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Dynamical Symmetries and Classical Mechanics*

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It is shown that all classical dynamical problems involving n degrees of freedom automatically possess invariances under the O_{n+1} and SU_n algebras, independent of the functional form of the Hamiltonian. For spherically symmetric systems with three degrees of freedom, the existence of a vector constant of motion is established.

INTRODUCTION

As a result of the many successful applications of group theory to elementary-particle phenomena, renewed attention has been given to an examination of classical dynamical problems from a group theoretical point of view.[1] In this connection, the systems discussed most often are the nonrelativistic Kepler problem, and the three-dimensional isotropic harmonic oscillator. Both these systems possess rotational invariance, but in addition are invariant under the larger groups $O(4)$ and $SU(3)$, respectively. These higher symmetries are usually called accidental symmetries, and for these two systems the corresponding higher symmetries carry over to quantum mechanics.

With respect to classical spherically symmetric potentials, the following remarkable result has recently been proved: All such Hamiltonians automatically possess both an $O(4)$ and an $SU(3)$ symmetry.[2] This has been established for either expression for the kinetic energy, relativistic or nonrelativistic. Thus, in the context of classical mechanics, the existence of these higher symmetries has been shown to be a characteristic of all spherically symmetric potentials, and not really special to the Kepler and oscillator systems. Bacry *et al.* (Ref. 2) have stated that invariance under the Lie algebras of $O(4)$ and $SU(3)$ should obtain for all three-dimensional systems.

In this paper, we give an explicit construction which demonstrates that the above results are valid for all classical dynamical systems with three degrees of freedom. Specifically, we shall show that, given any Hamiltonian as a function of three (generalized) coordinates and their conjugate momenta, we can find (i) a set of constants of motion whose Poisson bracket algebra coincides with the Lie algebra of the group $O(4)$; (ii) a set of constants of motion whose Poisson bracket algebra coincides with the Lie algebra of the group $SU(3)$. The existence of these symmetries will be related to the fact that there are three degrees of freedom; and these results will be generalized to systems involving an arbitrary, but finite, number of degrees of freedom. For the special class of systems possessing spherical symmetry (but whose Hamiltonians are not necessarily of the form "kinetic energy" plus "potential energy") we will demonstrate the existence of a vector constant of the motion. We will throughout restrict attention to Hamiltonians not explicitly dependent on the time.

I. SYMMETRIES OF A GENERAL HAMILTONIAN SYSTEM

We consider a classical mechanical system described by n canonical coordinates q_i, and their conjugate momenta, p_i, $i=1, 2, \cdots, n$. The Hamiltonian of the system is a function of the q's and p's:

$$H=H(q_i,p_i). \tag{1}$$

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[1] A. Loinger, Ann. Phys. (N. Y.) **20**, 132 (1962); E. C. G. Sudarshan, University of Rochester Report No. NYO-10250, 1962 (unpublished); H. Bacry, Nuovo Cimento **41A**, 221 (1966); E. C. G. Sudarshan, and N. Mukunda, *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colorado, 1966), Vol. VIII-B.
[2] D. M. Fradkin, Progr. Theoret. Phys. (Kyoto) (to be published); H. Bacry, H. Ruegg, and J. M. Souriau, Commun. Math. Phys. (Germany) **3**, 323 (1966).

Given such a Hamiltonian, we can ask whether there exists a dynamical variable (that is, a function of q_i and p_i) canonically conjugate to H, namely a variable $\Omega(q_i,p_i)$ obeying the equation

$$\{H,\Omega\}\equiv\sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\frac{\partial\Omega}{\partial p_i}-\frac{\partial H}{\partial p_i}\frac{\partial\Omega}{\partial q_i}\right)=1 \qquad (2)$$

and whether H and Ω can be chosen as the first pair of canonical variables of a new canonical set.[3] The answer to this question is in the affirmative, as is clear from the following theorem concerning functions of $2n$ canonical variables q_i,p_i[4]:

