested as a method of adapting the results to free fields, but it is pointed out that for discussion of a single mode, a free field is physically less satisfactory than a damped field. It is concluded that, within a reasonable approximation scheme, the source field may be described classically (the "vacuum" field furnishing all the necessary quantum-mechanical properties of the total field).

INTRODUCTION

THERE has arisen considerable interest recently in quantum optics, optics in which the field is described quantum-mechanically. Although the interaction between the quantum-mechanical electromagnetic field and matter forms the subject of quantum electrodynamics, one usually deals in optics with a class of phenomena which can be described by certain approximations in a greatly simplified manner. The basis for these approximations is the fact that in optics the behavior of the source is largely independent of the processes associated with the detection, measurement, and utilization of the field; these processes produce very little effect on the source. One can say that the part of the field which is of interest in optics reacts negligibly back on the source, no matter how it is affected by conceivable optical experiments. This isolation of the pertinent part of the field from the source may be described formally in several ways, and different experimental situations may lend themselves most conveniently to different descriptions. Thus, the part of the field that is of interest may be very far from the source, so that isolation—or weak coupling—between this part of the field and the source is of spatial origin. On the other hand, the interesting part of the field may not be localized in ordinary space but in wave-vector space; it may consist of one or more modes that are weakly coupled to the source. In the present article, only the latter type of isolation will be considered, and for simplicity of discussion, attention will be focused on a single mode. Generalization of the results to the case of a larger number of modes will be obvious.

A theorem has been proposed[1] which simplifies the concepts and formalism of quantum optics, and demonstrates explicitly the extent of the difference between quantum optics and classical optics. In the present context, this theorem states that the effect of weakly coupled sources on a radiation mode is approximately the same as that of classical sources, so that the field of modes weakly coupled to the source may be described as the sum of a classical field and the "vacuum" field.[2] It is the purpose of the present article to exhibit a method by which one may find the equivalent classical source (as far as the field is concerned) for an arbitrary, weakly coupled, quantum-mechanical source, and to examine the extent of the approximations involved in this equivalency. Incidentally, some refinements in the quantum mechanics of a damped radiation mode will be presented first.

I. QUANTUM MECHANICS OF A DAMPED RADIATION MODE

As is well known, the quantum mechanics of the field of a radiation mode is the same as the quantum me-

[1] I. R. Senitzky, Phys. Rev. Letters **15**, 233 (1965); **16**, 619 (1966).

[2] In the presence of damping, the true vacuum field is replaced by the field arising from the fluctuations of the loss mechanism, as indicated by the following discussion; the latter field will be referred to as the "vacuum" field.

chanics of a harmonic oscillator. Setting

$$\mathbf{E}=-(4\pi\hbar\omega)^{1/2}\mathbf{u}(\mathbf{r})p(t), \tag{1.1a}$$

$$\mathbf{H}=(4\pi c^2\hbar/\omega)^{1/2}\nabla\times\mathbf{u}(\mathbf{r})q(t), \tag{1.1b}$$

where $\mathbf{u}(\mathbf{r})$ describes the spatial dependence of the field for the mode under consideration and is normalized over a suitable volume, we have the result that q and p are the (dimensionless) coordinate and momentum, respectively, of a harmonic oscillator—the radiation oscillator—of (angular) frequency ω. The method of analysis of the damped oscillator to be used presently is, basically, that developed in two previous articles.[3] Certain refinements in the treatment of the damping mechanism (some of which were developed in a subsequent analysis of the damped two-level system[4]) not present there, however, will be included in the present discussion.

The Hamiltonian of the coupled systems under consideration is given by

$$H=H_{\rm osc}+H_{\rm LM}+H_S+\hbar p(\alpha S+F), \tag{1.2a}$$

where

$$H_{\rm osc}=\tfrac{1}{2}\hbar\omega(q^2+p^2) \tag{1.2b}$$

and

$$[q,p]=i. \tag{1.2c}$$

$H_{\rm LM}$ is the Hamiltonian of the loss mechanism (LM), H_S is the Hamiltonian of the source, S and F are the dynamical variables of the source and LM, respectively, through which these systems couple to the radiation oscillator, and α is a coupling constant. The coupling to the oscillator has been chosen to occur through p, which makes S and F the effective electric-dipole moments (in appropriate units). No significant change in the results would be obtained if source and LM coupled to the oscillator through q (by their magnetic-dipole moments), since q and p are symmetrical, or through both q and p. In the discussion of the present Section, only the coupling to the LM will be under consideration. The coupling of the oscillator to the source is, so far, arbitrary.

Some of the properties of the LM have been studied in detail previously and will be only summarized here. For the free (uncoupled) LM—indicated by the superscript zero—the expectation values of a product of F's is given by[3]

$$\langle F^{(0)}(t_1)F^{(0)}(t_2)\cdots F^{(0)}(t_n)\rangle=0,\quad n\ \text{odd} \tag{1.3a}$$

and

$$\langle F^{(0)}(t_1)F^{(0)}(t_2)\cdots F^{(0)}(t_n)\rangle =\sum\langle F^{(0)}(t_{j_1})F^{(0)}(t_{j_2})\rangle\cdots \langle F^{(0)}(t_{j_{n-1}})F^{(0)}(t_{j_n})\rangle,\quad n\ \text{even}, \tag{1.3b}$$

where $j_{2k-1}<j_{2k}$ (the order is the same in each pair as that in the original product), and where the summation is taken over all the different arrangements into pairs. Equations (1.3) describe a Gaussian stochastic variable, and the order in each pair is significant because $F^{(0)}(t)$ is a quantum-mechanical variable. The expectation value of a single pair is given by

$$\langle F^{(0)}(t_1)F^{(0)}(t_2)\rangle=\frac{2}{\pi}\int_0^\infty d\omega'[\eta(\omega')\cos\omega'(t_1-t_2) -i\xi(\omega')\sin\omega'(t_1-t_2)], \tag{1.4a}$$

where

$$\xi(\omega')=\tfrac{1}{2}\pi\hbar Z^{-1}B(\omega')[1-\exp(-\hbar\omega'/kT)], \tag{1.4b}$$

$$\eta(\omega')=\tfrac{1}{2}\pi\hbar Z^{-1}B(\omega')[1+\exp(-\hbar\omega'/kT)]; \tag{1.4c}$$

$$Z=\int_0^\infty dE\,\rho(E)\exp(-E/kT), \tag{1.4d}$$

$$B(\omega')=\int_0^\infty dE\,\rho(E+\hbar\omega')\rho(E)\tilde{F}^2(E+\hbar\omega',E)\times\exp(-E/kT), \tag{1.4e}$$

$\rho(E)$ being the density of energy states of the LM (assumed closely spaced), $\tilde{F}^2(E_i,E_k)$ being the average of $|F_{ik}{}^{(0)}(0)|^2$ over small ranges of E_i and E_k, and T being the LM temperature. Since

$$\frac{\xi(\omega')}{\eta(\omega')}=\frac{1-\exp(-\hbar\omega'/kT)}{1+\exp(-\hbar\omega'/kT)}, \tag{1.5}$$

Eqs. (1.3), (1.4a), and (1.5) are sufficient to describe the LM provided $\xi(\omega')$ and T are specified. $\xi(\omega')$ may be regarded, if one does not want to delve into the details of the LM, as a phenomenological function describing the LM. [As will become apparent, $\xi(\omega')$ is approximately the exponential decay constant of the expectation value of the amplitude of an initially excited oscillator of (angular) frequency ω'.] It is assumed to be a slowly varying function of ω' and much smaller than ω. All final results will be expectation values with respect to the LM. (This procedure can be justified by the assumption that the LM may be considered to consist of a large number of essentially independent systems, each constituent system itself having the properties of a LM.)

For simplicity, we introduce a shorthand notation in writing expectation values of products of $F^{(0)}$'s of different arguments; we replace $F^{(0)}(t_j)$ by the number j. Thus, the left side of Eq. (1.3a) is written in shorthand, as $\langle 12\cdots n\rangle$. The following theorem concerning the expectation value of products will be useful:

$$\langle 1\,2\cdots(n-1)\,[n,n+1](n+2)\cdots N\rangle =\langle[n,n+1]\rangle\langle 1\,2\cdots(n-1)\,(n+2)\cdots N\rangle, \tag{1.6}$$

[3] I. R. Senitzky, Phys. Rev. **119**, 670 (1960); **124**, 642 (1961). Other discussions of the damped harmonic oscillator include those of Julian Schwinger, J. Math. Phys. **2**, 407 (1961), and M. Lax, Phys. Rev. **145**, 110 (1966).

[4] I. R. Senitzky, Phys. Rev. **137**, A1635 (1965).

that is, a commutator may be replaced by its expectation value in the expectation value of any algebraic expression of $F^{(0)}$'s. The proof of this theorem is given in Appendix A. Since all final results will be expectation values with respect to the LM, this theorem implies that all commutators are effectively c numbers; we thus have

$$[F^{(0)}(t_i),F^{(0)}(t_j)]\rightarrow\langle[F^{(0)}(t_i),F^{(0)}(t_j)]\rangle$$
$$=-\frac{4i}{\pi}\int_0^\infty d\omega'\xi(\omega')\sin\omega'(t_i-t_j)\,, \quad (1.7)$$

using Eq. (1.4a).

We proceed now to the equations of motion based on the Hamiltonian of Eqs. (1.2). In the Heisenberg picture, which will be used throughout, these are

$$\dot{q}=\omega p+\alpha S+F\,, \quad (1.8a)$$
$$\dot{p}=-\omega q\,, \quad (1.8b)$$
$$\dot{S}=(i\hbar)^{-1}[S,H_S]\,, \quad (1.8c)$$
$$\dot{H}_S=-i\alpha p[H_S,S]\,, \quad (1.8d)$$
$$\dot{F}=(i\hbar)^{-1}[F,H_{\mathrm{LM}}]\,, \quad (1.8e)$$
$$\dot{H}_{\mathrm{LM}}=-ip[H_{\mathrm{LM}},F]. \quad (1.8f)$$

If we consider the coupling between oscillator and LM to begin at $t=0$, then the last two equations are equivalent to the integral equation

$$F(t)=F^{(0)}(t)+\frac{1}{\hbar}\int_0^t dt_1\int_0^{t_1} dt_2 U(t-t_1)$$
$$\times[F(t_1),[F(t_2),H_{\mathrm{LM}}(t_2)]p(t_2)]U^{-1}(t-t_1)\,, \quad (1.9a)$$

where

$$U(\tau)\equiv\exp[(i/\hbar)H_{\mathrm{LM}}(0)\tau]. \quad (1.9b)$$

Approximations based on the assumption that the LM is disturbed only slightly by the oscillator and that quantum-mechanical correlation between oscillator and LM in interaction terms of higher order than the second may be neglected (these approximations have been discussed in detail previously[3]) yield

$$F(t)\approx F^{(0)}(t)+\frac{1}{\hbar}\int_0^t dt_1\int_0^{t_1} dt_2$$
$$\times U(t-t_1)[F^{(0)}(t_1),[F^{(0)}(t_2),H_{\mathrm{LM}}{}^{(0)}]]$$
$$\times U^{-1}(t-t_1)p(t_2). \quad (1.10)$$

Interchanging the order of integration and utilizing the result—derived in Appendix B—that

$$\int_{t_2}^t dt_1 U(t-t_1)[F^{(0)}(t_1),[F^{(0)}(t_2),H_{\mathrm{LM}}{}^{(0)}]]U^{-1}(t-t_1)$$
$$=-i\hbar[F^{(0)}(t),F^{(0)}(t_2)]\,, \quad (1.11)$$

we obtain

$$F(t)=F^{(0)}(t)-i\int_0^t dt_1[F^{(0)}(t),F^{(0)}(t_1)]p(t_1)\,, \quad (1.12)$$

which, in view of Eq. (1.7), becomes

$$F(t)=F^{(0)}(t)$$
$$-\frac{4}{\pi}\int_0^t dt_1\int_0^\infty d\omega'\xi(\omega')\sin\omega'(t-t_1)p(t_1). \quad (1.13)$$

From Eqs. (1.8a) and (1.8b) one obtains

$$\ddot{p}+\omega^2 p=-\omega(\alpha S+F) \quad (1.14)$$

and, after substitution from Eq. (1.13),

$$\ddot{p}+\omega^2 p=-\omega(\alpha S+F^{(0)})$$
$$-\frac{4\omega}{\pi}\int_0^t dt_1\int_0^\infty d\omega'\xi(\omega')\sin\omega'(t-t_1)p(t_1). \quad (1.15)$$

We concern ourselves only with p, for the moment, since Eq. (1.8b) gives q immediately, once p is known. The integro-differential Eq. (1.15) can be rewritten as an integral equation:

$$p=P+\int_0^t dt_1 K(t,t_1)p(t_1)\,, \quad (1.16a)$$

where

$$K(t,t_1)\equiv\frac{4}{\pi}\int_0^\infty d\omega'\xi(\omega')\int_{t_1}^t dt_2$$
$$\times\sin\omega(t-t_2)\sin\omega'(t_2-t_1)\,, \quad (1.16b)$$

$$P\equiv p^{(0)}-\int_0^t dt_1[F^{(0)}(t_1)+\alpha S(t_1)]$$
$$\times\sin\omega(t-t_1)\,, \quad (1.16c)$$

and $p^{(0)}$ satisfies the free-harmonic-oscillator equation

$$\ddot{p}^{(0)}+\omega^2 p^{(0)}=0\,, \quad (1.16d)$$

as well as the initial conditions. The kernel $K(t,t_1)$ is evaluated—with some approximations based, essentially on $\xi(\omega)\ll\omega$—in Appendix C, to yield

$$K(t,t_1)\approx-2\xi(\omega)\cos\omega(t-t_1)+2\epsilon(\omega)\sin\omega(t-t_1)\,, \quad (1.17a)$$

where

$$\epsilon(\omega)=\frac{2}{\pi}\mathcal{P}\int_0^\infty \xi(\omega')\frac{\omega'}{\omega'^2-\omega^2}d\omega'. \quad (1.17b)$$

The integral Eq. (1.16a) is now equivalent to the more familiar differential equation

$$\ddot{p}+2\xi\dot{p}+\omega^2(1-2\epsilon/\omega)p=-\omega(\alpha S+F^{(0)})\,, \quad (1.18)$$

where ξ and ϵ stand for $\xi(\omega)$ and $\epsilon(\omega)$, respectively. The damping and reactive effects of the LM (contained in

ξ and ϵ, respectively) have been separated from the fluctuation effects (contained in $F^{(0)}$), and the dynamical variables of the LM are no longer unknown operators to be determined by the equations of motion. The complete set of these equations may now be written as

$$\dot{q}=-2\xi q+\omega(1-2\epsilon/\omega)p+\alpha S+F^{(0)}\,, \tag{1.19a}$$

$$\dot{p}=-\omega q\,, \tag{1.19b}$$

$$\dot{S}=(i\hbar)^{-1}[S,H_S]\,, \tag{1.19c}$$

$$\dot{H}_S=-i\alpha p[H_S,S]\,, \tag{1.19d}$$

where $\xi(\omega')$ (which, together with T, determines both ϵ and $F^{(0)}$) is assumed known.

For purposes of the following discussion, we ignore the reactive shift in frequency, assuming that ϵ/ω is negligible compared to unity. We will also neglect ξ/ω compared to unity. Furthermore, we shift the time origin to $-\infty$, so that the coupling to the LM begins then. As far as the source is concerned, however, we let $\alpha=0$ for $t\leq 0$, so that the coupling to the source still begins at $t=0$. With the above approximations and time shift, Eqs. (1.19a) and (1.19b) yield

$$q=q_F+q_S\,, \tag{1.20a}$$

$$p=p_F+p_S\,, \tag{1.20b}$$

where

$$q_F=\int_{-\infty}^{t}dt_1e^{-\xi(t-t_1)}F^{(0)}(t_1)\cos\omega(t-t_1)\,, \tag{1.20c}$$

$$q_S=\alpha\int_{0}^{t}dt_1e^{-\xi(t-t_1)}S(t_1)\cos\omega(t-t_1)\,, \tag{1.20d}$$

$$p_F=-\int_{-\infty}^{t}dt_1e^{-\xi(t-t_1)}F^{(0)}(t_1)\sin\omega(t-t_1)\,, \tag{1.20e}$$

$$p_S=-\alpha\int_{0}^{t}dt_1e^{-\xi(t-t_1)}S(t_1)\sin\omega(t-t_1)\,. \tag{1.20f}$$

It is to be noted that q_F and p_F are the coordinates in the absence of a source, and for zero LM temperature describe the ground state of the (damped) radiation oscillator, or the zero-point field of the mode—the "vacuum" field. Equations (1.20c) and (1.20e) show that the expectation value of a product of q_F's and p_F's can be expanded in the same manner as that of the $F^{(0)}$'s, namely, in terms of a product of expectation values of pairs, as described in Eqs. (1.3). The expectation value of these pairs can be derived by use of Eq. (1.4a). This derivation is carried out in Appendix D, and yields the relationships

$$\langle q_F(t_1)q_F(t_2)\rangle=\langle p_F(t_1)p_F(t_2)\rangle=\tfrac{1}{2}e^{-\xi|t_1-t_2|}[i\sin\omega(t_2-t_1)+(1+2\varphi)\cos\omega(t_1-t_2)]\,, \tag{1.21a}$$

$$\langle q_F(t_1)p_F(t_2)\rangle=\tfrac{1}{2}e^{-\xi|t_1-t_2|}[i\cos\omega(t_1-t_2)+(1+2\varphi)\sin\omega(t_1-t_2)]\,, \tag{1.21b}$$

$$\langle p_F(t_1)q_F(t_2)\rangle=\tfrac{1}{2}e^{-\xi|t_1-t_2|}[-i\cos\omega(t_1-t_2)-(1+2\varphi)\sin\omega(t_1-t_2)]\,, \tag{1.21c}$$

where

$$\varphi\equiv[\exp(\hbar\omega/kT)-1]^{-1}. \tag{1.21d}$$

It is seen that, for $T=0$, these expectation values become identical with those for a lossless harmonic oscillator in the ground state as either ξ or t_1-t_2 approaches zero. A further consequence of Eqs. (1.20) is the fact that the commutators of the q_F's and p_F's are to be regarded as c numbers (equal to their expectation values), in accordance with the reasoning of Eq. (1.7). Thus, we have

$$[q_F(t_1),q_F(t_2)]=e^{-\xi|t_1-t_2|}i\sin\omega(t_2-t_1)\,, \tag{1.22a}$$

$$[p_F(t_1),p_F(t_2)]=e^{-\xi|t_1-t_2|}i\sin\omega(t_2-t_1)\,, \tag{1.22b}$$

$$[q_F(t_1),p_F(t_2)]=e^{-\xi|t_1-t_2|}i\cos\omega(t_1-t_2)\,. \tag{1.22c}$$

These commutators approach the corresponding ones for the lossless oscillator as either ξ or t_1-t_2 approaches zero.

It is convenient at this point to introduce the frequently used non-Hermitian operators

$$a=2^{-1/2}(q+ip),\quad a^\dagger=2^{-1/2}(q-ip)\,. \tag{1.23}$$

From Eqs. (19), we have

$$[a_F(t_1),a_F(t_2)]=[a_F^\dagger(t_1),a_F^\dagger(t_2)]=0\,, \tag{1.24a}$$

$$[a_F(t_1),a_F^\dagger(t_2)]=\exp[-\xi|t_1-t_2|-i\omega(t_1-t_2)]\,. \tag{1.24b}$$

Also,

$$\langle a_F(t_1)a_F(t_2)\rangle=\langle a_F^\dagger(t_1)a_F^\dagger(t_2)\rangle=0\,, \tag{1.25a}$$

$$\langle a_F(t_1)a_F^\dagger(t_2)\rangle=(1+\varphi)\exp[-\xi|t_1-t_2|-i\omega(t_1-t_2)]\,, \tag{1.25b}$$

$$\langle a_F^\dagger(t_1)a_F(t_2)\rangle=\varphi\exp[-\xi|t_1-t_2|+i\omega(t_1-t_2)]\,. \tag{1.25c}$$

It is obvious that the expectation value of a product of a_F's and $a_F^\dagger$'s can be expanded in terms of products of the expectation values of pairs, in the same manner as the q_F's and p_F's, or as the $F^{(0)}$'s [that is, according to Eqs. (1.3)]. It follows from Eqs. (1.25a) that unless there are an equal number of a_F's and $a_F^\dagger$'s in any product, its expectation value will be zero. Furthermore, for $T=0$, we have $\varphi=0$, and obtain

$$\langle a_0^\dagger(t_1)a_0(t_2)\rangle=0\,, \tag{1.26}$$

where the subscript F has been replaced by the subscript 0 to indicate that $T=0$. Since the order of a particular $a_F(t_i)$ and $a_F^\dagger(t_j)$ in a pair must be the same as in the original product, Eqs. (1.25a) and (1.26) show that the expectation value of any product in which there is an $a_F(t_i)$ at the extreme right or an $a_F^\dagger(t_j)$ at the extreme left is zero for $T=0$.[5] We are not interested in thermal effects in the present article; for the sake of simplicity, therefore, we will consider only the case $T=0$ in the following discussion (but retain the notation a_F).

II. CONTRIBUTION OF THE SOURCE

We come now to a consideration of the part of the field due to the source—that is, q_s and p_s given by Eqs. (1.20d) and (1.20f), or a_s and $a_s^\dagger$ given by

$$a_S=\frac{\alpha}{\sqrt{2}}\int_0^t dt_1 S(t_1)e^{-i\Omega(t-t_1)}, \tag{2.1a}$$

$$a_S^\dagger=\frac{\alpha}{\sqrt{2}}\int_0^t dt_1 S(t_1)e^{i\Omega^*(t-t_1)}, \tag{2.1b}$$

where

$$\Omega\equiv\omega-i\xi. \tag{2.1c}$$

We want to compare the contribution of a quantum-mechanical source with that of a classical source. As far as the contribution of a quantum-mechanical source is concerned, there is no difficulty except a computational one; all that is needed is a solution of the equations of motion, Eqs. (1.19). As far as a classical source is concerned, however, the equations of motion themselves become inconsistent. It is not the *form* of Eqs. (1.19c) and (1.19d) that is troublesome, since the commutator bracket would be replaced, for a classical source, by the Poissom bracket multiplied by $i\hbar$.[6] It is the fact that the source is coupled to the field, and that its time development is affected by the field, which leads to inconsistencies, since, if the source is initially classical, it will acquire quantum-mechanical properties from the quantum-mechanical field as time progresses. Thus, the coupling of quantum-mechanical and classical systems results, in general, in inconsistencies.[7] There are, however, approximations with which one can treat a classical source coupled to a quantum-mechanical field. If the effect of the field (of the mode under consideration) on the source is negligible, then $\dot{H}_S$ in Eq. (1.19d) becomes negligible, and a classical source will remain classical, since its equations of motion involve only classical variables. This is the situation that will be considered in the present article; that is, we will consider the case in which the mode under consideration affects the source only slightly.[8] One should not conclude that this is the *only* situation in which a source may be treated consistently as classical. Where the quantum-mechanical aspects of the field are negligible, the equations of motion for the (classical) source and field become classical equations of motion that form the basis of classical radiation theory, of course. In our equations of motion for a single mode, Eqs. (1.19), this situation occurs when $F^{(0)}$ (for $\mathrm{T}=0$) is negligible compared to αS for nonvanishing ξ, or when the "vacuum" field is negligible compared to the part of the field due to the source. The case we consider in the present article, however, is that in which the source field is small compared to the "vacuum" field, a case in which the total field must be treated quantum-mechanically if, ultimately, its interaction with quantum-mechanical systems is investigated.

The reaction of the generated field back on the source is assumed to be negligible. Second-order perturbation theory may therefore be used, and our calculation will neglect all terms in q and p containing powers of α higher than the second.[9] Just as Eqs. (1.8e) and (1.8f) led to Eq. (1.9), Eqs. (1.8c) and (1.8d) lead to

$$S=S^{(0)}+\frac{\alpha}{\hbar}\int_0^t dt_1\int_0^{t_1} dt_2 V(t-t_1)\times[S(t_1),[S(t_2),H_S(t_2)]p(t_2)]V^{-1}(t-t_1), \tag{2.2}$$

where

$$V(\tau)\equiv\exp[(i/\hbar)H_S(0)\tau].$$

We are interested in S only for the purpose of inserting it into the expression for q_s and p_s, and can therefore neglect higher orders than the first in α. Thus,

$$S\approx S^{(0)}+\frac{\alpha}{\hbar}\int_0^t dt_1\int_0^{t_1} dt_2 V(t-t_1)\times[S^{(0)}(t_1),[S^{(0)}(t_2),H_S^{(0)}]p_F(t_2)]V^{-1}(t-t_1). \tag{2.3}$$

Since p_F commutes with the other variables in the integrand, and since the result of Eq. (1.11) can be ap-

[5] It is interesting to note the resemblance between the present results and field theory of a single (lossless) mode. For instance, the expansion in terms of the expectation value of pairs corresponds to Wick's theorem, and the vanishing of the expectation value of a product for a_0 at the extreme right or $a_0^\dagger$ at the extreme left corresponds to $a|0\rangle=\langle 0|a^\dagger=0$. The present results, however, are not derived from consideration of the lossless harmonic oscillator, but, mainly, from consideration of the LM.

[6] Note that $\hbar$ is inserted in the coupling term of the original Hamiltonian, Eq. (1.2), for dimensional reasons only.

[7] This is, in fact, the motivation for treating the field quantum mechanically when studying the mutual interaction between the field and quantum-mechanical systems.