Theorem: Given $2\nu+\kappa$ $(\leq 2n)$ independent functions

$$\begin{aligned}&\psi_1(q_i,p_i)\cdots\psi_\nu(q_i,p_i),\\&\phi_1(q_i,p_i)\cdots\phi_\nu(q_i,p_i)\cdots\phi_{\nu+\kappa}(q_i,p_i),\end{aligned} \qquad (3)$$

obeying the Poisson bracket relations:

$$\begin{aligned}&\{\psi_\alpha,\psi_\beta\}\equiv\sum_{i=1}^{n}\left(\frac{\partial\psi_\alpha}{\partial q_i}\frac{\partial\psi_\beta}{\partial p_i}-\frac{\partial\psi_\alpha}{\partial p_i}\frac{\partial\psi_\beta}{\partial q_i}\right)=0 \quad \alpha,\beta\leqslant\nu\\&\{\phi_\alpha,\phi_\beta\}=0 \quad \alpha,\beta\leqslant\nu+\kappa\\&\{\psi_\alpha,\phi_\beta\}=\delta_{\alpha\beta} \quad \alpha,\beta\leqslant\nu\\&\qquad\quad=0 \quad \alpha\leqslant\nu,\ \nu<\beta\leqslant\nu+\kappa\end{aligned} \qquad (4)$$

one can always find additional functions

$$\begin{aligned}&\psi_{\nu+1}(q_i,p_i)\cdots\psi_n(q_ip_i),\\&\phi_{\nu+\kappa+1}(q_i,p_i)\cdots\phi_n(q_i,p_i),\end{aligned} \qquad (5)$$

such that

$$\begin{aligned}&\{\psi_\alpha,\psi_\beta\}=0\\&\{\phi_\alpha,\phi_\beta\}=0,\\&\{\psi_\alpha,\phi_\beta\}=\delta_{\alpha\beta}, \quad \alpha,\beta\leq n.\end{aligned} \qquad (6)$$

We remark that the $2n$ functions ψ_α, ϕ_α so determined will be functionally independent, and the passage from the variables q_i, p_i to the variables ψ_α, ϕ_α constitutes a canonical transformation.

Let us apply this theorem to the case $\nu=0$, $\kappa=+1$. We choose the (negative of the) Hamiltonian H as the function $\phi_{\nu+1}\equiv\phi_1(q,p)$ of the theorem. Then we are assured of the existence of additional functions Ω, $Q_1, Q_2, \cdots, Q_{n-1}, P_1, P_2, \cdots, P_{n-1}$ of q_i, p_i, obeying the Poisson bracket relations[5]:

$$\begin{aligned}&\{H,\Omega\}=+1,\\&\{H,Q_i\}=\{H,P_i\}=\{\Omega,Q_i\}=\{\Omega,P_i\}=0, \quad i\leq n-1\\&\{Q_i,Q_j\}=\{P_i,P_j\}=0, \quad i,j\leq n-1\\&\{Q_i,P_j\}=\delta_{ij}, \quad i,j\leq n-1.\end{aligned} \qquad (7)$$

The functions H, Ω, Q_i, P_i will all be functionally independent. Before returning to the case $n=3$, we note two facts. (i) For any system with n degrees of freedom, there always exist $2n$-1 time-independent dynamical variables (including the Hamiltonian), which are constants of the motion, and also independent functions[3]; (ii) since any Hamiltonian can be made the first coordinate of a canonical set of variables, and since the transition from one canonical set to another is by definition a canonical transformation, any two Hamiltonians involving the same number of degrees of freedom are (at least locally) canonical transforms of one another.[6]

Restricting ourselves to the case $n=3$, we have the additional independent constants of motion.

$$Q_1,\ Q_2,\ P_1,\ P_2 \qquad (8)$$

and every other constant of motion is some function of H, Q_i, and P_i. We will now show how to construct generators for the Lie algebras of the groups $O(4)$ and $SU(3)$ out of functions of H, Q_i, and P_i.[7]