[8] This is the case usually encountered in optics, where not the entire field (all modes) is of interest, but only that part of the field which couples significantly to the detector. It may be necessary to partition the field into modes in a judicious manner in order to end up with modes that are negligibly coupled to the detector and modes that are significantly coupled. The latter usually affect the source negligibly. For a "single mode" laser, for instance, the field should be resolved, approximately, into two modes, one inside the laser cavity and one outside. Only the outside mode is coupled significantly to the detector, and this mode is coupled weakly to the laser.

[9] Second-order perturbation theory will account for the action (formally) of the "vacuum" field on the source, but not for the reaction of the generated field (the part of the field due to the source) back on the source. As indicated later, a quantum-mechanical source without the "vacuum" field is too "bare" to be physically meaningful.

plied to any system, we have, up to first order,

$$S=S^{(0)}-i\alpha\int_0^t dt_1[S^{(0)}(t),S^{(0)}(t_1)]p_F(t_1). \quad (2.4)$$

Substituting into Eqs. (2.1), we obtain the first- and second-order parts of a and $a^\dagger$ (the zeroth-order parts are a_F and $a_F{}^\dagger$),

$$a_S=a^{(1)}+a^{(2)}, \quad (2.5a)$$

$$a_S{}^\dagger=a^{\dagger(1)}+a^{\dagger(2)}, \quad (2.5b)$$

where

$$a^{(1)}(t)=\frac{\alpha}{\sqrt{2}}\int_0^t dt_1 S^{(0)}(t_1)e^{-i\Omega(t-t_1)}, \quad (2.5c)$$

$$a^{\dagger(1)}(t)=\frac{\alpha}{\sqrt{2}}\int_0^t dt_1 S^{(0)}(t_1)e^{i\Omega^*(t-t_1)}, \quad (2.5d)$$

$$a^{(2)}(t)=-\tfrac{1}{2}\alpha^2\int_0^t dt_1\int_0^{t_1} dt_2 e^{-i\Omega(t-t_1)}[S^{(0)}(t_1),S^{(0)}(t_2)]\times[a_F(t_2)-a_F{}^\dagger(t_2)], \quad (2.5e)$$

$$a^{\dagger(2)}(t)=-\tfrac{1}{2}\alpha^2\int_0^t dt_1\int_0^{t_1} dt_2 e^{i\Omega^*(t-t_1)}[S^{(0)}(t_1),S^{(0)}(t_2)]\times[a_F(t_2)-a_F{}^\dagger(t_2)]. \quad (2.5f)$$

Utilizing Eqs. (1.25), one can show that

$$\langle a_F a^{\dagger(2)}\rangle+\langle a^{(2)}a^\dagger{}_F\rangle+\langle a^{(1)}a^{\dagger(1)}\rangle=\langle a^{\dagger(1)}a^{(1)}\rangle, \quad (2.6)$$

a relationship that will be useful later.[10]

The physical interpretation of a quantum-mechanical result must be made in statistical terms, and such an interpretation is most easily obtained by an examination of the pertinent characteristic function. We consider, therefore, the characteristic function

$$f(\mu,\nu)\equiv\langle\exp i(\mu q+\nu p)\rangle=\langle\exp i(\zeta a+\zeta^* a^\dagger)\rangle\equiv\tilde{f}(\zeta,\zeta^*), \quad (2.7)$$

where

$$\zeta=2^{1/2}(\mu-i\nu). \quad (2.8)$$

This function contains all the statistical properties of the oscillator (at a given time), since the expectation value of a product of m q's and n p's, in any order, can be obtained from the commutation relations and the terms up to $\mu^m\nu^n$ in the power series of (μ,ν). We will derive an expression for the characteristic function in terms of the source variable up to second order.

We begin with the expansion

$$f(\mu,\nu)=\tilde{f}(\zeta,\zeta^*)\approx\langle\exp i(A_F+A^{(1)}+A^{(2)})\rangle, \quad (2.9a)$$

where

$$A\equiv\mu q+\nu p\equiv\zeta a+\zeta^* a^\dagger. \quad (2.9b)$$

[10] This relationship is consistent with the requirement $[a(t),a^*(t)]=1$.

Now, $A^{(1)}$ commutes with A_F, and the commutator of $A^{(1)}$ and $A^{(2)}$ is a third-order term and negligible in our approximation scheme. We can therefore write

$$\exp i(A_F+A^{(1)}+A^{(2)})=\exp iA^{(1)}\exp i(A_F+A^{(2)}). \quad (2.10)$$

One notes that $[A_F,A^{(2)}]$ is a second-order term that contains only source variables; it commutes with A_F, and its commutator with $A^{(2)}$ is of order higher than second, and negligible. We utilize the fact that if $[O_1,O_2]$ commutes with both O_1 and O_2, there exists the relationship

$$e^{(O_1+O_2)}=\tfrac{1}{2}(e^{O_1}e^{O_2}e^{-1/2[O_1,O_2]}+e^{O_2}e^{O_1}e^{1/2[O_1,O_2]}). \quad (2.11)$$

Applying this relationship to the second exponent on the right of Eq. (2.10), and dropping terms of higher order than the second, we obtain

$$\exp i(A_F+A^{(2)})=e^{iA_F}+\tfrac{1}{2}i\{e^{iA_F},A^{(2)}\}, \quad (2.12)$$

where the symmetrized product notation, $\{A,B\}=AB+BA$, is used. Thus up to second order,

$$\exp i(A_F+A^{(1)}+A^{(2)})=\tfrac{1}{2}\{\exp iA_F,(1+iA^{(1)}-\tfrac{1}{2}A^{(1)2}+iA^{(2)})\}, \quad (2.13)$$

where the symmetrization with the first three terms has no significance, of course.

To obtain the characteristic function, we must take the expectation value of the right side of Eq. (2.13). Since A_F and $A^{(1)}$ are expressed in terms of variables of different (uncoupled) systems, the expectation value of a product of a function of A_F and a function of $A^{(1)}$ is given by the product of the expectation values of the functions. A_F and $A^{(2)}$, however, both contain a_F and $a_F{}^\dagger$, and care must be exercised in evaluating the expectation value of the product of $\exp(iA_F)$ and $A^{(2)}$. It is shown in Appendix E that one obtains the result

$$\langle\{e^{iA_F},A^{(2)}\}\rangle=\langle e^{iA_F}\rangle i\langle\{A_F,A^{(2)}\}\rangle, \quad (2.14)$$

so that

$$\langle\exp i(A_F+A^{(1)}+A^{(2)})\rangle=\langle\exp iA_F\rangle(1+i\langle A^{(1)}\rangle-\tfrac{1}{2}\langle A^{(1)2}\rangle-\tfrac{1}{2}\langle\{A_F,A^{(2)}\}\rangle). \quad (2.15)$$

Equations (2.5) and (2.9b) yield

$$\langle A^{(1)}\rangle=\frac{\alpha}{\sqrt{2}}\int_0^t dt_1 e^{-\xi(t-t_1)}\langle S^{(0)}(t_1)\rangle\times[\zeta e^{-i\omega(t-t_1)}+\zeta^* e^{i\omega(t-t_1)}]; \quad (2.16)$$

taking also Eq. (1.25) into consideration, we obtain, with some calculation,

$$\langle A^{(1)2}\rangle+\langle\{A_F,A^{(2)}\}\rangle=\alpha^2\int_0^t dt_1\int_0^{t_1} dt_2 \times e^{-\xi(2t-t_1-t_2)}[\zeta e^{-i\omega(t-t_1)}+\zeta^* e^{i\omega(t-t_1)}]\times[\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle\zeta e^{-i\omega(t-t_2)}+\langle S^{(0)}(t_2)S^{(0)}(t_1)\rangle\zeta^* e^{i\omega(t-t_2)}]. \quad (2.17)$$

Equations (2.15)–(2.17) constitute the expression of the characteristic function in terms of the source variable up to second order.

We consider only those cases where the source has frequencies of oscillation that lie in the neighborhood of the oscillator frequency ω. In that event, there will be resonant contributions in the integrations involved in Eqs. (2.16) and (2.17); that is, there will be terms in the integrands which do not oscillate, or oscillate very slowly, with respect to the variables of integration, and yield the main contribution toward the integrals. The contributions of rapidly oscillating terms in the integrand will be neglected. Furthermore, in Eq. (2.16) the resultant integral will oscillate with frequency ω, and in Eq. (2.17) the integral will have, in general, a nonoscillating part, and may also have a part which oscillates with frequency 2ω. We assume that the double-frequency part is of no interest, and can be neglected; in other words, we obtain the average over a half cycle of the expression in Eq. (2.17) or Eq. (2.15). (Note that $\langle \exp A_F\rangle$ has no time variation.) This leads to the simpler form

$$\langle A^{(1)2}\rangle+\langle\{A_F,A^{(2)}\}\rangle=\tilde{\alpha}^2\int_0^t dt_1\int_0^{t_1} dt_2 \times e^{-\xi(2t-t_1-t_2)}[\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle \times e^{-i\omega(t_1-t_2)}+\langle S^{(0)}(t_2)S^{(0)}(t_1)\rangle e^{i\omega(t_1-t_2)}], \quad (2.18\text{a})$$

where, for simplicity of notation, we have set

$$\tilde{\alpha}\equiv\alpha|\zeta|. \quad (2.18\text{b})$$

Since the integrand is invariant with respect to an interchange of t_1 and t_2, we can write Eq. (2.18) in an alternative form as

$$\langle A^{(1)2}\rangle+\langle\{A_F,A^{(2)}\}\rangle=\tilde{\alpha}^2\int_0^t dt_1\int_0^{t} dt_2 \times\exp[-\xi(2t-t_1-t_2)-i\omega(t_1-t_2)] \times\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle. \quad (2.19)$$

Equations (2.15), (2.16), and (2.19) constitute the description of the characteristic function $\tilde{f}(\zeta,\zeta^*)$ (with double-frequency terms discarded), carried as far as possible without going into the details of the source. Special cases in which the source is a two-level system, a harmonic oscillator, and a blackbody will be considered later.

Before concluding the discussion of a general source, we calculate the expression $\langle A^{(1)2}\rangle+\langle\{A_F,A^{(2)}\}\rangle-\langle A^{(1)}\rangle^2$, which will be of interest in connection with the following discussion. Using Eq. (2.16), and again dropping double-frequency terms (or averaging over a half cycle), we obtain

$$\begin{aligned}\langle A^{(1)}\rangle^2&=\tfrac{1}{2}\tilde{\alpha}^2\int_0^t dt_1\int_0^t dt_2 e^{-\xi(2t-t_1-t_2)}\langle S^{(0)}(t_1)\rangle \\ &\quad\times\langle S^{(0)}(t_2)\rangle[e^{-i\omega(t_1-t_2)}+e^{i\omega(t_1-t_2)}], \\ &=\tilde{\alpha}^2\int_0^t dt_1\int_0^t dt_2 \exp[-\xi(2t-t_1-t_2) \\ &\quad -i\omega(t_1-t_2)]\langle S^{(0)}(t_1)\rangle\langle S^{(0)}(t_2)\rangle, \end{aligned}\quad (2.20)$$

which, together with Eq. (2.19) yields

$$\begin{aligned}&\langle A^{(1)2}\rangle+\langle\{A_F,A^{(2)}\rangle-\langle A^{(1)}\rangle^2 \\ &=\tilde{\alpha}^2\int_0^t dt_1\int_0^t dt_2[\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle \\ &\quad -\langle S^{(0)}(t_1)\rangle\langle S^{(0)}(t_2)\rangle] \\ &\quad\quad \times\exp[-\xi(2t-t_1-t_2)-i\omega(t_1-t_2)], \\ &=\langle DD^\dagger\rangle\geqslant 0, \end{aligned}\quad (2.21\text{a})$$

where

$$D\equiv\tilde{\alpha}\int_0^t dt_1[S^{(0)}(t_1)-\langle S^{(0)}(t_1)\rangle] \times\exp[-\xi(t-t_1)-i\omega t_1]. \quad (2.21\text{b})$$

III. COMPARISON WITH CLASSICAL SOURCE

It is our purpose to compare the above fields to those generated by classical sources, and to find classical sources which produce equivalent fields. Since the physical meaning of the above fields (and sources) can only be stated in statistical terms, it is clear that the classical sources will also have to be described in statistical terms. Let us consider, therefore, a radiation mode driven by a weakly coupled classical source which is described statistically. The coordinates may be expressed as

$$q'=q_F+q_c,\quad p'=p_F+p_c, \quad (3.1)$$

where q_c and p_c are (stochastic) c-number terms of first order, and represent the contribution of the classical source. q_c and p_c must, of course, be described in terms of an ensemble, but this ensemble is a classical statistical ensemble and is unrelated to the quantum-mechanical ensemble with which q_F and p_F are described. Our expectation-value notation will refer to either or both ensembles, depending on the quantities which are being averaged. In accordance with the earlier discussion, we assume that the effect of the radiation mode under consideration on the classical source is negligible, and we regard the source as prescribed. One might argue that our classical source is more "prescribed" than the quantum-mechanical source, for which we not only considered $q^{(1)}$, $p^{(1)}$

(which correspond to q_c, p_c), but also $q^{(2)}$, $p^{(2)}$. In a sense, this is true. $q^{(2)}$ and $p^{(2)}$, describe the effect of the "vacuum" field on the source. In the description of the behavior of a classical source, the "vacuum" field (for $T=0$) may not and need not be considered, as discussed previously, but in the case of a quantum-mechanical source, the effect of the "vacuum" field on the source must be considered even in lowest order. For example, the spontaneous emission power radiated by a quantum-mechanical source cannot be calculated correctly without consideration of the "vacuum" field. (Note that $\langle\{A_F,A^{(2)}\}\rangle$ does not vanish, in general, and is of the same order of magnitude as $\langle A^{(1)2}\rangle$.)

We construct, next, the characteristic function of q' and p':

$$\begin{aligned}\varphi(\mu,\nu)&\equiv\langle\exp i(\mu q'+\nu p')\rangle\\&=\langle\exp i(\zeta a'+\zeta^* a^{\dagger\prime})\rangle\\&\equiv\tilde{\varphi}(\zeta,\zeta^*).\end{aligned}\tag{3.2}$$

It is clear that, up to second order, we have the relationship

$$\varphi(\mu,\nu)\equiv\tilde{\varphi}(\zeta,\zeta^*)=\langle e^{iA_F}\rangle(1+i\langle A_c\rangle-\tfrac{1}{2}\langle A_c^2\rangle),\tag{3.3}$$

where

$$A_c\equiv\mu q_c+\nu p_c\equiv\zeta a_c+\zeta^* a_c^*.$$

If we now choose A_c in such a manner that

$$\langle A_c\rangle=\langle A^{(1)}\rangle,\tag{3.4a}$$

and

$$\langle A_c^2\rangle=\langle A^{(1)2}\rangle+\langle\{A_F,A^{(2)}\}\rangle\tag{3.4b}$$

[the double-frequency terms are considered dropped from the left side as well as from the right side of Eq. (3.4b); henceforth, double-frequency terms will be neglected in all second-order quantities], then the two characteristic functions are identical within our approximation scheme. In order that Eqs. (3.4) have (classical) statistical meaning as far as A_c is concerned, we must have

$$\langle A_c^2\rangle\geqslant\langle A_c\rangle^2.\tag{3.5}$$

This inequality is assured by Eq. (2.21). Since (without the double-frequency terms)

$$\langle A_c^2\rangle=2|\zeta|^2\langle a_c a_c^*\rangle,\tag{3.6a}$$

and

$$\begin{aligned}\langle A^{(1)2}\rangle+\langle\{A_F,A^{(2)}\}\rangle=|\zeta|^2[\langle\{a^{(1)},a^{\dagger(1)}\}\rangle\\+\langle a_F a^{\dagger(2)}\rangle+\langle a^{(2)}a_F^\dagger\rangle]=2|\zeta|^2\langle a^{\dagger(1)}a^{(1)}\rangle\end{aligned}\tag{3.6b}$$

from Eq. (2.6), we can see that Eqs. (3.4) are equivalent to

$$\langle a_c\rangle=\langle a^{(1)}\rangle,\tag{3.7a}$$

$$\langle a_c a_c^*\rangle=\langle a^{\dagger(1)}a^{(1)}\rangle.\tag{3.7b}$$

(Note that a_c and a_c^* commute, but $a^{(1)}$ and $a^{\dagger(1)}$ do not.) The physical interpretation of Eqs. (3.7) is very simple: The complex field amplitudes and the energy contributions must have the same expectation values for both sources at any given time. If the classical variable corresponding to S is designed by S_c (a real c number), we have

$$a_c=\frac{\alpha}{\sqrt{2}}\int_0^t dt_1 e^{-i\Omega(t-t_1)}S_c(t_1).\tag{3.8}$$

Equation (3.7a) requires that

$$\int_0^t dt_1 e^{-i\Omega(t-t_1)}\langle S_c(t_1)\rangle=\int_0^t dt_1 e^{-i\Omega(t-t_1)}\langle S^{(0)}(t_1)\rangle,\tag{3.9}$$

and Eq. (3.7b) requires that

$$\begin{aligned}&\int_0^t dt_1\int_0^t dt_2 e^{-\xi(2t-t_1-t_2)}\langle S_c(t_1)S_c(t_2)\rangle e^{-i\omega(t_1-t_2)}\\&=\int_0^t dt_1\int_0^t dt_2 e^{-\xi(2t-t_1-t_2)}\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle e^{-i\omega(t_1-t_2)}.\end{aligned}\tag{3.10}$$

Equations (3.9) and (3.10) are the only conditions that need be imposed on the specification of S_c, and, if they are satisfied, the fields produced by S and by S_c are equivalent (within our approximation framework). If we expand the real function $\langle S^{0)}(t_1)\rangle$ in the manner

$$\langle S^{(0)}(t_1)\rangle=\textstyle\sum_j(s_j e^{i\omega_j t_1}+s_j^* e^{-i\omega_j t_1}),\tag{3.11}$$

where the summation may be replaced by an integration, if necessary, it becomes clear that the requirements of Eq. (3.9) are met by

$$\langle S_c(t_1)\rangle=\textstyle\sum_j{}'(s_j e^{i\omega_j t_1}+s_j^* e^{-i\omega_j t_1}),\tag{3.12}$$

where the primed summation need be extended, for $t\gg\omega^{-1}$, only over those frequencies which lie in the neighborhood of ω. This neighborhood can be defined by

$$\left(1-\frac{\xi}{\omega}\right)\leqslant\frac{\omega_j}{\omega}\leqslant\left(1+\frac{\xi}{\omega}\right).\tag{3.13}$$

It might be said that Eq. (3.9) requires that $\langle S_c(t_1)\rangle=\langle S^{(0)}(t_1)\rangle$ in a narrow frequency range about ω. Equation (3.10), however, should not be interpreted as imposing a similar relationship between $\langle S_c(t_1)S_c(t_2)\rangle$ and $\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle$. In fact, $\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle$ is complex, in general [although the integral of Eq. (3.10) is real], while $\langle S_c(t_1)S_c(t_2)\rangle$ is real.

The requirement on $\langle S_c(t_1)S_c(t_2)\rangle$ imposed by Eq. (3.10) may be obtained from the following consideration. Since $S^{(0)}(t)$ is Hermitian, $\langle\{S^{(0)}(t_1),S^{(0)}(t_2)\}\rangle$ is real and $\langle[S^{(0)}(t_1),S^{(0)}(t_2)]\rangle$ is pure imaginary. Furthermore, the symmetrized product is even in t_1-t_2 while the commutator is odd. We can therefore expand these expressions in the form

$$\begin{aligned}&\langle\{S^{(0)}(t_1),S^{(0)}(t_2)\}\rangle\\&\qquad=\textstyle\sum_j G_j^{(+)}(e^{i\omega_j(t_1-t_2)}+e^{-i\omega_j(t_1-t_2)}),\end{aligned}\tag{3.14a}$$

and

$$\langle [S^{(0)}(t_1),S^{(0)}(t_2)]\rangle = \sum_j G_j^{(-)}(e^{i\omega_j(t_1-t_2)}-e^{-i\omega_j(t_1-t_2)}), \quad (3.14b)$$

where the G_j's are real, the ω_j's are a set of (positive) frequencies in which both the symmetrized product and commutator can be expanded, and where the summation may be replaced by an integration if necessary. In general, the G_j's are functions of t_1+t_2, but there will always exist terms in which the G_j's are constant, as shown in Appendix F. Since $2\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle$ is the sum of the symmetrized product and the commutator, we can write

$$\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle = \sum_j [\sigma_j^{(+)}e^{i\omega_j(t_1-t_2)}+\sigma_j^{(-)}e^{-i\omega_j(t_1-t_2)}], \quad (3.15)$$

where $\sigma_j^{(+)}$ and $\sigma_j^{(-)}$ are real quantities given by

$$\sigma_j^{(+)}=\tfrac{1}{2}(G^{(+)}+G^{(-)}), \quad \sigma_j^{(+)}=\tfrac{1}{2}(G^{(+)}-G^{(-)}). \quad (3.16)$$

It is clear that, for $t\gg\omega^{-1}$, the $\sigma_j^{(-)}$ terms make no significant contribution to the integral on the right side of Eq. (3.10), the main contribution coming from the constant (or approximately constant) $\sigma_j^{(+)}$ terms for which ω_j is sufficiently close to ω. Now, $\langle S_c(t_1)S_c(t_2)\rangle$ is a real quantity, and is even with respect to t_1-t_2. We must therefore have

$$\langle S_c(t_1)S_c(t_2)\rangle = \sum_j{}'' \sigma_j^{(+)}[e^{i\omega_j(t_1-t_2)}+e^{-i\omega_j(t_1-t_2)}], \quad (3.17)$$

where the double prime indicates that only the terms with $\sigma_j^{(+)}$ approximately constant and ω_j near ω need be included in the summation.[11] Thus, the characteristic function for the field in presence of the classical source is the same (in accordance with our approximation scheme) as that in presence of the quantum-mechanical source, provided Eqs. (3.12) and (3.17) are satisfied. These two equations may be regarded as describing the equivalent classical source.[12] It is seen that the only information needed about the quantum-mechanical source is $\langle S^{(0)}(t)\rangle$ and $\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle$ (which furnish s_j and $\sigma_j^{(+)}$).

Before we go on to specific illustrations, it is of interest to examine the approximation of discarding double-frequency terms. Such an approximation makes the computations simpler, of course, but is it significant from a physical viewpoint? It should be pointed out, first of all, that if we were to ask for a classical source that yields the same expectation values for the (complex) amplitude and energy of the field, then the above results would be obtained without reference to double-frequency terms, since neither the amplitude nor the energy contain such terms for sources near resonance. Equations (3.7), the equations which essentially define the equivalent classical source, are, in fact, the equality conditions for instantaneous amplitude and energy expectation values, as mentioned previously. As far as the equality of characteristic functions is concerned, however, the situation is somewhat different. It is best illustrated by a calculation for a simple quantum-mechanical source in which the double-frequency terms are retained, which follows.

We consider a two-level system with the matrix elements of $S^{(0)}(t)$ given by

$$S_{12}^{(0)}=S_{21}^{(0)*}=e^{-i\omega t}, \quad S_{11}^{(0)}=S_{22}^{(0)}=0, \quad (3.18)$$

and the state of the system specified by the density matrix ρ. This gives us

$$\langle S^{(0)}(t)\rangle=\rho_{12}e^{i\omega t}+\rho_{21}e^{-i\omega t}, \quad (3.19)$$

and

$$\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle=\rho_{11}e^{-i\omega(t_1-t_2)}+\rho_{22}e^{i\omega(t_1-t_2)}. \quad (3.20)$$

Substituting into Eq. (2.16) and retaining in the integrand only terms which do not oscillate with respect to the variable of integration (resonant terms), we obtain

$$\langle A^{(1)}\rangle^2=(\tilde{\alpha}/\xi)^2|\rho_{12}|^2(1-e^{-\xi t})^2 \times[1+\cos 2(\omega t+\theta)], \quad (3.21)$$

where θ is determined by the phases of ρ_{12} and ζ. Similarly, we obtain from Eq. (2.17)

$$\langle A^{(1)2}\rangle+\langle\{A_F,A^{(2)}\}\rangle=(\tilde{\alpha}/\xi)^2\rho_{22}(1-e^{-\xi t})^2. \quad (3.22)$$

Now, the equivalence of the classical and the quantum-mechanical source is obtained by setting $\langle A_c\rangle=\langle A^{(1)}\rangle$ and $\langle A_c^2\rangle=\langle A^{(1)2}\rangle+\langle\{A_F,A^{(2)}\}\rangle$. As pointed out previously, this is statistically meaningful, that is, the classical source can be described statistically, only if $\langle A_c^2\rangle\geq\langle A_c\rangle^2$. In the present instance, we have

$$\langle A^{(1)2}\rangle+\langle\{A_F,A^{(2)}\}\rangle-\langle A^{(1)}\rangle^2=(\tilde{\alpha}/\xi)^2(1-e^{-\xi t})^2 \times[\rho_{22}-|\rho_{12}|^2(1+\cos 2\omega t)], \quad (3.23)$$

where we have set $\theta=0$ for simplicity. Now, the inequality $\rho_{22}\geq|\rho_{12}|^2$ always holds, but the inequality $\rho_{22}\geq 2|\rho_{12}|$ does not always hold. Thus, Eqs. (3.4) will give statistically meaningful results for all states (pure states as well as mixtures) only if we consider Eq. (3.4b) to be a relationship for quantities averaged over a half cycle. (This procedure amounts to dropping the double-frequency terms. There are no higher frequency

[11] In the present notation,

$$\langle a^{(1)}\rangle\approx 2^{-1/2}(\alpha/\xi)\sum_j{}' s_j[1+i(\omega_j-\omega)/\xi] \times e^{-i\omega_j t}(1-e^{-\xi t+i(\omega_j-\omega)t}),$$

and

$$\langle a^{\dagger(1)}a^{(1)}\rangle\approx\tfrac{1}{2}(\alpha^2/\xi^2)\sum_j{}''\sigma_j^{(+)}[1-2e^{-\xi t}\cos(\omega_j-\omega)t+e^{-2\xi t}],$$

where $(\omega_j-\omega)^2/\xi^2$ has been neglected compared to unity, and where a possible small time variation in the $\sigma_j^{(+)}$'s has been ignored.