A. Generators of $O(4)$

The Lie bracket relations for the group $O(4)$ are

$$\begin{aligned}&\{L_i,L_j\}=\epsilon_{ijk}L_k,\\&\{L_i,A_j\}=\epsilon_{ijk}A_k,\\&\{A_i,A_j\}=\epsilon_{ijk}L_k, \quad i,j,k=1,2,3.\end{aligned} \qquad (9)$$

We seek functions of q_i, p_i for the generators L_i, A_i, and interpret the brackets on the left-hand side of (9) as Poisson brackets. The L_i generate an $O(3)$ subalgebra of the $O(4)$ algebra, while the A_i transform as a vector with respect to this $O(3)$ algebra. It is more convenient to write (9) in the following form:

$$\begin{aligned}&M_i=\tfrac{1}{2}(L_i+A_i), \quad N_i=\tfrac{1}{2}(L_i-A_i),\\&\{M_i,M_j\}=\epsilon_{ijk}M_k, \quad \{Ni,N_j\}=\epsilon_{ijk}N_k,\\&\{M_i,N_j\}=0,\end{aligned} \qquad (10)$$

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[3] All these considerations are of a local nature; for instance, if there was a point in phase space where H had an absolute minimum, and if Eq. (2) were to remain valid in the neighborhood of that point, we expect Ω to become singular or undefined there.

[4] See, for instance, L. P. Eisenhart, *Continuous Groups of Transformation* (Dover Publications, Inc., New York, 1963), Chap. VI, pp. 281–291.

[5] Strictly speaking, all this is possible as long as we are not in a region of phase space where H is stationary, i.e., where all the derivatives $\partial H/\partial q_i$, $\partial H/\partial p_i$ vanish. We will explicitly exclude this possibility.

[6] E. C. G. Sudarshan, University of Rochester Report No. NYO-9680, 1961 (unpublished). In this connection it must be remembered that the most general canonical transformation consists of an arbitrary change of variables from one set of coordinates q_r, p_r to a new set Q_r, P_r, the latter being arbitrary functions of the former subject only to the requirement that the Poisson brackets of the new coordinates with one another have the standard values. Under such a general canonical transformation, the possible values of an old coordinate and the corresponding new one may not be the same, as is clear from the transformation

$$[q,\ p\rightarrow\tfrac{1}{2}(q^2+p^2),\ \tan^{-1}(q/p)],$$

which is familiar from the theory of the harmonic oscillator.

[7] We concern ourselves only with constructing realizations of Lie algebras, not of the Lie groups. Even when we talk of invariance under the $O(4)$ group, for example, we really intend invariance under the algebra.

making explicit the $O(3)\times O(3)$ structure of the $O(4)$ algebra.

A particularly simple solution for M_i, N_i as functions of H, Q_1, Q_2, P_1, and P_2 is the following:

$$\begin{aligned} M_1&=(j_1^2-Q_1^2)^{1/2}\sin P_1,\\ M_2&=(j_1^2-Q_1^2)^{1/2}\cos P_1,\\ M_3&=Q_1,\\ N_1&=(j_2^2-Q_2^2)^{1/2}\sin P_2,\\ N_2&=(j_2^2-Q_2^2)^{1/2}\cos P_2,\\ N_3&=Q_2. \end{aligned} \qquad (11)$$

Here we may choose j_1^2 and j_2^2 to be any two real positive functions of the Hamiltonian H. That the expressions (11) obey the Poisson bracket relations (10) may be verified by using the basic Poisson bracket relations of Eq. (7).

An algebraically different-looking solution for the generators of $O(4)$ will be described later on.

B. Generators of $SU(3)$

The Lie bracket relations for the generators of $SU(3)$ are

$$-i\{A_i{}^j,A_k{}^l\}=\delta_k{}^jA_i{}^l-\delta_i{}^lA_k{}^j,$$

$$(A_i{}^j)^*=A_j{}^i,\quad \sum_{i=1}^{3}A_i{}^i=0. \qquad (12)$$

A particular solution for the $A_i{}^j$ may be obtained as follows: Define the variables