[12] It should be noted that not *all* the statistical properties of the equivalent classical source are determined by Eqs. (3.12) and (3.17), but only the first two moment functions. Statistical distributions satisfying these two equations will, therefore not be unique, in general.

terms.) In other words, we set the requirement on the classical source to be

$$\langle A_c\rangle=\langle A^{(1)}\rangle, \tag{3.24}$$

$$\langle A_c{}^2\rangle_{\mathrm{av}}=\langle A^{(1)2}+\{A_F,A^{(2)}\}\rangle_{\mathrm{av}}, \tag{3.25}$$

where the average is defined by

$$\langle X(t)\rangle_{\mathrm{av}}=\frac{\omega}{\pi}\int_{-\pi/2\omega}^{\pi/2\omega} d\tau\langle X(t+\tau)\rangle. \tag{3.26}$$

In the present example, averaging on the right side of Eq. (3.25) has no effect, of course, since the expression contains no double-frequency terms. If $\langle A_c\rangle$ is not zero, however, $\langle A_c{}^2\rangle$ will contain double-frequency terms as well as a nonoscillating term. It is only the latter that is prescribed by $\langle A^{(1)2}+\{A_F,A^{(2)}\}\rangle$. The manner in which this occurs will be shown explicitly in the next section.

The neglect of the double-frequency terms may be built into the equivalence condition for the classical and quantum-mechanical sources by approximating our characteristic functions for both sources with an average over a half cycle. Thus, the equivalence requirement in terms of the characteristic function may be stated as

$$\langle \exp i(\mu q+\nu p)\rangle_{\mathrm{av}}=\langle \exp i(\mu q'+\nu p')\rangle_{\mathrm{av}}, \tag{3.27}$$

according to the definition of Eq. (3.26). It is obvious that the relationships among slowly varying and among single-frequency terms is essentially unaffected by this averaging.

IV. ILLUSTRATIONS

The application of the preceding theory will be illustrated by the consideration of three types of sources: (1) a system with only a single pair of energy levels in resonance with the mode; (2) a harmonic oscillator; (3) a blackbody.

1. Source with Single Resonant Energy Interval

Let the state of the system be described by the density matrix ρ. The matrix elements of $S^{(0)}$ are given by

$$S_{jk}{}^{(0)}(t)=S_{jk}e^{i\omega_{jk}t},\quad \hbar\omega_{jk}=E_j-E_k. \tag{4.1}$$

Only a single pair of levels, E_a and E_b are related by

$$E_b-E_a=\hbar\omega, \tag{4.2}$$

all other frequencies falling outside the neighborhood of ω, as described by Eq. (3.13). Some calculation shows that, for use in Eqs. (3.12) and (3.17),

$$s_j=\rho_{ab}S_{ba}, \tag{4.3}$$

$$\sigma_j{}^{(+)}=\rho_{bb}|S_{ab}|^2. \tag{4.4}$$

Only the a and b levels enter into the result, and the effect of the source on the mode is identical to that of a two-level system with levels a and b. Equations (3.12) and (3.17) show that we can consider the classical system to be described by

$$\langle S_c(t)\rangle=\rho_{ab}S_{ba}e^{i\omega t}+\rho_{ba}S_{ab}e^{-i\omega t}, \tag{4.5}$$

and

$$\langle S_c(t_1)S_c(t_2)\rangle=2\rho_{bb}|S_{ab}|^2\cos\omega(t_1-t_2), \tag{4.6}$$

where, it should be recalled, double-frequency oscillations are neglected. Setting

$$\rho_{ab}S_{ba}=|\rho_{ab}S_{ba}|e^{i\theta} \tag{4.7}$$

we can rewrite Eq. (4.5) as

$$\langle S_c(t)\rangle=2|\rho_{ab}S_{ba}|\cos(\omega t+\theta). \tag{4.8}$$

If we take $S_c(t)$ to be given by

$$S_c(t)=B\cos(\omega t+\varphi), \tag{4.9}$$

where φ is a random variable with a probability distribution $P(\varphi)$, then it is easily seen that[12]

$$B=2\rho_{bb}{}^{1/2}|S_{ab}|, \tag{4.10}$$

and

$$P(\varphi)=\rho_{bb}{}^{-1/2}[|\rho_{ab}|\delta(\varphi-\theta)+(2\pi)^{-1}(\rho_{bb}{}^{1/2}-|\rho_{ab}|)], \tag{4.11}$$

will give the expectation values of Eqs. (4.8) and (4.6). [$P(\varphi)$ meets all the requirements of a probability distribution; it is positive, since $\rho_{bb}\geq|\rho_{ab}|^2$, and normalized.] We have thus displayed a classical, statistically described, sinusoidal oscillator which has essentially the same effect on the radiation mode as a quantum-mechanical system with a single resonant pair of levels.

2. Harmonic-Oscillator Source

We consider the source to be a lossless harmonic oscillator of frequency ω, with (dimensionless) co-ordinates Q_s and P_s. The variable $S(t)$ should be identified (except for an irrelevant constant which can be chosen to be unity) with $Q(t)$. There is, obviously, only one frequency, $\omega_j=\omega$, involved in the determination of s_j and $\sigma_j{}^{(+)}$, and the summations of Eqs. (3.12) and (3.17) need only single terms. It is easily seen that

$$s_j=\tfrac{1}{2}\langle Q_S(0)-iP_S(0)\rangle\equiv 2^{-1/2}\langle \mathcal{a}_S{}^\dagger(0)\rangle, \tag{4.12}$$

and

$$\begin{aligned}\sigma_j{}^{(+)}&=\tfrac{1}{4}\langle Q_S{}^2(0)+P_S{}^2(0)-1\rangle\\&=\tfrac{1}{2}\langle \mathcal{a}_S{}^\dagger(0)\mathcal{a}_S(0)\rangle.\end{aligned} \tag{4.13}$$

Consider now a classical oscillator

$$S_c(t)=2^{-1/2}(B^*e^{i\omega t}+Be^{-i\omega t}), \tag{4.14}$$

where B is a random complex variable. Equations (3.12), (3.17), (4.12), and (4.13) imply that

$$\langle B^*\rangle=\langle \mathcal{a}_S{}^\dagger(0)\rangle, \tag{4.15a}$$

$$\langle B^*B\rangle\ (=\langle BB^*\rangle)=\langle \mathcal{a}_S{}^\dagger(0)\mathcal{a}_S(0)\rangle. \tag{4.15b}$$

Since

$$\langle \mathcal{a}_S{}^\dagger(0)\mathcal{a}_S(0)\rangle\geqslant\langle \mathcal{a}_S{}^\dagger(0)\rangle\langle \mathcal{a}_S(0)\rangle, \tag{4.16}$$

Eqs. (4.15) describe a classical stochastic variable B. Thus, if we know the state or density matrix of the quantum-mechanical harmonic-oscillator source, the right sides of Eqs. (4.15) are known, and we can obtain a statistical description of the equivalent classical oscillator from the *classical* expectation values of Eqs. (4.15). As is to be expected, B vanishes with the excitation energy; the theory leads to a natural elimination of the zero-point energy in Eq. (4.13).

Although the problem of finding an equivalent classical oscillator has been reduced to a purely classical problem, it is of some interest to look at two simple types of quantum-mechanical states and find their classical equivalents. For an energy state $|n\rangle$ we have

$$\langle B^*\rangle = \langle n|\,\mathfrak{a}_S^\dagger(0)\,|n\rangle = 0\,, \tag{4.17a}$$

$$\langle BB^*\rangle = \langle n|\,\mathfrak{a}_S^\dagger(0)\mathfrak{a}_S(0)\,|n\rangle = n\,, \tag{4.17b}$$

which shows that the amplitude of the equivalent classical oscillator is given by

$$|B| = n^{1/2}, \tag{4.18}$$

and all phases are equally probable. For an oscillating wave packet—or "coherent"—state, defined by[13]

$$\mathfrak{a}_S(0)\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \tag{4.19}$$

where α is a complex number, we have

$$\langle B^*\rangle = \langle \alpha|\,\mathfrak{a}_S^\dagger(0)\,|\alpha\rangle = \alpha^*\,, \tag{4.20a}$$

$$\langle BB^*\rangle = \langle \alpha|\,\mathfrak{a}_S^\dagger(0)\mathfrak{a}_S(0)\,|\alpha\rangle = |\alpha|^2, \tag{4.20b}$$

which shows that the equivalent classical oscillator is described by

$$B = \alpha\,, \tag{4.21}$$

that is, it has a precisely defined phase as well as a precisely defined amplitude. In this case, all members of the (classical) statistical ensemble describing the source are identical, or, better yet, no statistical description of the source is necessary.

3. Blackbody Source

In the case of a blackbody, the source has the same properties as the LM, and $\alpha S(t)$ is similar to $F(t)$, except for the coupling strength.[14] If we replace $F(t)$ by $S(t)$ in Eqs. (1.4) [α drops out when $\tilde{F}^2$ in Eq. (1.4e) is replaced by S^2], we obtain

$$\langle S^{(0)}(t_1)\rangle = 0\,, \tag{4.22a}$$

$$\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle = \frac{1}{\pi}\int_0^\infty d\omega'\{[\tilde{\eta}(\omega') - \tilde{\xi}(\omega')] \times e^{i\omega'(t_1-t_2)} + [\tilde{\eta}(\omega') + \tilde{\xi}(\omega')]e^{-i\omega'(t_1-t_2)}\}\,, \tag{4.22b}$$

where $\tilde{\eta}(\omega')$ and $\tilde{\xi}(\omega')$ are now defined in terms of the source properties in the same manner as $\eta(\omega')$ and $\zeta(\omega')$ are defined by Eqs. (1.4b)–(1.4e) in terms of the LM properties. We have thus,

$$\langle S_c(t)\rangle = 0\,, \tag{4.23}$$

and, by comparing Eq. (4.22b) with Eqs. (3.15) and (3.17), we can write immediately

$$\begin{aligned}\langle S_c(t_1)S_c(t_2)\rangle &= \frac{2}{\pi}\int_0^\infty d\omega'[\tilde{\eta}(\omega') - \tilde{\xi}(\omega')]\cos\omega'(t_1-t_2)\,,\\ &= \frac{4}{\pi}\int_0^\infty d\omega'\frac{\tilde{\xi}(\omega')\cos\omega'(t_1-t_2)}{\exp(\hbar\omega'/kT)-1}\,,\end{aligned} \tag{4.24}$$

where the relationship between $\tilde{\xi}(\omega')$ and $\tilde{\eta}(\omega')$ given by Eq. (1.5) has been utilized. As far as the effect on the radiation oscillator is concerned, only the ω' interval in the neighborhood of ω is significant, and we also have

$$\langle S_c(t_1)S_c(t_2)\rangle \approx 4\tilde{\xi}(\omega)[\exp(\hbar\omega/kT)-1]^{-1}\delta(t_1-t_2)\,. \tag{4.25}$$

Equations (4.23) and (4.24) [or (4.25)] are just the equations for a "classical" blackbody that obeys Planck's radiation law.

V. CORRELATION FUNCTIONS

So far, we have studied the characteristic function of the field in the presence of a (arbitrary) quantum-mechanical source, compared it to the characteristic function in the presence of a classical source, and found classical sources which yielded approximately the same expression for the characteristic function as the quantum-mechanical sources. In other words, we have replaced the quantum-mechanical source by an approximately equivalent classical source. The approximations consisted in going only up to second order in perturbation theory, and in discarding double-frequency terms in the characteristic function (or in averaging over a half cycle). Now, the characteristic function involves the various moments of the field, which, in the notation of Eq. (2.9b), may be written as $\langle A^n\rangle$. As Eq. (2.15) shows, however, the contribution of the source, because of the weak coupling, is involved only in the factors $\langle A^{(1)}\rangle$, $\langle A^{(1)2}\rangle$, and $\langle\{A_F,A^{(2)}\}\rangle$. All other factors in $\langle A^n\rangle$ are due only to the "vacuum" field. We can express this fact by the statement that only $\langle A\rangle$ and $\langle A^2\rangle$ need concern us, and these only up to second order.

One should note that the moments of A are not the most general statistical expressions referring to A. They may be regarded as equal-time correlation functions of A, and special cases of the general correlation functions

[13] E. Schrödinger, Naturwiss. 14, 664 (1927); Julian Schwinger, Signal Corps Report, Contract No. SC64531, 1956 (unpublished); R. J. Glauber, Phys. Rev. 131, 2766 (1963).