$$a_i=(P_i-iQ_i)/\sqrt{2},\quad a_i^*=(P_i+iQ_i)/\sqrt{2},\quad i=1,2$$

$$N=\sum_{i=1}^{2}a_i^*a_i. \qquad (13)$$

The a_i obey the Poisson bracket relations

$$\{a_i,a_j\}=0,\quad \{a_i,a_j^*\}=-i\delta_{ij}. \qquad (14)$$

Set

$$\begin{aligned} &A_i{}^j=a_j^*a_i-\lambda/3\delta_i{}^j,\quad ij=1,2\\ &A_i{}^3=(\lambda-N)^{1/2}a_i,\quad A_3{}^i=(\lambda-N)^{1/2}a_i^*,\\ &A_3{}^3=\tfrac{2}{3}\lambda-N. \end{aligned} \qquad (15)$$

Here λ can be chosen to be any (real, positive) function of the Hamiltonian H. The Poisson bracket relations (12) can now be checked to be true, by using the basic relations (7).

We have thus shown the existence of both $O(4)$ and $SU(3)$ symmetries for any system with three degrees of freedom. The solution (15) for the $SU(3)$ generators can be thought of as the generators of the *noninvariance group*[8] $SU(3)$ for an isotropic two-dimensional harmonic oscillator in the variables Q_1, Q_2, P_1, and P_2. While the generators for the $O(4)$ algebra given by Eq. (11) are a particular solution to that problem, one could have proceeded in a slightly different fashion. One could consider a Hamiltonian for the two-dimensional Kepler problem in the variables Q_1, Q_2, P_1, and P_2, and one could then construct the generators of the *noninvariance group* $O(4)$ for this system. [Note that though in both the $SU(3)$ case and the $O(4)$ case one considers the noninvariance group for an auxiliary Hamiltonian in the variables Q_1, Q_2, P_1, and P_2, one ends up with generators of *invariance groups* for the original Hamiltonian, since all these generators are functions of Q_1, Q_2, P_1, P_2 (and H), alone]. This method of construction of invariance groups permits an immediate extension to systems with any number n of degrees of freedom. According to the theorem quoted earlier, there are in the general case the $2n$-2 additional constants of motion $Q_1\cdots P_{n-1}$. By considering an $(n-1)$ dimensional isotropic harmonic-oscillator Hamiltonian in the variables $Q_1\cdots P_{n-1}$, we could construct the generators of the noninvariance group $SU(n)$ for this system. Similarly, by considering a Kepler-type Hamiltonian in $(n-1)$ dimensions in $Q_1\cdots P_{n-1}$, we could construct the generators of the noninvariance group $O(n+1)$ for this system. As before, these generators give rise to *invariance* groups of the original Hamiltonian H. Thus we reach the conclusion that all systems with n degrees of freedom certainly possess invariance under both the $SU(n)$ and the $O(n+1)$ groups. (Strictly speaking, we have only realizations of the Lie algebras of these groups by means of functions that are constants of motion; in general it does not immediately follow that this can be used to generate finite canonical transformations yielding a realization of the group).

It appears plausible from the above that there is a relation between the number of canonical degrees of freedom, and the Lie algebras for which we may obtain realizations in terms of functions of those canonical variables. For example, we can show that a Hamiltonian system with three degrees of freedom cannot possess invariance under the groups $O(5)$ or G_2. [These algebras, with $O(4)$ and $SU(3)$, are the only four semi-simple Lie algebras of rank two.] More details on this and related points will be discussed elsewhere.[9]

II. SYSTEM WITH SPHERICAL SYMMETRY

We now consider Hamiltonians involving three degrees of freedom, and which are spherically symmetric. We denote for definiteness the Cartesian

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[8] H. Bacry (Ref. 1); E. C. G. Sudarshan and N. Mukunda (Ref. 1); N. Mukunda, L. O'Raifeartaigh, and E. C. G. Sudarshan, Phys. Rev. Letters **15**, 1041 (1965); E. C. G. Sudarshan, N. Mukunda, and L. O'Raifeartaigh, Phys. Letters **19**, 322 (1965).