[14] The perturbation theory applied to the source coupling is not of high enough order to affect the damping of the radiation oscillator. The source coupling must therefore be assumed to be considerably smaller than the LM coupling; sufficiently smaller, in fact, so as to play a negligible role in the damping.

$\langle A(t_1)\cdots A(t_n)\rangle$. Up to second order, the source will be involved only in the factors $\langle A^{(1)}(t_i)\rangle$, $\langle A^{(1)}(t_i)A^{(1)}(t_j)\rangle$, and $\langle A_F(t_i)A^{(2)}(t_j)+A^{(2)}(t_i)A_F(t_j)\rangle$, which is somewhat similar to the case of the moments. It is of interest, therefore, to investigate whether the equivalence of the classical and quantum-mechanical sources that holds for $\langle A^2(t)\rangle$ also holds for $\langle A(t)A(t+\tau)\rangle$.

Since the first-order part of $\langle A(t)A(t+\tau)\rangle$ vanishes, we have

$$\langle A(t)A(t+\tau)\rangle=\langle A(t)A(t+\tau)\rangle^{(0)}+\langle A(t)A(t+\tau)\rangle^{(2)}, \quad (5.1a)$$

where, neglecting double-frequency terms,

$$\langle A(t)A(t+\tau)\rangle^{(0)}=\langle A_F(t)A_F(t+\tau)\rangle=|\zeta|^2\langle a_F(t)a_F^\dagger(t+\tau)\rangle, \quad (5.1b)$$

and

$$\langle A(t)A(t+\tau)\rangle^{(2)}=|\zeta|^2[\langle a_F(t)a^{\dagger(2)}(t+\tau)\rangle+\langle a^{(2)}(t)a_F^\dagger(t+\tau)\rangle+\langle a^{(1)}(t)a^{\dagger(1)}(t+\tau)\rangle+\langle a^{\dagger(1)}(t)a^{(1)}(t+\tau)\rangle]. \quad (5.1c)$$

The computation is aided by the following decomposition:

$$a^{(1)}(t+\tau)=e^{-i\Omega\tau}a^{(1)}(t)+\frac{\alpha}{2^{1/2}}\int_t^{t+\tau}dt_1e^{-i\Omega(t+\tau-t_1)}S^{(0)}(t_1), \quad (5.2a)$$

$$a^{(2)}(t+\tau)=e^{-i\Omega\tau}a^{(2)}(t)-\tfrac{1}{2}\alpha^2\int_t^{t+\tau}dt_1\int_0^{t_1}dt_2\times e^{-i\Omega(t+\tau-t_1)}[S^{(0)}(t_1),S^{(0)}(t_2)]\times[a_F(t_2)-a_F^\dagger(t_2)], \quad (5.2b)$$

and the corresponding conjugate equations. Substituting from Eqs. (5.2) and utilizing Eq. (2.6), one obtains, after some calculation, the result,

$$\langle A(t)A(t+\tau)\rangle^{(2)}=C_1+C_2+C_3, \quad (5.3a)$$

where

$$C_1\equiv|\zeta|^2\langle a^{\dagger(1)}(t)a^{(1)}(t)\rangle(e^{i\Omega^*\tau}+e^{-i\Omega\tau})=\tilde{\alpha}^2\cos\omega\tau\int_0^t dt_1\int_0^t dt_2\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle\times\exp[-\xi(2t+\tau-t_1-t_2)-i\omega(t_1-t_2)], \quad (5.3b)$$

$$C_2\equiv\tilde{\alpha}^2\cos\omega\tau\int_t^{t+\tau}dt_1\int_0^t dt_2\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle\times\exp[-\xi(2t+\tau-t_1-t_2)-i\omega(t_1-t_2)], \quad (5.3c)$$

$$C_3\equiv\tfrac{1}{2}\tilde{\alpha}^2\int_t^{t+\tau}dt_1\int_t^{t_1}dt_2\langle[S^{(0)}(t_1),S^{(0)}(t_2)]\rangle\times\exp[i\Omega^*(\tau-t_1+t_2)], \quad (5.3d)$$

for $\tau\geq 0$. We have, of course, the relationship

$$\langle A(t+\tau)A(t)\rangle=\langle A(t)A(t+\tau)\rangle^*, \quad (5.4)$$

which can be used to obtain an expression for $\langle A(t)A(t-\tau)\rangle$. The symmetrized product and commutator are also of interest. C_1 is obviously real. The resonant contribution to C_2 (we ignore the other contributions), as shown by Eq. (3.15), is also real. The symmetrized product is therefore given by

$$\langle\{A(t),A(t+\tau)\}\rangle=2C_1+2C_2+2C_3', \quad (5.5a)$$

where

$$C_3'=\tfrac{1}{2}(C_3+C_3^*)=\tfrac{1}{4}\tilde{\alpha}^2\int_t^{t+\tau}dt_1\int_t^{t_1}dt_2\langle[S^{(0)}(t_1),S^{(0)}(t_2)]\rangle\times[e^{i\Omega^*(\tau-t_1+t_2)}-e^{-i\Omega(\tau-t_1+t_2)}], \quad (5.5b)$$

and the commutator is given by

$$\langle[A(t),A(t+\tau)]\rangle=C_3-C_3^*. \quad (5.6)$$

We consider now the correlation function of the field A' in the presence of a classical source. From

$$A'=A_F+A_c, \quad (5.7)$$

we obtain

$$\langle A'(t)A'(t+\tau)\rangle=\langle A_F(t)A_F(t+\tau)\rangle+\langle A_c(t)A_c(t+\tau)\rangle. \quad (5.8)$$

The relationship $A_c=\zeta a_c+\zeta^* a_c^*$ together with Eq. (3.8) and the neglect of double-frequency terms yields

$$\langle A_c(t)A_c(t+\tau)\rangle=K_1+K_2, \quad (5.9a)$$

where

$$K_1\equiv|\zeta|^2\langle a_c(t)a_c^*(t)\rangle(e^{i\Omega^*\tau}+e^{-i\Omega^*\tau}) \quad (5.9b)$$

and

$$K_2\equiv\tfrac{1}{2}\tilde{\alpha}^2\int_t^{t+\tau}dt_1\int_0^t dt_2\langle S_c(t_1)S_c(t_2)\rangle\times e^{-\xi(2t+\tau-t_1-t_2)}[e^{i\omega(\tau-t_1+t_2)}+e^{-i\omega(\tau-t_1+t_2)}]. \quad (5.9c)$$

Equation (3.7b) shows that

$$C_1=K_1. \quad (5.10)$$

Furthermore, Eqs. (3.15) and (3.17) show that the resonant contributions in the integrals of C_2 and K_2 are the same. This is particularly easy to see in the case of exact resonance, where both C_2 and K_2 are equal to

$$\tilde{\alpha}^2e^{-2\xi\tau}\cos\omega\tau\int_t^{t+\tau}dt_1\int_0^t dt_2e^{-\xi(2t-t_1-t_2)}\sigma^{(+)}. \quad (5.11)$$

Neglecting other than resonant contributions, we have, therefore,

$$C_2=K_2. \quad (5.12)$$

Thus, the only difference between $\langle A(t)A(t+\tau)\rangle$ and $\langle A'(t)A'(t+\tau)\rangle$, that is, between the correlation functions in the presence of the quantum-mechanical and the classical source, is the term C_3.

It is significant to look at the orders of magnitude of the terms C_1, C_2, and C_3. This can be done very simply in the case of a source with one frequency at resonance and no other frequencies near resonance, the case we will now consider. The essential aspects of the argument which follows also apply to the somewhat more complicated situation where there may be more than one frequency near resonance. Utilizing the expansion of Eq. (3.15), we obtain from Eqs. (5.3), with the usual approximations,

$$C_1=(\tilde{\alpha}^2/\xi^2)\sigma^{(+)}e^{-\xi\tau}(1-e^{-\xi t})^2\cos\omega\tau\,, \tag{5.13a}$$

$$C_2=(\tilde{\alpha}^2/\xi^2)\sigma^{(+)}(1-e^{-\xi\tau})(1-e^{-\xi t})\cos\omega\tau\,, \tag{5.13b}$$

$$C_3=\tfrac{1}{2}(\tilde{\alpha}^2/\xi^2)(\sigma^{(+)}-\sigma^{(-)})[1-e^{-\xi\tau}(1+\xi\tau)]e^{i\omega\tau}, \tag{5.13c}$$

where the index has been dropped from the σ's for obvious reasons. Now, the physical significance of a product such as $A(t)A(t+\tau)$ (or its symmetrized version) lies in the fact that it occurs in the expression for an elementary detection processes of the field. Thus, the lowest-order expression for the rate at which a number of atoms absorb energy from the field (which might be considered the description of an idealized detection process) depends on the field through the term

$$\int_{-\infty}^{t} dt_1 f_D(t,t_1)\langle\{p(t),p(t_1)\}\rangle\,, \tag{5.14}$$

where $f_D(t,t_1)$ is determined by the atoms.[15] A physically meaningful measuring device needs a response time short compared to ξ^{-1}, which implies that $f_D(t,t_1)$ should be such that the significant contribution to the above integral comes from the values of t_1 for which $\xi(t-t_1)\ll 1$. The important values of τ, therefore, are those for which $\xi\tau\ll 1$, and in this range we can write

$$C_1\approx(\tilde{\alpha}^2/\xi^2)\sigma^{(+)}(1-e^{-\xi t})^2\cos\omega\tau\,, \tag{5.15a}$$

$$C_2\approx\xi\tau(\tilde{\alpha}^2/\xi^2)\sigma^{(+)}(1-e^{-\xi t})\cos\omega\tau\,, \tag{5.15b}$$

$$C_3\approx\tfrac{1}{4}(\xi\tau)^2(\tilde{\alpha}^2/\xi^2)(\sigma^{(+)}-\sigma^{(-)})e^{i\omega\tau}. \tag{5.15c}$$

It is seen that the two correlation functions, $\langle A(t)A(t+\tau)\rangle$ and $\langle A'(t)A'(t+\tau)\rangle$, differ only by a term of second order in $\xi\tau$, a difference that is negligible as far as field detection processes are concerned.

It is instructive, nevertheless, to examine the physical origin of the difference of the two correlation functions, C_3 (or, rather, C_3', since a classical product should really be compared to a symmetrized quantum-mechanical product), which we proceed to do as follows: Consider a classical field which is suddenly generated at time t_0 and is allowed to decay freely in presence of the quantum-mechanical source that has been under consideration all along. This field is described in zeroth order (that is, in the absence of coupling to the source) by

$$\begin{aligned}\tilde{\alpha}(t)&=\tilde{\alpha}(t_0)e^{-i\Omega(t-t_0)}, && t\geqslant t_0\\ \tilde{\alpha}(t)&=0, && t<t_0.\end{aligned} \tag{5.16}$$

We take $\tilde{\alpha}(t_0)$ to be a random number with all phases having equal probability. Let us now derive an expression for that part of $\langle A(t_0)A(t_0+\tau)\rangle$ which is due to the interaction between $\tilde{\alpha}(t)$ and the source. [The generation of $\tilde{\alpha}(t_0)$ is assumed to be independent of the source under consideration.] This will be a second-order term given by $\langle\tilde{A}(t_0)\tilde{A}^{(2)}(t_0+\tau)\rangle$, where $\tilde{A}^{(2)}$ is determined by

$$\tilde{\alpha}^{(2)}(t_0+\tau)=-\tfrac{1}{2}\alpha^2\int_{t_0}^{t_0+\tau}dt_1\int_{t_0}^{t_1}dt_2 \times e^{-i\Omega(t_0+\tau-t_1)}[S^{(0)}(t_1),S^{(0)}(t_2)][\tilde{\alpha}(t_2)-\tilde{\alpha}^*(t_2)], \tag{5.17}$$

[from Eq. (2.5e)] and the corresponding conjugate equation. Substitution yields

$$\begin{aligned}\langle\tilde{A}(t_0)\tilde{A}^{(2)}(t_0+\tau)\rangle=&-\tfrac{1}{2}\tilde{\alpha}^2\int_{t_0}^{t_0+\tau}dt_1\int_{t_0}^{t_1}dt_2\\ &\times\langle[S^{(0)}(t_1),S^{(0)}(t_2)]\rangle[\zeta e^{-i\Omega(t_0+\tau-t_1)}\\ &+\zeta^*e^{i\Omega^*(t_0+\tau-t_1)}]\langle[\zeta\tilde{\alpha}(t_0)+\zeta^*\tilde{\alpha}^*(t_0)]\\ &\times[\tilde{\alpha}(t_0)e^{-i\Omega(t_2-t_0)}-\tilde{\alpha}^*(t_0)e^{i\Omega^*(t_2-t_0)}]\rangle,\end{aligned} \tag{5.18}$$

where the last expectation value calls for the averaging over the $\tilde{\alpha}$'s. Due to the randomness of the phases, the squares of $\tilde{\alpha}$ and $\tilde{\alpha}^*$ drop out. Furthermore, noting from Eq. (3.14b) that $\langle[S^{(0)}(t_1),S^{(0)}(t_2)]\rangle$ involves the exponentials of $\pm i\omega(t_1-t_2)$, we retain only the same exponentials of the remaining factors of the integrand [that is, we drop the exponentials of $\pm i\omega(t_1+t_2)$], for only these will give a resonant contribution to the integral. The result is

$$\begin{aligned}\langle\tilde{A}(t_0)\tilde{A}^{(2)}(t_0+\tau)\rangle&=\tfrac{1}{2}\langle\{\tilde{A}(t_0),\tilde{A}^{(2)}(t_0+\tau)\}\rangle\\ &=\tfrac{1}{2}\tilde{\alpha}^2\langle\tilde{\alpha}(t_0)\tilde{\alpha}^*(t_0)\rangle\int_{t_0}^{t_0+\tau}dt_1\int_{t_0}^{t_1}dt_2\langle[S^{(0)}(t_1),S^{(0)}(t_2)]\rangle\\ &\qquad\times[e^{i\Omega^*(\tau-t_1+t_2)}-e^{-i\Omega(\tau-t_1+t_2)}].\end{aligned} \tag{5.19}$$

We compare, now, the expression of Eq. (5.19) with that of C_3' in Eq. (5.5). We note that the two are identical if we identify t_0 with t, provided

$$\langle\tilde{\alpha}(t_0)\tilde{\alpha}^*(t_0)\rangle=\tfrac{1}{2}. \tag{5.20}$$

Now, $\tilde{\alpha}\tilde{\alpha}^*$ is the energy in units of $\hbar\omega$ of the classical field described by $\tilde{\alpha}$. Furthermore, an energy of $\frac{1}{2}\hbar\omega$ is just the minimum energy which may be associated with

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[15] See, for instance, I. R. Senitzky, Phys. Rev. **119**, 1807 (1960), Eqs. (72) and (21) ($P^{[0]}$ in these equations has the meaning of the present p as far as the detection process is concerned), which show that $f_D(t,t_1)$ has the form $\sum_m c_m\cos\omega_m(t-t_1)$, where ω_m is the atomic frequency associated with the absorption by the mth atom.

the disturbance of the (quantum-mechanical) field produced by a measurement at t_0. We can therefore interpret the difference between the expression for $\langle\{A(t),A(t+\tau)\}\rangle$ and $\langle\{A'(t),A'(t+\tau)\}\rangle$ with the statement that the quantum-mechanical source responds to the disturbance of the field produced by a measurement at t—this response accounting for C_3'—while the classical source does not. In other words, the difference between the two correlation functions may be regarded as being due to a disturbance of the source by an ideal measurement of the field. Since the field is treated quantum-mechanically, the disturbance of the field by a measurement is unavoidable, of course. Furthermore, the absence of a response of the classical source to the field is to be expected, since it was built into the specification of the source in the present treatment, in accordance with our earlier discussion.

VI. FREE FIELDS

If one wants to take advantage of the simplicity afforded by the discussion of a single mode, consideration of a lossless field is less satisfactory, from a physical viewpoint, than that of a damped field. In any real situation, losses usually play an important role. Also, the choice of initial conditions is not clearly indicated in the lossless case, since any disturbance produced in the remote past will remain in existence indefinitely. Furthermore, the consideration of steady-state resonant sources is impossible, since these will produce an indefinite increase in field strength, a completely unrealistic situation. As a matter of fact, most of the discussion, in the literature, of the lossless field in quantum optics leaves sources entirely out of consideration, and treats various states of the free field without reference to their generation.

In this connection it is worth noting that the field of a free mode, uncoupled from both sources and LM, cannot exhibit properties which are frequently of interest in optics, no matter what the state of the radiation oscillator may be. Thus, (random) fluctuations in time, which are often referred to as noise or incoherence, can be exhibited by the oscillator only while it is under the influence of external systems. In order to describe a free mode, we set α and F equal to zero in Eqs. (1.8a) and (1.8b) to obtain

$$q(t)_{\text{free}}=q(t_0)\cos\omega(t-t_0)+p(t_0)\sin\omega(t-t_0)\,, \tag{6.1a}$$

$$p(t)_{\text{free}}=-q(t_0)\sin\omega(t-t_0)+p(t_0)\cos\omega(t-t_0)\,, \tag{6.1b}$$

which leads to the relationship

$$\langle\{q(t)_{\text{free}},q(t+\tau)_{\text{free}}\}\rangle_{\text{av}}=\tfrac{1}{2}\langle q^2(t_0)+p^2(t_0)\rangle\cos\omega\tau\,, \tag{6.2}$$

where the averaging in accordance with Eq. (3.26) merely removes double-frequency terms. Now, this is a correlation function for a purely sinusoidal oscillation, regardless of the state of the free field. On the other hand, the corresponding expression for a damped mode in equilibrium with the LM at temperature T is, from Eq. (1.21a),

$$\langle\{q_F(t),q_F(t+\tau)\}\rangle =2\{[\exp(\hbar\omega/kT)-1]^{-1}+\tfrac{1}{2}\}e^{-\xi\tau}\cos\omega\tau\,, \tag{6.3}$$

just the type of correlation function expected in the presence of Gaussian noise.

The description of the field of an undamped mode coupled to a source can be obtained immediately from the present results by taking the limit as ξ vanishes, provided the mode is initially unexcited and the generated field does not become sufficiently large to react significantly back on the source. q_F and p_F then describe the true vacuum field. The arguments concerning the equivalent classical source remain unchanged. It is also possible to obtain a free (excited) field in this case by terminating the coupling to the source at some time t_0, with the field being specified from then on by Eqs. (6.1). It is clear from these equations that the free field is described by the superposition of the true vacuum field and a classical field.

Another, and perhaps preferable, method of approaching the idealization of a free field is to view it as one that is very weakly coupled to a source and very weakly damped. We let α and ξ become very small (but not zero) in such a manner that α/ξ remains constant. After the source has been acting for a sufficiently long time, the field is *almost* the same as an excited free field. The arguments of the present article apply, of course, with q_F and p_F having properties that are almost those of the true vacuum field.

VII. CONCLUSION

The present analysis may be regarded as a discussion of spontaneous emission in a general sense. Only the lowest-order processes are involved in the weak interaction between the source and the radiation mode under study, but the source may be arbitrarily complex and be coupled strongly to other radiation modes, so that high-order interactions play an important role in its behavior. The above results show that spontaneous emission from a quantum-mechanical system may be described classically, within a reasonable approximation framework, the total field being the superposition of a classical field due to the source and the "vacuum" field. The total field is, of course, fully quantum-mechanical, since the "vacuum" field furnishes the necessary quantum mechanical properties.[16] The classical field may be regarded as being generated by a classical source and a description of this equaivalent source is obtained from Eqs. (3.12) and (3.17); the only information needed about the quantum-mechanical source are the values of

[16] It is clear that if one considers expressions in which the "vacuum" field makes no contribution—such as those for (lowest-order) induced emission and absorption—then only the classical field need be considered.

$\langle S^{(0)}(t)\rangle$ and $\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle$, quantities that are independent of the interaction between the source and the mode under study. The description of the classical source (as well as the classical field) will, in general, be statistical.

APPENDIX A

The theorem of Eq. (1.6), which states that

$$\langle 1\ 2 \cdots (n-1)\ [n, n+1]\ (n+2) \cdots N\rangle = \langle [n, n+1]\rangle\langle 1\ 2 \cdots (n-1)\ (n+2) \cdots N\rangle \quad \text{(A1)}$$

will be proved. We assume N to be even, since for odd N both sides vanish, according to Eq. (1.3a). Equation (1.3b) may be written, in the shorthand notation, as

$$\langle 1\ 2 \cdots N\rangle = \sum \langle j_1j_2\rangle\langle j_3j_4\rangle\cdots\langle j_{N-1}j_N\rangle, \quad \text{(A2)}$$

where the summation is over all the different arrangements $j_1j_2,\cdots j_{2k-1}j_{2k},\ \cdots j_{N-1}j_N$, into pairs, with $j_{2k-1}<j_{2k}$. The order in which the pairs themselves are arranged is immaterial. Two arrangements are considered different only if at least one pair in one arrangement is different from any pair in the other arrangement; all numbers from 1 to N must occur once, and only once, in each arrangement. The requirement $j_{2k-1}<j_{2k}$ provides that the order of the two numbers in each pair be the same as the order of these two numbers in the expectation value $\langle 12\cdots k\cdots N\rangle$.

Consider now the two expectation values

$$X_1 \equiv \langle 1\ 2\cdots(n-1)\ n\ (n+1)\ (n+2)\cdots N\rangle, \quad \text{(A3)}$$

and

$$X_2 \equiv \langle 1\ 2\cdots(n-1)\ (n+1)\ n\ (n+2)\cdots N\rangle, \quad \text{(A4)}$$

and let them be expanded according to Eq. (A2). The only terms in one sum that will be different from the terms in the other sum are those containing the pairs $\langle n(n+1)\rangle$ and $\langle(n+1)n\rangle$. The sum corresponding to X_1 will contain terms with the former and the sum corresponding to X_2 will contain terms with the latter. Thus,

$$X_1-X_2 = [\langle n(n+1)\rangle - \langle(n+1)n\rangle] \times \sum{}' \langle j_1j_2\rangle\cdots\langle j_{N-1}j_N\rangle, \quad \text{(A5)}$$

where the primed summation indicates a summation over all different arrangements into pairs with n and $n+1$ omitted. Since

$$\sum{}' \langle j_1j_2\rangle\cdots\langle j_{N-1}j_N\rangle = \langle 1\ 2\cdots(n-1)\ (n+2)\cdots N\rangle, \quad \text{(A6)}$$

the relationship in Eq. (A1) is proved.

APPENDIX B

The derivation of the relationship

$$\int_{t_2}^{t} dt_1 U(t-t_1)[F^{(0)}(t_1),[F^{(0)}(t_2),H_{\mathrm{LM}}^{(0)}]]U^{-1}(t-t_1) = -i\hbar[F^{(0)}(t),F^{(0)}(t_2)],$$

where

$$U(\tau)\exp[(i/\hbar)H_{\mathrm{LM}}(0)\tau], \quad \text{(B1)}$$

is presented in this Appendix. The Hamiltonian for the uncoupled LM, $H_{\mathrm{LM}}^{(0)}$, is identical with $H_{\mathrm{LM}}(0)$. (The coupling is assumed to begin at $t=0$). The double commutator under the integral sign is an operator referring to the uncoupled LM only. $U(t-t_1)$ is therefore the time displacement operator for the double commutator which adds $t-t_1$ to all the time arguments. Thus,

$$\begin{aligned}
&\int_{t_2}^{t} dt_1 U(t-t_1)[F^{(0)}(t_1),[F^{(0)}(t_2),H_{\mathrm{LM}}^{(0)}]]U^{-1}(t-t_1) \\
&= \int_{t_2}^{t} dt_1[F^{(0)}(t),[F^{(0)}(t+t_2-t_1),H_{\mathrm{LM}}^{(0)}]] \\
&= i\hbar\int_{t_2}^{t} dt_1[F^{(0)}(t),\dot{F}^{(0)}(t+t_2-t_1)] \\
&= -i\hbar[F^{(0)}(t),F^{(0)}(t_2)-F^{(0)}(t)] \\
&= -i\hbar[F^{(0)}(t),F^{(0)}(t_2)]. \qquad \text{(B2)}
\end{aligned}$$

APPENDIX C

We evaluate the kernel

$$K(t,t_1) = \frac{4}{\pi}\int_0^\infty d\omega'\xi(\omega')\int_{t_1}^{t} dt_2 \times \sin\omega(t-t_2)\,\sin\omega'(t_2-t_1), \quad \text{(C1)}$$

in the integral equation (1.16a). Making use of the relationship

$$\begin{aligned}
&\int_{t_1}^{t} dt_2\,\sin\omega(t-t_2)\,\sin\omega'(t_2-t_1) \\
&= (\omega'+\omega)^{-1}[\tfrac{1}{2}\sin(\omega'+\omega)\tau\,\cos\omega\tau \\
&\quad +\sin^2\tfrac{1}{2}(\omega'+\omega)\tau\,\sin\omega\tau] - (\omega'-\omega)^{-1} \\
&\quad \times[\tfrac{1}{2}\sin(\omega'-\omega)\tau\,\cos\omega\tau - \sin^2\tfrac{1}{2}(\omega'-\omega)\tau\,\sin\omega\tau],
\end{aligned} \quad \text{(C2)}$$

where $\tau\equiv t-t_1$, we can write

$$K(t,t_1) = K_1\cos\omega(t-t_1) + K_2\sin\omega(t-t_1), \quad \text{(C3)}$$

where

$$K_1 = \frac{2}{\pi}\int_0^\infty d\omega'\xi(\omega') \times\left[\frac{\sin(\omega'+\omega)(t-t_1)}{\omega'+\omega} - \frac{\sin(\omega'-\omega)(t-t_1)}{\omega'-\omega}\right] \quad \text{(C4a)}$$

and

$$K_2=\frac{2}{\pi}\int_0^\infty d\omega'\xi(\omega')\left[\frac{1-\cos(\omega'+\omega)(t-t_1)}{\omega'+\omega}+\frac{1-\cos(\omega'-\omega)(t-t_1)}{\omega'-\omega}\right]. \quad \text{(C4b)}$$