[9] N. Mukunda, J. Math. Phys. (to be published); P. Chand, C. L. Mehta, N. Mukunda, and E. C. G. Sudarshan (to be published).

coordinates and momenta by q_i, p_i, respectively. In this case, we can prove on general grounds the existence of three constants of motion which transform as a vector under rotations, and are linearly independent of the angular momenta. No assumption is made for the functional form of the Hamiltonian beyond its rotational invariance; for instance, it need not have the form

$$p^2/2m+V(q) \tag{16}$$

or the relativistic analog of this expression. In previous work, the existence of such a vector constant of motion for spherically symmetric potentials was used as a starting point for the construction of the symmetries $O(4)$ and $SU(3)$ of the Hamiltonian.[2]

We note first that for a spherically symmetric H, we can choose the canonically conjugate variable Ω to be likewise spherically symmetric. (In practice this may be achieved by starting with any variable canonically conjugate to H, and then averaging it over the rotation group.) The generators of rotations are

$$L_i=\epsilon_{ijk}q_jp_k \tag{17}$$

and obey the Poisson bracket relations

$$\{L_i,L_j\}=\epsilon_{ijk}L_k. \tag{18}$$

In addition we have

$$\{L_i,H\}=\{L_i,\Omega\}=\{L_i,L^2\}=0, \qquad L^2=L_1^2+L_2^2+L_3^2. \tag{19}$$

Now consider the five functions of q_ip_i:

$$H,\ \Omega,\ L_3,\ \tan^{-1}(L_1/L_2),\ L. \tag{20}$$

We may use them in the theorem in Sec. I, taking the case $\nu=2$, $\kappa=1$, $n=3$, and setting

$$\psi_1=H,\quad \psi_2=L_3, \qquad \Phi_1=\Omega,\quad \Phi_2=\tan^{-1}(L_1/L_2),\quad \Phi_3=L. \tag{21}$$

One may verify that all the conditions of the theorem are satisfied; therefore the existence of a variable $\psi_3(\mathbf{q},\mathbf{p})$ canonically conjugate to L is guaranteed. It obeys the Poisson bracket relations

$$\{\psi_3,H\}=\{\psi_3,\Omega\}=\{\psi_3,L_3\}=\{\psi_3,\tan^{-1}(L_1/L_2)\}=0, \qquad \{\psi_3,L\}=+1. \tag{22}$$

With the help of ψ_3, we construct the following three quantities:

$$K_1=-\frac{LL_2\tan\psi_3+L_1L_3}{L(L^2-L_3^2)^{1/2}\sec\psi_3},\qquad K_2=\frac{LL_1\tan\psi_3+L_2L_3}{L(L^2-L_3^2)^{1/2}\sec\psi_3},\qquad K_3=-\frac{(L^2-L_3^2)^{1/2}}{L}\cos\psi_3. \tag{23}$$

It is clear that the K_i's are constants of the motion. In addition one finds

$$\{L_i,K_j\}=\epsilon_{ijk}K_k, \qquad \{K_i,K_j\}=0, \qquad K^2=1,\quad \mathbf{L}\cdot\mathbf{K}=0. \tag{24}$$

Thus the K_i transform as a vector under rotations, and together with the angular momenta L_i they generate the algebra of the Euclidean group $E(3)$ in three dimensions.

III. CONCLUSIONS

It has been shown that all classical mechanical problems involving three degrees of freedom automatically possess both $O(4)$ and $SU(3)$ symmetry. For centrally symmetric systems, a vector constant of motion (distinct from the angular momentum) has been explicitly constructed. It seems clear that these facts are purely consequences of rather detailed properties of Poisson brackets which are the Lie brackets relevant for classical mechanical realizations of Lie algebras. Therefore, it is unlikely that these symmetries survive when one makes the transition to quantum mechanics, except for some special systems. It would be interesting to explore the extra conditions that must be imposed on a classical Hamiltonian in order that some of the higher symmetries carry over to the quantum-mechanical problem. It is also interesting to examine under what circumstances a Poisson bracket realization of a Lie algebra could lead to a realization of the corresponding group by finite canonical transformations, and whether this is related to the quantum-mechanical problem.

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