For $t-t_1 \gg \omega^{-1}$ (which implies that we are considering $t \gg \omega^{-1}$) K_1 and K_2 can be easily approximated. In the expression for K_1 the contribution of the first term in the square bracket is negligible, and the contribution of the second term lies mainly in the neighborhood of ω. Since $\xi(\omega')$ is a slowly varying function, we have

$$K_1\approx\frac{2}{\pi}\xi(\omega)\int_{-\infty}^\infty d\omega'\frac{\sin(\omega'-\omega)(t-t_1)}{\omega'-\omega}=-2\xi(\omega). \quad \text{(C5)}$$

A good approximation for K_2 is given by

$$K_2\approx\frac{2}{\pi}\int_0^\infty d\omega'\xi(\omega')\left[\frac{1}{\omega'+\omega}+\frac{\mathcal{P}}{\omega'-\omega}\right]=2\epsilon(\omega), \quad \text{(C6)}$$

where

$$\epsilon(\omega)\equiv\frac{2}{\pi}\mathcal{P}\int_0^\infty d\omega'\xi(\omega')\frac{\omega'}{\omega'^2-\omega}. \quad \text{(C7)}$$

We obtain, thus,

$$K(t,t_1)\approx-2\xi(\omega)\cos\omega(t-t_1)+2\epsilon(\omega)\sin\omega(t-t_1). \quad \text{(C8)}$$

The important assumption underlying the use of Eq. (C8) is that the main contribution to the integral in the integral equation (1.16a) comes from values of t_1 such that $t-t_1\gg\omega^{-1}$. Since the kernel oscillates with frequency ω, this assumption is justified only if the significant frequencies in $p(t_1)$ lie near ω. Anticipating the solution of the integral equation [or of the equivalent differential equation (1.18)], we can say that the significant frequencies will lie near ω when resonance effects play an important role, or when the damping constant $\xi(\omega)$ is sufficiently small compared to ω.

APPENDIX D

We consider the derivation of Eqs. (1.21). From Eqs. (1.20) one has

$$\langle q(\tau_1)q(\tau_2)\rangle=\int_{-\infty}^{\tau_1}dt_1\int_{-\infty}^{\tau_2}dt_2\exp[-\xi(\tau_1+\tau_2-t_1-t_2)]\times\cos\omega(\tau_1-t_1)\cos\omega(\tau_2-t_2)\langle F^{(0)}(t_1)F^{(0)}(t_2)\rangle, \quad \text{(D1)}$$

where, according to Eq. (1.4a),

$$\langle F^{(0)}(t_1)F^{(0)}(t_2)\rangle=\frac{2}{\pi}\int_0^\infty d\omega'\times[\eta(\omega')\cos\omega'(t_1-t_2)-i\xi(\omega')\sin\omega'(t_1-t_2)]. \quad \text{(D2)}$$

If one thinks of the time integration as being carried out before the frequency integration, it is clear that main contribution to the frequency integration will come from the neighborhood of $\omega'=\omega$. Recalling that $\xi(\omega')$ and $\eta(\omega')$ are slowly varying functions, we approximate by replacing them with $\xi(\omega)$ and $\eta(\omega)$, respectively. Equation (D2) then becomes,[3]

$$\langle F^{(0)}(t_1)F^{(0)}(t_2)\rangle\approx2\left[\eta(\omega)\delta(t_1-t_2)-\frac{i}{\pi}\xi(\omega)\frac{\mathcal{P}}{t_1-t_2}\right]. \quad \text{(D3)}$$

We can approximate the second term on the right side of Eq. (D2) somewhat differently than in Eq. (D3). Instead of replacing $\xi(\omega')$ by $\xi(\omega)$, we replace $\xi(\omega')/\omega'$ by $\xi(\omega)/\omega$. Thus, for purposes of later time integration,

$$-\frac{2i}{\pi}\int_0^\infty d\omega'[\xi(\omega')/\omega']\omega'\sin\omega'(t_1-t_2)$$
$$\approx-\frac{2i}{\pi}[\xi(\omega)/\omega]\int_0^\infty d\omega'\omega'\sin\omega'(t_1-t_2)$$
$$=2i[\xi(\omega)/\omega]\delta'(t_1-t_2),$$

and

$$\langle F^{(0)}(t_1)F^{(0)}(t_2)\rangle\approx2\{\eta(\omega)\delta(t_1-t_2)+i[\xi(\omega)/\omega]\delta'(t_1-t_2)\}. \quad \text{(D4)}$$

Equation (D4) is not only simpler to use for some computational purposes than Eq. (D3), but, as has been pointed out by Lax[3], yields the relationship $[q_F(t), p_F(t)]=i$ exactly, while Eq. (D3) yields the same relationship approximately, the approximation being the neglect of ξ/ω compared to unity.

Using either Eq. (D3) or (D4), noting the relationship between η and ξ given by Eq, (1.5), and neglecting terms of the order ξ/ω compared to unity, we obtain

$$\langle q_F(t_1)q_F(t_2)\rangle=\tfrac{1}{2}e^{-\xi|t_1-t_2|}[(1+2\varphi)\cos\omega(t_1-t_2)+i\sin\omega(t_2-t_1)], \quad \text{(D5)}$$

which is Eq. (1.21a). The other relationships in Eqs. (1.21) are obtained similarly.

APPENDIX E

The equation

$$\langle\{e^{iA_F},A^{(2)}\}\rangle=\langle e^{iA_F}\rangle i\langle\{A_F,A^{(2)}\}\rangle, \quad \text{(E1)}$$

which is Eq. (2.14) of the text, will now be derived. We recall that

$$a_F=2^{-1/2}\int_0^t dt_1F^{(0)}(t_1)e^{-i\Omega(t-t_1)}, \quad \text{(E2)}$$

and

$$A=\zeta a+\zeta^*a^\dagger. \quad \text{(E3)}$$

Equations (E2) and (E3) yield

$$A_F(t)=\int_0^t dt_1\psi(t_1)F^{(0)}(t_1)\,, \tag{E4}$$

where

$$\psi(t_1)\equiv 2^{-1/2}[\zeta e^{-i\Omega(t-t_1)}+\zeta^* e^{i\Omega^*(t-t_1)}]\,,$$

and from Eqs. (2.5) we obtain

$$A^{(2)}(t)=\int_0^t dt_{\mathrm{I}}\int_0^{t_{\mathrm{I}}} dt_{\mathrm{II}}\int_0^{t_{\mathrm{II}}} dt_{\mathrm{III}} \times\varphi(t_{\mathrm{I}},t_{\mathrm{II}},t_{\mathrm{III}})F^{(0)}(t_{\mathrm{III}})\,, \tag{E5}$$

where

$$\varphi(t_{\mathrm{I}},t_{\mathrm{II}},t_{\mathrm{III}})=2^{-3/2}\alpha^2[\zeta e^{-i\Omega(t-t_{\mathrm{I}})}+\zeta^* e^{i\Omega^*(t-t_{\mathrm{I}})}] \times[e^{i\Omega^*(t_{\mathrm{II}}-t_{\mathrm{III}})}-e^{-i\Omega(t_{\mathrm{II}}-t_{\mathrm{III}})}][S^{(0)}(t_{\mathrm{I}}),S^{(0)}(t_{\mathrm{II}})].$$

Substituting from Eqs. (E4) and (E5) into the relationship

$$\langle e^{iA_F}A^{(2)}\rangle=\sum_{n=0}^{\infty}\frac{i^n}{n!}\langle A_F{}^nA^{(2)}\rangle\,, \tag{E6}$$

we have

$$\langle e^{iA_F}A^{(2)}\rangle=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_0^t dt_1\cdots\int_0^t dt_n\int_0^t dt_{\mathrm{I}} \times\int_0^{t_{\mathrm{I}}}dt_{\mathrm{II}}\int_0^{t_{\mathrm{II}}}dt_{\mathrm{III}}\psi(t_1)\cdots\psi(t_n)\langle\varphi(t_{\mathrm{I}},t_{\mathrm{II}},t_{\mathrm{III}})\rangle \times\langle F^{(0)}(t_1)\cdots F^{(0)}(t_n)F^{(0)}(t_{\mathrm{III}})\rangle. \tag{E7}$$

From the expansion rule for expectation values of products of $F^{(0)}$'s, Eq. (1.3), we obtain [using the shorthand notation introduced in connection with Eq. (1.6)]

$$\langle 1\cdots n\mathrm{III}\rangle=\sum_{j=1}^{n}\langle j\mathrm{III}\rangle\langle 1\cdots(j-1)(j+1)\cdots n\rangle. \tag{E8}$$

Now, for purposes of integration in Eq. (E7) all t_j's are equivalent, that is, they may be interchanged arbitrarily. We can therefore write under the integral sign

$$\langle 1\cdots n\ \mathrm{III}\rangle=n\langle 1\cdots(n-1)\rangle\langle n\ \mathrm{III}\rangle\,, \tag{E9}$$

so that

$$\begin{aligned}\langle e^{iA_F}A^{(2)}\rangle&=i\sum_{n=1}^{\infty}\frac{i^{n-1}}{(n-1)!}\int_0^t dt_1\cdots\int_0^t dt_n\int_0^t dt_{\mathrm{I}}\times\int_0^{t_{\mathrm{I}}}dt_{\mathrm{II}}\int_0^{t_{\mathrm{II}}}dt_{\mathrm{III}}\psi(t_1)\cdots\psi(t_n)\langle\varphi(t_{\mathrm{I}},t_{\mathrm{II}},t_{\mathrm{III}})\rangle\langle 1\cdots(n-1)\rangle\langle n\mathrm{III}\rangle\,,\\ &=i\sum_{n=1}^{\infty}\frac{i^{n-1}}{(n-1)!}\langle A_F{}^{n-1}\rangle\langle A_FA^{(2)}\rangle\,,\\ &=\langle e^{iA_F}\rangle i\langle A_FA^{(2)}\rangle.\end{aligned} \tag{E10}$$

Similarly, we have

$$\langle A^{(2)}e^{iA_F}\rangle=\langle e^{iA_F}\rangle i\langle A^{(2)}A_F\rangle\,, \tag{E11}$$

and thus obtain Eq. (E1).

APPENDIX F

We consider expressions for $\langle\{S^{(0)}(t_1),S^{(0)}(t_2)\}\rangle$ and $\langle[S^{(0)}(t_1),S^{(0)}(t_2)]\rangle$. For the sake of simplicity, we assume that the source has a discrete spectrum. Let the density matrix ρ describe the state of the source. Using the relationship

$$\langle S_{jk}{}^{(0)}(t)\rangle=S_{jk}e^{i\omega_{jk}t}\,, \tag{F1}$$

where

$$S_{jk}\equiv S_{jk}{}^{(0)}(0),\quad \hbar\omega_{jk}=E_j-E_k\,,$$

one obtains

$$\langle S^{(0)}(t_1)S^{(0)}(t_2)\rangle=\sum_{ijk}\rho_{ij}S_{jk}S_{ki} \times e^{1/2i(\omega_{jk}-\omega_{ik})(t_1+t_2)}e^{1/2i(\omega_{jk}+\omega_{ik})(t_1-t_2)}. \tag{F2}$$

If we set

$$\omega_l=\tfrac{1}{2}|\omega_{jk}+\omega_{ik}|\,, \tag{F3}$$

then it is easily seen that, according to the definitions of Eqs. (3.14),

$$G_l{}^{(+)}(ijk)=\rho_{ij}S_{jk}S_{ki}e^{1/2i(\omega_{jk}-\omega_{ik})(t_1+t_2)}+\text{c.c.} \tag{F4}$$

$$G_l{}^{(-)}(ijk)=\pm\rho_{ij}S_{jk}S_{ki}e^{1/2i(\omega_{jk}-\omega_{ik})(t_1+t_2)}+\text{c.c.} \tag{F5}$$

the plus or minus sign depending on whether $\omega_{jk}+\omega_{ik}$ is positive or negative. The diagonal elements of the density matrix yield constants for $G_l{}^{(+)}$ and $G_l{}^{(-)}$. These are

$$G_l{}^{(+)}(iik)=2\rho_{ii}|S_{ik}|^2,\quad G_l{}^{(-)}(iik)=\pm 2\rho_{ii}|S_{ik}|^2, \tag{F6}$$

with $\omega_l=|\omega_{ik}|$, and the plus or minus sign depending on whether ω_{ik} is positive or negative.